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Abstract

The A. d. s. Salam International Center for Theoretical Physics, Trieste, Italy.

Aissa Wade

A GENERALIZATION OF POISSON-NIJENHUIS STRUCTURES

THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

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1 Introduction

Jacobi structures, which are natural generalizations of Poisson structures, have been studied by A. Lichnerowicz and his collaborators [L], [D-L-M], [G-L], etc. A Jacobi structure on a manifold $M$ is defined by a pair $(\Lambda, E)$, where $\Lambda$ is a bivector field, $E$ is a vector field such that $[E, \Lambda] = 0$ and $[\Lambda, \Lambda] = 2E \wedge \Lambda$. Two Jacobi structures $(\Lambda_1, E_1)$ and $(\Lambda_2, E_2)$ are said to be compatible if $(\Lambda_1 + \Lambda_2, E_1 + E_2)$ is also a Jacobi structure (see [N]). Here, we give compatibility conditions between a Jacobi structure $(\Lambda, E)$ and a $(1,1)$-tensor field $J$ whose Nijenhuis torsion $N_J$ vanishes ($J$ is called a Nijenhuis tensor). When these compatibility conditions are satisfied, we get another Jacobi structure denoted by $(J\Lambda, JE)$, which is compatible with $(\Lambda, E)$. These conditions generalize the notion of Poisson-Nijenhuis structures introduced by Magri in [M-M]. Recently, J. Monterde et al. (see [M-M-P]) considered Jacobi-Nijenhuis structures. This work contributes to further generalization of Jacobi-Nijenhuis structures.

Poisson-Nijenhuis structures play a central role in the study of integrable systems. In [V], the author defined the Poisson-Nijenhuis structures in the general algebraic framework of Gel’fand and Dorfman. Moreover, Y. Kosmann-Schwarzbach gave in [K] a characterization of Poisson-Nijenhuis structures in terms of Lie algebroids. Another one is given in [B-M].

The paper is organized as follows. In Section 2, we recall some definitions and basic results about Jacobi structures. Furthermore, inspired by the construction of Magri et al. (see [C-M-P]), we establish that certain compatible Jacobi structures define a sequence of functions in involution.

In Section 3, we give necessary and sufficient conditions for a Nijenhuis tensor $J$ and a Jacobi structure $(\Lambda, E)$ to define, in a natural way, a new Jacobi structure which is compatible with $(\Lambda, E)$. Moreover, we prove that the main property of the Poisson-Nijenhuis manifolds holds for the Jacobi ones endowed with a compatible Nijenhuis tensor. Namely, they determine a sequence of Jacobi structures which are pairwise compatible (see Theorem 3.9).

Section 4 is devoted to the analysis of homogeneous Poisson structures, which are compatible with a Nijenhuis tensor. Such structures are called homogeneous Poisson-Nijenhuis structures. It is well known that homogeneous Poisson structures are related to Jacobi ones, their relations being already established in [D-L-M]. We give sufficient conditions to have homogeneous Poisson-Nijenhuis structures and deduce some consequences for Jacobi structures.

2 Preliminaries

In the sequel, all manifolds, multi-vector fields and forms are assumed to be differentiable of class $C^\infty$. 

\[\text{2 Preliminaries}\]
2.1 Jacobi structures

Definition 2.1 A Jacobi manifold $(M, \{ , \})$ is a manifold $M$ equipped with a $\mathbb{R}$-bilinear and skew-symmetric map $\{ , \} : C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$, called a Jacobi bracket, which satisfies the following properties:

1) the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \quad \forall f, g, h \in C^\infty(M, \mathbb{R});$$

2) the bracket is local (i.e. the support of $\{f, g\}$ is a subset of the intersection of the supports of $f$ and $g$).

The definition of a Jacobi structure is equivalent to giving a pair $(\Lambda, E)$ formed by a bivector field $\Lambda$ and a vector field $E$ such that

$$[E, \Lambda] = 0 \quad \text{and} \quad [\Lambda, \Lambda] = 2E \wedge \Lambda,$$

where $[ , ]$ is the Schouten-Nijenhuis bracket on the space of multivector fields (see [Kz]). The Jacobi bracket is then given by

$$\{f, g\} = \Lambda(df, dg) + \langle fdg - gdf, E \rangle.$$

When $E$ is zero, we obtain a Poisson structure. In other words, a Poisson structure on a manifold $M$ is given by a bivector field $\Lambda$ such that the Schouten-Nijenhuis bracket $[\Lambda, \Lambda]$ vanishes. Then $(M, \Lambda)$ is called a Poisson manifold. In [L], Lichnerowicz has shown that to any Jacobi structure $(\Lambda, E)$ on a manifold $M$, one may associate a Poisson structure $\pi$ on $M \times \mathbb{R}$ defined by

$$\pi(x, t) = e^{-t}(\Lambda(x) + \frac{\partial}{\partial t} \wedge E).$$

Then, $\pi$ is called the Poissonization of $(\Lambda, E)$. Let us recall other examples of Jacobi structures (see [L] for example).

Example 1: locally conformal symplectic manifolds. Let $M$ be a $2n$-dimensional manifold. A locally conformal symplectic structure on $M$ is given by a pair $(F, \omega)$, where $F$ is a nondegenerate 2-form and $\omega$ is 1-form such that

$$d\omega = 0 \quad \text{and} \quad dF + \omega \wedge F = 0.$$ 

We define a bivector field $\Lambda$ and a vector field $E$ by:

$$i_E F = \omega \quad \text{and} \quad i_\Lambda F = -\alpha.$$
Then \((\Lambda, E)\) defines a Jacobi structure. In fact, for any \(x \in M\), there exist a neighborhood \(U_x\) and a function \(f\) defined on \(U_x\) such that \(\omega = df\) and \(\Omega = e^F\) is symplectic.

**Example 2: contact manifolds.** Let \(M\) be a \((2n+1)\)-dimensional manifold. A differential 1-form \(\eta\) on \(M\) defines a contact structure if \(\eta \wedge (d\eta)^n\) does not vanish at any point of \(M\). So, the map \(\varphi : \chi(M) \to \Omega^1(M)\) defined by \(\varphi(X) = i_X d\eta + \eta(X)\eta\) is an isomorphism of \(C^\infty(M, \mathbb{R})\)-modules, where \(\chi(M)\) is the space of vector fields and \(\Omega^1(M)\) is the space of differential 1-forms on \(M\). Consider the vector field \(E\) and the bivector field \(\Lambda\) such that

\[
\Lambda(\alpha, \beta) = d\eta(\varphi^{-1}(\alpha), \varphi^{-1}(\beta)) \quad \text{and} \quad E = \varphi^{-1}(\eta).
\]

The pair \((\Lambda, E)\) defines a Jacobi structure on \(M\).

### 2.2 Characteristic distribution of a Jacobi manifold

Let \((M, \Lambda, E)\) be a Jacobi manifold. For any \(f \in C^\infty(M, \mathbb{R})\), the vector field given by

\[
X_f = \Lambda(df) + fE
\]

is called Hamiltonian vector field associated with \(f\). We have the following proposition (see [G-L]):

**Proposition 2.2** The pair \((\Lambda, E)\) defines a Jacobi structure on \(M\) if and only if

\[
X_{\{f, g\}} = [X_f, X_g], \quad \forall \ f, g \in C^\infty(M, \mathbb{R}),
\]

where \(\{f, g\} = \Lambda(f, g) + fE(df) - gE(dg)\). Moreover,

\[
X_f = 0 \iff \{f, g\} = 0, \quad \forall g \in C^\infty(M, \mathbb{R}).
\]

The characteristic distribution of a Jacobi manifold \((M, \Lambda, E)\) is the subbundle \(C\) of \(TM\) spanned by all the Hamiltonian vectors fields. Thus, \(C_x = \text{Span}\{E(x), (\Lambda\alpha)(x), \alpha\text{ is a 1-form}\}\) is the fiber at the point \(x\). The characteristic distribution of \((M, \Lambda, E)\) is completely integrable in the sense of Stefan-Sussmann (see [St] [Su]); it defines a singular foliation on \(M\). The leaves of this foliation are contact manifolds or locally conformal symplectic manifolds, according to their dimension.

A Jacobi structure is said to be transitive if \(C = TM\). It is known (see [L], [G-L]) that a transitive Jacobi manifold is either a contact manifold (when its dimension is odd) or a locally conformal symplectic manifold (when its dimension is even).
2.3 Jacobi pencils

A manifold \( M \) is said to be a bihamiltonian manifold if \( M \) is endowed with two Poisson tensors \( \pi_1 \) and \( \pi_2 \) such that \( \pi_1 - \lambda \pi_2 \) is a Poisson tensor for any \( \lambda \in \mathbb{R} \). Then \( \pi_1 - \lambda \pi_2 \) is called a Poisson pencil. By analogy, if \( \{.,.\}_1 \) and \( \{.,.\}_2 \) are two Jacobi structures such that \( \{.,.\}_\lambda = \{.,.\}_1 - \lambda \{.,.\}_2 \), defines a Jacobi structure for any \( \lambda \) in \( \mathbb{R} \), then \( \{.,.\}_\lambda \) will be called Jacobi pencil. In this case, the two Jacobi structures are said to be compatible.

**Proposition 2.3** (see [N]) Let \((\Lambda_1,E_1)\) and \((\Lambda_2,E_2)\) be two Jacobi structures on \( M \). Denote by \( \pi_i = e^{-t}(\Lambda_i + \partial / \partial t \wedge E_i) \), with \( i = 1, 2 \), the associated Poisson tensors on \( M \times \mathbb{R} \). Then the following assertions are equivalent:

1. \((\Lambda_1,E_1)\) and \((\Lambda_2,E_2)\) define a Jacobi pencil on \( M \).
2. \([\Lambda_1,\Lambda_2] = E_1 \wedge \Lambda_2 + E_2 \wedge \Lambda_1 \) and \([E_1,\Lambda_2] + [E_2,\Lambda_1] = 0\).
3. The pair \((\pi_1,\pi_2)\) defines a Poisson pencil on \( M \times \mathbb{R} \).

From the classical Liouville theory, it follows that the integrability of a Hamiltonian system is related to the number and the independence of its first integrals in involution (i.e. commuting first integrals). Therefore, the methods of construction of functions in involution play an important role in integrable systems. We shall see that the one given in [C-M-P] using the Casimir of a Poisson pencil holds for Jacobi structures. Denoting by \( N[[\lambda]] = C^\infty(M,\mathbb{R}) \otimes \mathbb{R}[[\lambda]] \) the space of formal power series in \( \lambda \) over \( C^\infty(M,\mathbb{R}) \), we may extend a Jacobi bracket \( \{.,.\}_i \) defined on \( C^\infty(M,\mathbb{R}) \) to \( N[[\lambda]] \) by

\[
\left\{ \sum_{i=0}^\infty \lambda^i f_i, \sum_{j=0}^\infty \lambda^j g_j \right\} := \sum_{r=0}^\infty \lambda^r \left( \sum_{p+q=r} \left\{ f_p, g_q \right\} \right).
\]

Now, assume that \( \{.,.\}_\lambda = \{.,.\}_1 - \lambda \{.,.\}_2 \) is a Jacobi pencil. If \( (\Lambda_j,E_j) \), with \( j = 1, 2 \), are the tensors associated to the Jacobi brackets \( \{.,.\}_j \), we consider the mapping \( \sigma_\lambda \) defined by

\[
\sigma_\lambda f = (\Lambda_1 - \lambda \Lambda_2) df + f (E_1 - \lambda E_2),
\]

which can be extended to \( N[[\lambda]] \). If \( h = \sum_{i=0}^\infty \lambda^i h_i \in N[[\lambda]] \) is such that \( \sigma_\lambda (h) = 0 \), then for any \( f \in C^\infty(M,\mathbb{R}) \) we have

\[
\{ h_{i+1}, f \}_1 = \{ h_i, f \}_2.
\]

We deduce that

\[
\{ h_i, h_{i+j} \}_1 = \{ h_i, h_{i+j} \}_2 = 0, \quad \forall \ i, j.
\]

So this gives a sequence of functions in involution for the Jacobi brackets \( \{.,.\}_i \), with \( \ell = 1, 2 \).
2.4 The Lie algebroid of a Jacobi manifold

It was proven in [Ke-SB] that there is a Lie algebroid associated with an arbitrary Jacobi manifold \((M, \Lambda, E)\). Let us recall that a vector bundle \(A\) over a differentiable manifold \(M\) is said to be a Lie algebroid if there is a Lie bracket \([\, , \,]_A\) on the space \((A)\) of smooth sections of \(A\) and a bundle map \(\varrho : A \to TM\), extended to a map between sections of these bundles, such that

1) \(\varrho([X, Y]_A) = [\varrho(X), \varrho(Y)]\),

2) \([X, fY]_A = f[X, Y]_A + (\varrho(X)f)Y\),

for any \(X, Y\) smooth sections of \(A\) and for any smooth function \(f\) on \(M\). Then \(\varrho\) is called the anchor of the Lie algebroid.

Consider the vector bundle \(T^*M \oplus \mathbb{R}\). The space \((T^*M \oplus \mathbb{R})\) of smooth sections may be identified with \(\Omega^1(M) \times C^\infty(M, \mathbb{R})\). The Lie algebroid associated with a Jacobi manifold \((M, \Lambda, E)\) is \(T^*M \oplus \mathbb{R}\) with the Lie bracket \(\{ , \}_{(\Lambda, E)}\) on \((T^*M \oplus \mathbb{R})\), which is defined by

\[
\{ (\alpha, f), (\beta, g) \}_{(\Lambda, E)} = \left( L_{\Lambda\alpha} \beta - L_{\Lambda\beta} \alpha - d(\Lambda(\alpha, \beta)) + f L_E \beta - gL_E \alpha - i_E (\alpha \wedge \beta),
\right.
\]

\[
\left. -\Lambda(\alpha, \beta) + \Lambda(\alpha, dg) - \Lambda(\beta, df) + f E(df) - gE(dg) \right) ,
\]

where \(d\) is the exterior derivative and \(L_X = di_X + i_X d\) is the Lie derivation by \(X\), for any vector field \(X\). The anchor is given by the map \#_{(\Lambda, E)}\) such that

\[
#_{(\Lambda, E)}(\alpha, f) = \Lambda \alpha + f E.
\]

Notice that we have \#_{(\Lambda, E)}(df, f) = Xf.

**Proposition 2.4** The pair \((\Lambda, E)\) defines a Jacobi structure on \(M\) if and only if

\[
[#_{(\Lambda, E)}(\alpha, f), #_{(\Lambda, E)}(\beta, g)] = #_{(\Lambda, E)} \left( \{ (\alpha, f), (\beta, g) \}_{(\Lambda, E)} \right) .
\]

**Sketch of proof:** The operation \#_{(\Lambda, E)}\) is the unique \(\mathbb{R}\)-bilinear map which satisfies

\[
(R_1) \#_{(\Lambda, E)} \left( \{ (df, f), (dg, g) \}_{(\Lambda, E)} \right) = [#_{(\Lambda, E)}(df, f), #_{(\Lambda, E)}(dg, g)],
\]

\[
(R_2) \#_{(\Lambda, E)} \left( \{ (\alpha, f), h(\beta, g) \}_{(\Lambda, E)} \right) = h \left( #_{(\Lambda, E)} \{ (\alpha, f), (\beta, g) \}_{(\Lambda, E)} \right) + \left( #_{(\Lambda, E)}(\alpha, f) h \right) #_{(\Lambda, E)}(\beta, g),
\]

for any \(\alpha, \beta \in \Omega^1(M)\) and for any smooth functions \(f, g\). Since the map

\[
((\alpha, f), (\beta, g)) \mapsto #_{(\Lambda, E)}(\alpha, f), #_{(\Lambda, E)}(\beta, g)
\]

also satisfies these rules \((R_1)\) and \((R_2)\), they are equal.
Let $J$ be a $(1,1)$-tensor field of $M$. The Nijenhuis torsion $N_J$ of $J$ with respect to the Lie bracket $[.,.]$ on the space $\chi(M)$ of vector fields is defined by

$$N_J(X,Y) = [JX, JY] - J[JX,Y] - J[X, JY] + J^2[X,Y], \quad \forall X, Y \in \chi(M).$$

**Definition 3.1** $J$ is called a Nijenhuis tensor if its Nijenhuis torsion vanishes.

**Notations.** To any bivector field $\Lambda$ on $M$, we may associate the skew-symmetric linear map denoted also by $\Lambda : \Omega^1(M) \to \chi(M)$ and defined by:

$$\langle \beta, \Lambda \alpha \rangle = \langle \alpha \wedge \beta, \Lambda \rangle = \Lambda(\alpha, \beta).$$

Conversely, a linear map $\Lambda : \Omega^1(M) \to \chi(M)$ defines a bivector field on $M$ if and only if

$$\langle \alpha, \Lambda \beta \rangle + \langle \beta, \Lambda \alpha \rangle = 0.$$

In particular, when $J$ is a $(1,1)$-tensor field on $M$ and $\Lambda : \Omega^1(M) \to \chi(M)$ is a linear map, then $J \circ \Lambda$ defines a bivector field if and only if $J \circ \Lambda = \Lambda \circ t^J$. In this case, the associated bivector field is denoted by $J\Lambda$.

Furthermore, any bivector field $\Lambda$ gives a bracket defined on the differential 1-forms by

$$\{\alpha, \beta\}_\Lambda = L_{\Lambda \alpha} \beta - L_{\Lambda \beta} \alpha - d(\Lambda(\alpha, \beta)), \quad \forall \alpha, \beta \in \Omega^1(M),$$

where $L_X$ is the Lie derivation by $X$, for any vector field $X$.

Whenever $J \circ \Lambda = \Lambda \circ t^J$, we denote by $C(\Lambda, J)$ the $\mathbb{R}$-bilinear map given by

$$C(\Lambda, J)(\alpha, \beta) = \{\alpha, \beta\}_\Lambda - \left(\{t^J \alpha, \beta\}_\Lambda + \{\alpha, t^J \beta\}_\Lambda - t^J \{\alpha, \beta\}_\Lambda\right).$$

**Definition 3.2** (see [K-M]) A Poisson-Nijenhuis structure on a manifold $M$ is defined by a Poisson tensor $\pi$ and a Nijenhuis tensor $J$ on $M$ such that

(a) $J \circ \pi = \pi \circ t^J,$

(b) $C(\pi, J) = 0.$

In this case, we say that $\pi$ and $J$ are compatible.

To extend this definition to Jacobi structures, it is natural to think about the Poissonization method but the latter gives a weak generalization (see subsection 3.2). We propose the following definition.

**Definition 3.3** Let $(M, \Lambda, E)$ be a Jacobi manifold. A Nijenhuis tensor $J$ on $M$ is said to be compatible with the Jacobi structure $(\Lambda, E)$ if
(i) \(J \circ \Lambda = \Lambda \circ t J,\)

(ii) \(\Lambda(\alpha, \beta)JE - \Lambda(\alpha, tJ\beta)E = \Lambda(C(\Lambda, J)(\alpha, \beta)), \quad \forall \alpha, \beta \in \Omega^1(M);\)

(iii) \([J^k E, \Lambda] + [E, J^k \Lambda] = 0 \text{ for any } k \in \mathbb{N}^p.\)

When the property (iii) holds only for \(k \leq p,\) and the other properties are satisfied, we will say that \((\Lambda, E)\) are compatible up to the order \(p.\)

When \(E = 0\) (i.e. \(\Lambda\) defines a Poisson structure), the pair \((\Lambda, J)\) is said to be a weak Poisson-Nijenhuis structure (see [M-M-P]). In such a case, the compatibility conditions are reduced to (ii), which includes the one given in [M-M]. In other words, a Poisson-Nijenhuis structure is always a weak Poisson-Nijenhuis structure but the converse is false.

**Theorem 3.4** Let \((\Lambda, E)\) be a Jacobi structure on \(M.\) Assume that \(J\) is a \((1, 1)\)-tensor such that \(J \circ \Lambda = \Lambda \circ t J\) and

\[N_J(\Lambda \alpha + f E, \Lambda \beta + g E) = 0, \quad \forall \alpha, \beta \in \Omega^1(M) \quad \text{and} \quad \forall f, g \in C^\infty(M, \mathbb{R}),\]

where \(N_J\) is the Nijenhuis torsion of \(J.\) Then \((J \Lambda, JE)\) is a Jacobi structure on \(M\) if and only if the following properties are satisfied for all \(\alpha, \beta, \gamma \in \Omega^1(M):\)

(a) \(J([JE, \Lambda] \alpha + [E, J\Lambda] \alpha) = 0,\)

(b) \(\langle t J \gamma, \Lambda(C(\Lambda, J)(\alpha, \beta)) - \Lambda(\alpha, \beta)JE + \Lambda(\alpha, tJ\beta)E \rangle = 0.\)

In particular, if \(J\) is a Nijenhuis tensor compatible with \((\Lambda, E),\) then \((J \Lambda, JE)\) is a Jacobi structure on \(M.\)

The proof of Theorem 3.4 is based on the following three lemmas.

**Lemma 3.5** For any bivector field \(\Lambda,\) we have:

\[\langle \gamma, \Lambda \{\alpha, \beta\}_\Lambda \rangle = \langle \gamma, [\Lambda \alpha, \Lambda \beta] \rangle + \frac{1}{2} [\Lambda, \Lambda](\alpha, \beta, \gamma), \quad \forall \alpha, \beta, \gamma \in \Omega^1(M).\]  

(2)

This formula is proven in [G-D] and [K-M].

**Lemma 3.6** Consider a couple \((\Lambda, E)\) formed by a bivector field \(\Lambda\) and a vector field \(E\) on \(M\) such that \([\Lambda, \Lambda] = 2E \Lambda.\) Then, for any linear map \(J\) on \(\chi(M)\) satisfying \(J \circ \Lambda = \Lambda \circ t J,\) the following formula holds:

\[
\frac{1}{2} [J \Lambda, J\Lambda](\alpha, \beta, \gamma) = (JE \wedge J\Lambda)(\alpha, \beta, \gamma) + \langle J \gamma, \Lambda(C(\Lambda, J)(\alpha, \beta)) \rangle + E(tJ\gamma) \Lambda(\alpha, tJ\beta) - JE(tJ\gamma) \Lambda(\alpha, \beta) - \langle \gamma, N_J(\Lambda \alpha, \Lambda \beta) \rangle.
\]
Proof: We use Lemma 3.5 which gives:

\[
\frac{1}{2} [J\Lambda, J\Lambda](\alpha, \beta, \gamma) = \langle \gamma, C(\Lambda, J)(\alpha, \beta) \rangle - \langle \gamma, N_J(\Lambda\alpha, \Lambda\beta) \rangle.
\]

Next, we add and withdraw the following quantity:

\[
\langle \gamma, J(\Lambda, J)(\alpha, \beta) \rangle.
\]

Using again the relation (2), we obtain

\[
\frac{1}{2} [J\Lambda, J\Lambda](\alpha, \beta, \gamma) = \frac{1}{2} \left( [\Lambda, \Lambda](\alpha, \beta) + [\Lambda, \Lambda](\alpha, \beta) + [\Lambda, \Lambda](\alpha, \beta) \right)
\]

Since \([\Lambda, \Lambda] = 2E \wedge \Lambda \) it turns out that:

\[
\frac{1}{2} [J\Lambda, J\Lambda](\alpha, \beta, \gamma) = \langle \gamma, N_J(\Lambda\alpha, \Lambda\beta) \rangle.
\]

This is the formula wanted. 

Lemma 3.7 Let \( \Lambda \) and \( E \) be respectively a bivector field and a vector field on \( M \). Then the following relation holds for any linear map \( J \) on \( \chi(M) \):

\[
[J, E\Lambda](\alpha, \beta) = \langle \beta, N_J(E\Lambda, \alpha) \rangle + \langle \beta, J[E, \Lambda] \alpha + J[E, \Lambda] \alpha - J^2[E, \Lambda] \alpha \rangle.
\]

Proof: For any bivector field \( \Lambda \) and for all \( \alpha, \beta \in \Omega^1(M) \), we have:

\[
[J, E\Lambda](\alpha, \beta) = L_E(\Lambda(\alpha, \beta)) - \Lambda(L_E \alpha, \beta) - \Lambda(\alpha, L_E \beta).
\]

This is equivalent to the relation

\[
[J, E\Lambda](\alpha, \beta) = L_E(\Lambda(\alpha, \beta)) - \Lambda(L_E \alpha, \beta) - \Lambda(\alpha, L_E \beta).
\]

Using (4), we obtain for any \( \alpha \in \Omega^1(M) \):

\[
[J, E\Lambda](\alpha) = [J, E\Lambda] - J\Lambda L_E \alpha
\]

Replacing \([E, \Lambda] \alpha \) by \([E, \Lambda] \alpha + \Lambda L_E \alpha \), we deduce that

\[
[J, E\Lambda](\alpha) = [J, E\Lambda] + J([E, \Lambda] \alpha - \Lambda L_E \alpha) + J([E, \Lambda] \alpha - J^2[E, \Lambda] \alpha) - J\Lambda L_E \alpha.
\]
Proof of Theorem 3.4: Lemma 3.7 ensures that $[JE, J\Lambda] = 0$ is equivalent to (a). While Lemma 3.6 says that $[J\Lambda, J\Lambda] = 2JE \wedge J\Lambda$ if and only if property (b) is satisfied. So the theorem is proved.

Now, let us express the properties (a) and (b) of Theorem 3.4 using the Lie algebroid associated with the Jacobi structure (see Proposition 2.4).

**Proposition 3.8** Let $(\Lambda, E)$ be a Jacobi structure on $M$ and let $J$ be a $(1,1)$-tensor field such that

$$J \circ \Lambda = \Lambda \circ \ i_J \quad \text{and} \quad N_J(\Lambda\alpha + fE, \Lambda\beta + gE) = 0, \ \forall \alpha, \beta \in \Omega^1(M), \ \forall f, g \in C^\infty(M, \mathbb{R}).$$

Then we have the following equivalences:

(a) is satisfied $\iff [J\Lambda\alpha + fJE, gJE] = \#_{(J\Lambda, J\Lambda)}\{(\alpha, f), (0, g)\} \{J\Lambda, JE\}$

(b) is satisfied $\iff [J\Lambda\alpha, J\Lambda\beta] = \#_{(J\Lambda, J\Lambda)}\{(\alpha, 0), (\beta, 0)\} \{J\Lambda, JE\}$

Proof: we have

$$[J\Lambda\alpha + fJE, gJE] = g[J\Lambda\alpha, JE] + (J\Lambda(\alpha, dg) + \langle f dg - gdf, JE \rangle)JE.$$

On the other hand, we have

$$\#_{(J\Lambda, J\Lambda)}\{(\alpha, f), (0, g)\} \{J\Lambda, JE\} = -gJ\Lambda L_{JE} \alpha + (J\Lambda(\alpha, dg) + \langle f dg - gdf, JE \rangle)JE.$$

We deduce that

$$[J\Lambda\alpha + fJE, gJE] - \#_{(J\Lambda, J\Lambda)}\{(\alpha, f), (0, g)\} \{J\Lambda, JE\} = g\{J\Lambda\alpha, JE\} + J\Lambda L_{JE} \alpha$$

$$= g[J\Lambda, JE] \alpha.$$

But Lemma 3.7 says that

$$[J\Lambda, JE] \alpha = 0 \iff J([JE, \Lambda] \alpha + [E, J\Lambda] \alpha) = 0.$$

Hence we obtain the first equivalence. In the same way, we prove the second equivalence using Lemma 3.6.
3.1 Hierarchy of Jacobi structures

The following theorem is a generalization of a result proved in [M-M] and [K-M]:

**Theorem 3.9** For any Jacobi structure $(\Lambda, E)$ compatible with a Nijenhuis tensor $J$ on $M$ and for each $k \in \mathbb{N}^*$, the pair $(J^k \Lambda, J^k E)$ is a Jacobi structure on $M$. Furthermore for $k_1, k_2 \in \mathbb{N}^*$, $(J^{k_1} \Lambda, J^{k_1} E)$ and $(J^{k_2} \Lambda, J^{k_2} E)$ define a Jacobi pencil.

**Lemma 3.10** Let $J$ be a $(1,1)$-tensor field. Then, we have:

$$N_{J^{k+1}}(X,Y) = N_{J^k}(JX,JY) + J^k\left(N_J(J^kX,Y) + N_J(X,J^kY)\right) - J^2\left(N_{J^{k-1}}(JX,JY) - N_{J^k}(X,Y)\right), \quad \forall X,Y \in \chi(M).$$

The proof of this lemma is straightforward.

**Proof of Theorem 3.9:** Assume that $[J^\ell \Lambda, J^\ell \Lambda] = 2J^\ell E \wedge J^\ell \Lambda$, for any $\ell \leq k$. It follows from Lemma 3.6 that

$$\frac{1}{2}[J^{k+1} \Lambda, J^{k+1} \Lambda](\alpha, \beta, \gamma) = (J^{k+1} E \wedge J^{k+1} \Lambda)(\alpha, \beta, \gamma) + (t \gamma, J^k \Lambda C(J^k \Lambda, J)(\alpha, \beta)) + J^k E(t \gamma, J^k \Lambda(\alpha, \beta)) - J^{k+1} E(t \gamma, J^k \Lambda(\alpha, \beta)).$$

We shall prove that

$$J^k \Lambda C(J^k \Lambda, J)(\alpha, \beta) + J^k \Lambda(\alpha, t J^k E) = J^k \Lambda(\alpha, \beta)J^{k+1} E = 0.$$

In fact, for any bivector field $\Lambda$ and for any linear map $J$ on $\chi(M)$ such that $J \circ \Lambda = \Lambda \circ t J$, the following relation holds (see [M-M]):

$$\langle C(J \Lambda, J)(\alpha, \beta), X \rangle = \langle C(\Lambda, J)(t J \alpha, \beta), X \rangle + \langle \alpha, N_J(\Lambda \beta, X) \rangle.$$  \hspace{1cm} (5)

Hence, we obtain by induction that for any $k \geq 1$,

$$C(J^k \Lambda, J)(\alpha, \beta) = C(\Lambda, J)(t J^k \alpha, \beta).$$  \hspace{1cm} (6)

Since $J$ is compatible with $(\Lambda, E)$, we have

$$\Lambda \left(C(\Lambda, J)(\alpha, \beta)\right) = \Lambda(\alpha, \beta)JE - \Lambda(\alpha, t J^k E).$$  \hspace{1cm} (7)

We deduce that

$$J^k \Lambda C(J^k \Lambda, J)(\alpha, \beta) = J^k C(\Lambda, J)(t J^k \alpha, \beta) = J^k \left(\Lambda(t J^k \alpha, \beta)JE - \Lambda(t J^k \alpha, t J^k \beta)E\right) = J^k \Lambda(\alpha, \beta)J^{k+1} E - J^k \Lambda(\alpha, t J^k E).$$
So, we obtain the relation wanted. The latter implies that
\[ [J^k \Lambda, J^k \Lambda] = 2J^k E \wedge J^k \Lambda \quad \text{for any} \quad k \geq 1. \]

Moreover, replacing \( J \) by \( J^k \) in Lemma 3.7, we obtain:
\[ [J^k E, J^k \Lambda](\alpha, \beta) = \langle \beta, \, N_{J^k}(E, \Lambda \alpha) \rangle + \langle t^k J^k \beta, [J^k E, \Lambda \alpha + [E, J^k \Lambda] \alpha] \rangle. \]

From Lemma 3.10, we obtain by induction that the Nijenhuis torsion of \( J^k \) vanishes for any \( k \geq 1 \). Therefore,
\[ [J^k E, J^k \Lambda] = 0 \quad \text{for any} \quad k \geq 1. \]

Thus, \( (J^k \Lambda, J^k E) \) defines a Jacobi structure for any \( k \geq 1 \).

Now take two different pairs \( (J^{k_1} \Lambda, J^{k_1} E) \) and \( (J^{k_2} \Lambda, J^{k_2} E) \). We shall prove that they determine a Jacobi pencil. For any \( \lambda \in \mathbb{R} \), we have to prove that
\[ [J^{k_1} \Lambda - \lambda J^{k_2} \Lambda, J^{k_1} \Lambda - \lambda J^{k_2} \Lambda] = 2(J^{k_1} E - \lambda J^{k_2} E) \wedge (J^{k_1} \Lambda - \lambda J^{k_2} \Lambda). \]

Since we have
\[ [J^{k_i} \Lambda, J^{k_i} \Lambda] = 2J^{k_i} E \wedge J^{k_i} \Lambda, \quad \forall i = 1, 2, \]
thus we have only to prove that
\[ [J^{k_1} \Lambda, J^{k_2} \Lambda] = J^{k_1} E \wedge J^{k_2} \Lambda + J^{k_2} E \wedge J^{k_1} \Lambda. \]

Assume that \( k_1 = k_2 + \ell \), then we apply \( \ell \) times the result saying that, for arbitrary bivector fields \( \Lambda \) and \( \pi \) on \( M \), for any linear map \( J \) on \( \chi(M) \) the following formula holds (see [M-M]):
\[ [J \Lambda, \pi \Lambda](\alpha, \beta, \gamma) = [\Lambda, \pi \Lambda](\alpha, \beta, \, t J \gamma) + \langle C(\pi, J)(\alpha, \beta), \, \Lambda \beta \rangle \\
- \langle C(\pi, J)(\beta, \gamma), \, \Lambda \alpha \rangle - \langle C(\Lambda, J)(\alpha, \beta), \, \pi \gamma \rangle. \]

We apply this last relation and we calculate by recursion the \( \ell \) quantities \( [J^{k_2+\ell} \Lambda, J^{k_2} \Lambda], \ldots, [J^{k_2+1} \Lambda, J^{k_2} \Lambda] \).

It follows that:
\[ [J^{k_1} \Lambda, J^{k_2} \Lambda](\alpha, \beta, \gamma) = [J^{k_2} \Lambda, J^{k_2} \Lambda](\alpha, \beta, \, t J^{\ell} \gamma) \]
\[ + \sum_{r=1}^{\ell} \langle C(J^{k_2} \Lambda, J)(\alpha, \, t J^{\ell-1} \gamma), \, J^{k_1-r} \Lambda \beta \rangle \\
- \sum_{r=1}^{\ell} \langle C(J^{k_2} \Lambda, J)(\beta, \, t J^{\ell-1} \gamma), \, J^{k_1-r} \Lambda \alpha \rangle \\
- \sum_{r=1}^{\ell} \langle C(J^{k_1-r} \Lambda, J)(\alpha, \beta), \, J^{k_2+r-1} \Lambda \gamma \rangle. \]
Now, we use the relation (6) as well as (7) and the fact $[J^{k_2}\Lambda, J^{k_2}\Lambda] = 2J^{k_2}E \wedge J^{k_2}\Lambda$, we obtain after computations:

$$[J^{k_1}\Lambda, J^{k_2}\Lambda] = J^{k_1}E \wedge J^{k_2}\Lambda + J^{k_2}E \wedge J^{k_1}\Lambda.$$ 

The last step is to show that:

$$[J^{k_1}E - \lambda J^{k_2}E, J^{k_1}\Lambda - \lambda J^{k_2}\Lambda] = 0.$$ 

This is equivalent to showing that $[J^{k_1}E, J^{k_2}\Lambda] + [J^{k_2}E, J^{k_1}\Lambda] = 0$. By hypothesis this relation is true when $k_2 = 1$ and using Lemma 3.7, we can easily show by induction that this formula holds for any $k_1$ and $k_2$. 

Example 3. Let $\omega$ be a closed 1-form and let $F_1, F_2$ be two nondegenerate 2-forms on $M$. Assume that $(F_1, \omega)$ and $(F_2, \omega)$ are locally conformal symplectic structures on $M$. Let $(\Lambda_i, E_i)$ denote the Jacobi structures associated with $(F_i, \omega)$, where $i = 1, 2$. Assume that these two Jacobi structures are compatible. Define the isomorphism of $C^\infty(M, \mathbb{R})$-modules $\beta_i : \chi(M) \to \Omega^1(M)$ by

$$\beta_i(X) = -i_X F_i.$$ 

We have

$$E_i = -\beta_i^{-1}(\omega) \quad \text{and} \quad \Lambda_i \alpha = \beta_i^{-1}(\alpha), \quad \forall \alpha \in \Omega^1(M).$$ 

Then, the $(1, 1)$-tensor field $J = \beta_2^{-1} \circ \beta_1$ is compatible with $(\Lambda_1, E_1)$ at any order. Indeed, for any $x \in M$, there exist a neighborhood $U_x$ and a function $f$ defined on $U_x$ such that $\omega = df$. The 2-forms $\Omega_1 = e^f F_1$ and $\Omega_2 = e^f F_2$ are symplectic and the Poisson tensors associated with $\Omega_1$, $\Omega_2$ are respectively $\pi_1 = e^{-f} \Lambda_1$, $\pi_2 = e^{-f} \Lambda_2$.

We claim that the Jacobi structures $(\Lambda_1, E_1)$ and $(\Lambda_2, E_2)$ are compatible if and only if $\pi_1$ and $\pi_2$ are compatible. Let us prove this claim. Using the properties of the Schouten-Nijenhuis bracket, we get

$$[\pi_1, \pi_2] = e^{-2f} \left( [\Lambda_1, \Lambda_2] - [\Lambda_1, f] \wedge \Lambda_2 - [\Lambda_2, f] \wedge \Lambda_1 \right).$$

Since $E_i = [\Lambda_i, f] = -\Lambda_i(df)$, we have

$$[\pi_1, \pi_2] = e^{-2f} \left( [\Lambda_1, \Lambda_2] - E_1 \wedge \Lambda_2 - E_2 \wedge \Lambda_1 \right).$$

Therefore, $[\pi_1, \pi_2] = 0$ if and only if $[\Lambda_1, \Lambda_2] = E_1 \wedge \Lambda_2 + E_2 \wedge \Lambda_1$. Moreover, we may remark that the Jacobi identity of the Schouten-Nijenhuis bracket gives

$$[[\pi_1, \pi_2], e^f] = -[[\pi_2, e^f], \pi_1] - [[e^f, \pi_1], \pi_2] = -[[\Lambda_2, f], e^{-f} \Lambda_1] - [[f, \Lambda_1], e^{-f} \Lambda_2].$$

13
The fact that \( E_i = [\Lambda_i, f] \) implies 
\[
[[\pi_1, \pi_2], e^f] = -e^{-f}( [E_2, \Lambda_1] + [E_1, \Lambda_2] ) .
\]
Thus, \((\Lambda_i, E_i)_{i=1,2}\) form a Jacobi pencil if and only if the tensors \((\pi_i)_{i=1,2}\) define a Poisson pencil. So, we may deduce that the Nijenhuis torsion of \( J \) vanishes. Furthermore, the sequence \((J^k \pi_1)\) is formed by pairwise compatible Poisson tensors, while \((J^k \Lambda_1, J^k E_1)\) is a sequence of pairwise compatible Jacobi structures.

### 3.2 Compatibility and Poissonization

First, let us see why the method of Poissonization gives a weak generalization. Let \((\Lambda, E)\) be a Jacobi structure and let \( J \) be a \((1,1)\)-tensor field on \( M \). Denote by \( \pi \) the corresponding Poisson tensor on \( M \times \mathbb{R} \). Consider \( \tilde{J} : (M \times \mathbb{R}) \rightarrow (M \times \mathbb{R}) \) an extension of \( J \) of the form
\[
\tilde{J} = J + \alpha_0 \otimes \frac{\partial}{\partial t} + f_0 \, dt \otimes \frac{\partial}{\partial t},
\]
where \( \alpha_0 \in \Omega^1(M) \) and \( f_0 \) is a smooth function on \( M \). On the one hand, the relation \( \tilde{J} \circ \pi = \pi \circ \tilde{J} \) gives a strong condition, which is the following:
\[
JE = \Lambda \alpha_0 + f_0 E .
\]
For instance when \( \Lambda \) is zero, we must have \( JE = f_0 E \). Moreover, if we express the fact that the Nijenhuis torsion of \( \tilde{J} \) vanishes, we have other conditions on \( \alpha_0 \) and \( f_0 \). On the other hand, when \((\pi, J)\) is a Poisson-Nijenhuis structure on \( M \times \mathbb{R} \), we have necessarily the conditions of compatibility \((i), (ii)\) and \((iii)\) of Definition 3.3. Indeed, in such a case, the hierarchy of pairwise compatible Poisson structures \((\tilde{J}^k \pi)\) is given by
\[
\tilde{J}^k \pi = e^{-t}(J^k \Lambda + \frac{\partial}{\partial t} \wedge J^k E) .
\]
We know that \( \tilde{J} \pi \) is a Poisson tensor on \( M \times \mathbb{R} \) if and only if the pair \((J \Lambda, JE)\) is a Jacobi structure on \( M \). Hence, by Theorem 3.4 we have \((i)\) and \((ii)\). Furthermore, \((iii)\) is obtained by using the fact that the Poisson tensors \( \tilde{J}^k \pi \) are compatible with \( \pi \).

Now, let us make precise why many Jacobi-Nijenhuis structures are particular cases of Jacobi structures compatible with a Nijenhuis tensor (see Definition 3.3). For any bivector field \( \Lambda \) (resp. vector field \( E \)), we may define a mapping \( \bar{#}_{(\Lambda, E)} : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \rightarrow \chi(M) \times C^\infty(M, \mathbb{R}) \) by
\[
\bar{#}_{(\Lambda, E)}(\beta, g) = (\Lambda \beta + gE, \langle \beta, E \rangle) .
\]

**Definition 3.11** (see [M-M-P]) Let \( \tilde{J} : \chi(M) \times C^\infty(M, \mathbb{R}) \rightarrow \chi(M) \times C^\infty(M, \mathbb{R}) \) be a \( C^\infty(M, \mathbb{R}) \)-linear map and let \((\Lambda, E)\) be a Jacobi structure on \( M \). The triple \((\Lambda, E, \tilde{J})\) is said to be a
Jacobi-Nijenhuis structure on $M$ if we have $\tilde{J} \circ \#_{(\Lambda, E)} = \#_{(\Lambda, E)} \circ t^{\text{t}} \tilde{J}$ and $(\Lambda_1, E_1)$ is a Jacobi structure compatible with $(\Lambda, E)$, where $\Lambda_1$ and $E_1$ are characterized by the relation

$$\#_{(\Lambda_1, E_1)} = \tilde{J} \circ \#_{(\Lambda, E)}.$$ 

An extension $\tilde{J}$ of an endomorphism $J$ of $\chi(M)$ to $\chi(M \times \mathbb{R})$ is equivalent to giving a $C^\infty(M, \mathbb{R})$-linear map $\tilde{J} : \chi(M) \times C^\infty(M, \mathbb{R}) \to \chi(M) \times C^\infty(M, \mathbb{R})$. When $\tilde{J}$ sends $\{0\} \times C^\infty(M, \mathbb{R})$ to itself, then we may set

$$\tilde{J}(X, 0) = (JX, \langle \alpha_0, X \rangle) \quad \text{and} \quad \tilde{J}(0, 1) = (0, f_0).$$

We get

$$\tilde{J} = J + \alpha_0 \otimes \frac{\partial}{\partial t} + f_0 \frac{\partial}{\partial t}.$$ 

If $(\Lambda, E)$ is a Jacobi structure on $M$ and $\pi$ denotes the corresponding Poisson tensor on $M \times \mathbb{R}$, then

$$\tilde{J} \circ \pi_1 = \pi_1 \circ t^{\text{t}} \tilde{J} \iff \tilde{J} \circ \#_{(\Lambda, E)} = \#_{(\Lambda, E)} \circ t^{\text{t}} \tilde{J},$$

Moreover $(\Lambda, E, \tilde{J})$ is a Jacobi-Nijenhuis structure on $M$ iff $(\pi, \tilde{J})$ is a Poisson-Nijenhuis structure on $M \times \mathbb{R}$.

Suppose $(\Lambda, E, \tilde{J})$ is a Jacobi-Nijenhuis structure on $M$ such that $\tilde{J}(0, 1) = (0, f_0)$. Then, from what we have seen above, we may deduce that the $(1, 1)$-tensor field on $M$, which corresponds to $\tilde{J}$, is compatible with $(\Lambda, E)$.

4 Nijenhuis tensors and homogeneous Poisson structures

**Definition 4.1** A homogeneous Poisson manifold $(M, \pi, Z)$ is a Poisson manifold $(M, \pi)$ with a vector field $Z$ over $M$ such that

$$[Z, \pi] = -\pi.$$ 

**Theorem 4.2** Assume that $(M, \pi, Z)$ is a homogeneous Poisson manifold. Let $J$ be Nijenhuis tensor compatible with $\pi$. Then $(M, J\pi, Z)$ is a homogeneous Poisson manifold if and only if we have the following property

$$\pi \circ (L_Z \circ t^J - t^J \circ L_Z) = 0, \quad (8)$$

where $L_Z = i_Z d + d i_Z$ is the Lie derivation by $Z$. When this property holds, $J\pi - \lambda \pi$ defines a Poisson pencil which is homogeneous with respect to $Z$. 

15
Proof: Taking into account Theorem 3.9, we have only to prove that \([Z, J\pi] = -J\pi\). Let us compute \([Z, J\pi]\). We obtain

\[
[Z, J\pi](df, dg) = L_Z d(J\pi(df, dg)) - J\pi(L_Z df, dg) - J\pi(df, L_Z dg)
\]

\[
= L_Z d(\pi(t^t Jdf, dg)) - \pi(t J L_Z df, dg) - \pi(t Jdf, L_Z dg)
\]

Since

\[
L_Z d(\pi(t^t Jdf, dg)) = [Z, \pi](t^t Jdf, dg) + \pi(L_Z t^t Jdf, dg) + \pi(t Jdf, L_Z dg),
\]

We obtain

\[
[Z, J\pi](df, dg) = [Z, \pi](t^t Jdf, dg) + \pi(L_Z t^t Jdf, dg) - J\pi(L_Z df, dg)
\]

\[
= -\pi(t Jdf, dg) + \pi(L_Z t Jdf, dg) - J\pi(L_Z df, dg)
\]

Hence, the relation \([Z, J\pi] = -J\pi\) is equivalent to the following one:

\[
\pi \circ L_Z \circ t^t J = \pi \circ t^t J \circ L_Z.
\]

This proves the theorem. \(\blacksquare\)

**Definition 4.3** A homogeneous Poisson manifold \((M, \pi, Z)\) equipped with a Nijenhuis tensor \(J\) which is compatible with \(\pi\) and satisfies equation (8) is said to be a homogeneous Poisson-Nijenhuis manifold.

**Corollary 4.4** Let \((M, \pi, J)\) be a Poisson-Nijenhuis manifold. If \(\pi\) is homogeneous with respect to a vector field \(Z\) and if the following property holds

\[
[Z, JX] = J[Z, X], \quad \forall X \in \chi(M),
\]

then the triple \((M, \pi, J)\) is a homogeneous Poisson-Nijenhuis manifold with respect to \(Z\).

**Proof:** We obtain the corollary using the above theorem and the fact that

\[
[Z, JX] = J[Z, X], \quad \forall X \in \chi(M) \iff \quad L_Z \circ t^t J = t^t J \circ L_Z.
\]

**Definition 4.5** A map \(\psi : (M_1, \Lambda_1, E_1) \to (M_2, \Lambda_2, E_2)\) between two Jacobi manifolds is said to be a conformal Jacobi morphism if there exists a function \(a \in C^\infty(M_1, \mathbb{R})\) which vanishes nowhere such that for any \(f, g \in C^\infty(M_2, \mathbb{R})\) we have:

\[
\{a(f \circ \psi), a(g \circ \psi)\}_1 = a(\{f, g\}_2 \circ \psi),
\]

where the brackets \{,\}_1 and \{,\}_2 are the Jacobi brackets associated with \((\Lambda_1, E_1)\) and \((\Lambda_2, E_2)\) respectively.
Homogeneous Poisson manifolds are closely related to Jacobi manifolds and their relations were established in [D-L-M]. In terms of Poisson pencils, we have the following results.

**Proposition 4.6** Let \( \{\ldots\}_\lambda \) be a Jacobi pencil on \( M \), then there exists a Poisson pencil on \( M \times \mathbb{R} \) such that the projection \( P : M \times \mathbb{R} \to M \) is a conformal Jacobi morphism.

**Proof:** If \((\Lambda_i, E_i)\) denotes the Jacobi structure on \( M \) associated to \( \{\ldots\}_\lambda \), with \( i = 1, 2 \); then the Poisson pencil on \( M \times \mathbb{R} \) is given by \( \pi_1 = \lambda \pi_2 \) where

\[
\pi_i(x, t) = e^{-t} \left( \Lambda_i(x) + \frac{\partial}{\partial t} \Lambda_i(x) \right).
\]

One may easily verify that \( P : (M \times \mathbb{R}, \pi_\lambda) \to (M, \{\ldots\}_\lambda) \) is a conformal Jacobi morphism. □

Conversely, we may prove that homogeneous Poisson pencils give Jacobi pencils by using a proof done in [D-L-M]. Precisely we have:

**Proposition 4.7** Let \( \pi_\lambda \) be a homogeneous Poisson pencil on \( M \) with respect to the vector field \( Z \), and let \( N \) be a submanifold of \( M \) of codimension 1 which is transverse to \( Z \). Then there exists a Jacobi pencil on \( N \) such that for any pair of functions \((f, g)\) defined on an open set \( U \) of \( M \), satisfying \( <Z, df> = f \) and \( <Z, dg> = g \), we have

\[
\{f|_NU, g|_NU\}_\lambda = \pi_\lambda(df, dg)|_{NU}.
\]

**Corollary 4.8** Let \((M, \Lambda, E)\) be a Jacobi manifold and let \( J \) be a Nijenhuis tensor on \( M \), which is compatible with \((\Lambda, E)\). Then there exists a sequence of Poisson-Nijenhuis structures \((\pi_k)\) on \( M \times \mathbb{R} \) that the projection \( P_k : (M \times \mathbb{R}, \pi_k) \to (M, \Lambda, E) \) is a conformal Jacobi morphism, for each \( k \geq 1 \).

Conversely, if \((M, \pi, J)\) is a homogeneous Poisson-Nijenhuis manifold with respect to the vector field \( Z \) and if \( N \) is a submanifold of \( M \) of codimension 1, which is transverse to \( Z \), then there exists a sequence of pairwise compatible Jacobi structures on \( N \) determined by \( \pi, Z \) and \( J \).

This corollary is a direct consequence of Theorem 3.9 as well as Propositions 4.6 and 4.7.

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**References**


