Abstract

In this paper we explore some general aspects of the embeddings associated with brane-localized gravity. In particular we show that the consistency of such embeddings can require (or impose) very specific relations between all the involved bulk and brane matter source parameters. Given the fact that the 5-geometry associated with the embedding of a 3-brane with non-zero spatial 3-curvature $k$ into an empty 5-dimensional bulk is not conformal to flat, we find that in the presence of such $k$ non-zero 3-branes the addition of a non-zero cosmological constant to the bulk is then unable, at least in the static case which we study, to provide the exponential suppression of the geometry thought necessary to localize gravity to the brane.

I. GENERAL INTRODUCTORY REMARKS

It has been suggested recently [1–3] that it is possible for our 4-dimensional universe to be a brane embedded in some higher dimensional bulk spacetime whose spacelike extra dimensions need not in fact be as minuscule as their string theory Planck length expectation. And while the original motivation of these studies was an attempt to solve the hierarchy problem, nonetheless the potential existence of any large such extra dimension is a matter of great interest in and of itself.¹ Moreover, Randall and Sundrum [2,3] were able to show that the embedding of our universe into a 5-dimensional anti de Sitter spacetime ($AdS_5$) could then potentially localize gravity to our 4-dimensional world, thereby releasing the extra higher dimension from needing to be tiny. Now while this is a very nice property of the $AdS_5$ embedding, it is important to see just how generic it in fact is, and to what extent the embedding into a 5-dimensional spacetime of any given 4-dimensional set of matter fields in

¹For a recent compilation of some of the rapidly growing literature in this field see e.g. [4].
any given 4-dimensional geometry would in fact then result in a gravity that actually was localized to the 4-dimensional world. Moreover, it is equally necessary to see whether any given such 4 space configuration of matter fields can even be consistently embedded into a higher dimensional space at all. And indeed, in one sense the whole localization issue is initially somewhat puzzling, since gravitating material always produces a gravitational field in the empty space around it, and it is straightforward to produce source configurations in which the associated gravitational potentials can actually grow at distances far from such sources. While an immediately obvious example of such a source might be a uniform density sheet of non-relativistic static gravitating material, a source which produces a constant, non-declining, Newtonian gravitational force away from the sheet when the sheet is immersed in an otherwise flat, empty background, such a non-relativistic gravitational field is just a coordinate artifact, being equivalent to a uniform acceleration in flat space. Thus, as we will show below, an $AdS_5$ embedding of such a sheet does indeed produce a gravity which is localized to the sheet, though as we shall also see, the covariantizing of such a sheet (to then produce a true gravitational field) proves to be instructive, with the consistency of its embedding (even into a source free bulk) being found to only be achievable for very specific brane equations of state. Motivated by this analysis, we shall then extend our study to the case where the static sheet or brane is endowed with a non-zero spatial 3-curvature $k$ (a configuration whose flat embedding leads to a gravitational force which even grows with distance, one which is not a coordinate artifact), to then find that in this (admittedly somewhat idealized) case $AdS_5$ exponentially suppressed localization of the geometry to the brane does not in fact appear to occur.

To begin our analysis it is instructive to recall some general properties of $AdS_5$ spacetime. As well as being constructible as a constant surface in a flat 6-dimensional space, the $AdS_5$ metric can also be given by the convenient form

$$ds^2 = \left(\frac{R^2}{z^2}\right)(dz^2 - dt^2 + d\vec{x}^2)$$

where $R$ is the radius of curvature. To see that this metric is in fact an $AdS_5$ metric, we note that since this metric is conformal to flat, we can explicitly determine its associated curvature by conformally transforming the flat $\eta_{\mu\nu}(x)$ metric according to $\eta_{\mu\nu}(x) \rightarrow \Omega^2(z)\eta_{\mu\nu}(x) = g_{\mu\nu}(x)$ where $\Omega(z) = R/z$. Under such a transformation the initially zero 5-dimensional Ricci tensor is found to transform to

$$R^{\mu\nu} \rightarrow \Omega^{-5} \partial_\rho \partial^\rho (\Omega^3) \delta^{\mu}_\nu / 3 - 3 \Omega^{-1} \partial^\mu \partial_\nu (\Omega^{-1}) = 4 \delta^{\mu}_\nu / R^2.$$  

Moreover, since the Weyl tensor vanishes in geometries which are conformal to flat, the Riemann tensor associated with the metric of Eq. (1) is determinable from its associated Ricci tensor $R_{\mu\nu} = 4g_{\mu\nu}/R^2$ alone, with it then immediately being found to take the form $R_{\lambda\rho\sigma\nu} = -(g_{\nu\rho}g_{\lambda\sigma} - g_{\nu\sigma}g_{\lambda\rho})/R^2$. We thus recognize the spacetime associated with the metric of Eq. (1) to be that of a 5-space with constant negative curvature $K = -1/R^2$, viz. $AdS_5$. As the above analysis shows, we could construct metrics of the form $ds^2 = \left(\frac{R^2}{x^2}\right)(dz^2 - dt^2 + d\vec{x}^2)$ where $x$ is any one of the four spacelike coordinates and still have an $AdS_5$ spacetime. However, because of the signature change in the flat d’Alambertian operator $\partial_\rho \partial^\rho$, the metric $ds^2 = \left(\frac{R^2}{t^2}\right)(dz^2 - dt^2 + d\vec{x}^2)$ would have constant positive curvature $K = +1/R^2$ and thus be a de Sitter rather than an anti de Sitter space. For negative curvature spaces then the multiplying overall factor in the metric of Eq. (1) must only be associated with one of the
spacelike coordinates. Since for such coordinates the transformation \( z = R e^{y/R} \) allows us to rewrite the metric in the form
\[
ds^2 = dy^2 - e^{-2y/R}(dt^2 - d\bar{x}^2)
\]
we see that in \( AdS \) spaces the spatial exponential \( e^{-2y/R} \) factor acts just like its temporal analog \( e^{2t/R} \) in de Sitter spacetimes. Such exponential behavior in \( AdS \) spaces is thus the spatial analog of inflation, with the \( e^{-2y/R} \) factor leading to rapid suppression as we go out in \( y \) away from the 4-dimensional space associated with the metric \( dt^2 - d\bar{x}^2 \). Now since the multiplying factor in the metric of Eq. (1) is quadratic in \( z \) (and uniquely so for \( AdS \) spaces\(^2\)), we can make transformations of either of the forms \( z = Re^{y/R} \) and \( z = Re^{-y/R} \) on the metric of Eq. (1). Thus we can make the transformation \( z = Re^{y/R} \) in the \( y > 0 \) region and the transformation \( z = Re^{-y/R} \) in the \( y < 0 \) region to then give us \( e^{-2|y|/R} \) exponential suppression for every value of \( y \), positive or negative. However, now the two regions in \( y \) will be two separate patches of \( AdS_5 \) with there then necessarily being a discontinuity at \( y = 0 \) where the two patches meet. It is thus at just such a discontinuity that our 4-dimensional universe can be located with our universe then being a 3-brane\(^3\) embedded in a higher dimensional bulk space containing two separate geometrical patches; with gravity potentially then being localized to the brane through the \( e^{-2|y|/R} \) suppression \([2,3]\).\(^4\) It is thus to the implications of the embedding of such brane universes, and to the dynamical interplay of the bulk and the brane entailed by the very fact of such embeddings (even when the bulk itself contains no matter fields at all), to which we now turn, first in a source free flat background and then in \( AdS_5 \).

II. EMBEDDING A BRANE IN AN EMPTY BULK

For the embedding of a homogeneous, isotropic standard 4-dimensional static Robertson-Walker universe with spatial 3-curvature \( k \) and spatial metric tensor \( \gamma_{ij} = (1 + kr^2/4)^{-2}\delta_{ij} \) \((i, j = 1, 2, 3)\) into a 5-dimensional static bulk space the metric is taken to be of the form (a form which is more general than the \( AdS_5 \) metric while including it as a special case)
\[
ds^2 = -n^2(y)dt^2 + a^2(y)\gamma_{ij}dx^i dx^j + dy^2, \tag{4}
\]

\(^2\)Multiplying a flat space metric by any conformal factor will always lead to a new metric which is conformal to flat. However, it is only the choice \( \Omega^2(z) = R^2/z^2 \) which leads to a spacetime with the same maximal number of Killing vectors as the original flat spacetime itself.

\(^3\)The dimensionality of a brane is defined by the number of its spatial dimensions just like that of a sheet of material.

\(^4\)The dependence of the geometry away from the brane on \(|y|\) rather than on \( y \) itself is characteristic of the requirement made in all brane localized gravity studies (this one included) that there is to be a \( y \rightarrow -y \) symmetry of the metric around the \( y = 0 \) brane.
this being the static limit of the embedded cosmological metric which is studied in [5,6].

While in brane-localized studies it is desired to recover standard gravity only in the 4-dimensional world, it is conventional to assume that the full 5-space gravity is given simply by the 5-dimensional Einstein equations (rather than by some more complicated set of 5-dimensional equations), viz.

\[ G_{AB} = R_{AB} - g_{AB}R^C_C/2 = -\kappa_5^2[T_{AB} + T_{\mu\nu}\delta^\mu_A\delta^\nu_B\delta(y)] \]  \hspace{1cm} (5)

where \( T_{AB} \) \((A, B = 0, 1, 2, 3, 5)\) is due to sources in the bulk and \( T_{\mu\nu} \) \((\mu, \nu = 0, 1, 2, 3)\) is due to sources on the \( y = 0 \) brane. For the symmetry of Eq. (4) both of these energy-momentum tensors are given as perfect fluids, viz.

\[
T^A_B = \text{diag}(-\rho_B, P_B, P_B, P_B, P_T), \quad T^\mu_\nu = \text{diag}(-\rho_b, p_b, p_b, p_b)
\]  \hspace{1cm} (6)

\((B \text{ denotes bulk and } b \text{ denotes brane})\). For the metric of Eq. (4) it is straightforward to write the 5-dimensional Einstein equations, with the resulting expressions (see e.g. [5,6]) simplifying to (the prime denotes differentiation with respect to \( y \))

\[
G_{00} = 3e^2f''/2f^2 - 3e^2k/f^2 = -\kappa_5^2e^2[\rho_B + \rho_b\delta(y)]/f
\]  \hspace{1cm} (7)

\[
G_{ij} = [-f''/2 - fe''/e + k]\gamma_{ij} = -\kappa_5^2f[p_B + p_b\delta(y)]\gamma_{ij}
\]  \hspace{1cm} (8)

\[
G_{55} = -3f'e'/2fe + 3k/f = -\kappa_5^2P_T
\]  \hspace{1cm} (9)

when the identification \( a(y) = f^{1/2}(y) \), \( n(y) = e(y)/f^{1/2}(y) \) is made.

While we shall discuss the implications of these equations in various situations below, we note immediately that when the bulk \( T_{AB} \) is set to zero and when the spatial \( i, j \) coordinates are restricted to a flat 2-dimensional \((x, y)\) plane and \( y \) is replaced by the usual spatial \( z \), Eq. (4) then describes a uniform, infinite, flat 2-dimensional sheet of static matter embedded in ordinary empty spacetime, a system whose Newtonian limit is known to correspond to a gravitational potential which grows linearly with \( z \) and a gravitational force \( F(z) \) per unit mass which is independent of \( z \). Moreover, in such a case, the Newtonian gravitational force points toward the \( z = 0 \) sheet no matter which side it is on, with Gauss’ Law yielding \( F(z = 0^+) - F(z = 0^-) = 4\pi G\sigma \) and \( F(z = 0^+) = -F(z = 0^-) = 2\pi G\sigma \) for a sheet of surface matter density \( \sigma \), with the gravitational potential \( \phi = 2\pi G\sigma|z| \) thus being discontinuous across the surface.\(^5\) As such the relation \( F(z = 0^+) - F(z = 0^-) = 4\pi G\sigma \) is a non-relativistic analog of the fully covariant relativistic Israel junction conditions [7] (see e.g. [8] for a recent derivation)

\[
K_{\mu\nu}(y = 0^+) - K_{\mu\nu}(y = 0^-) = -\kappa_5^2(T_{\mu\nu} - q_{\mu\nu}T^\alpha_\alpha/3)
\]  \hspace{1cm} (10)

across a discontinuous surface with normal \( n^A \) where \( q_{AB} = g_{AB} - n_An_B \equiv q_{\mu\nu} \) is the induced metric on the surface and \( K_{\mu\nu} = q^\alpha_\mu q^\beta_\nu n_\beta n_\alpha \) is its extrinsic curvature. We shall thus expect

\(^5\)With brane studies always requiring the gravitational potential to only be a function of \(|z|\), we see that such a requirement nicely dovetails with the constraints of Gauss’ Law.
to see an analog of this Newtonian gravity discontinuity in the treatment of the relativistic case associated with Eq. (4), something to which we now turn.

For the simplest case first of a $k = 0$ spatially flat Robertson-Walker geometry embedded in a source free bulk, on taking the metric coefficient $a(y)$ to be a function of $|y| = y(\theta(y) - \theta(-y))$ (where $|y| = \theta(y) - \theta(-y)$, $|y|^2 = 1$, $|y|^n = 2\delta(y)$), integration of Eq. (7) is then found to yield

$$a^2(y) = \alpha(1 - \kappa_5^2\rho_b|y|/3)$$

(11)

where $\alpha$ is an arbitrary constant which can be absorbed in a redefinition of the spatial $x_i$ coordinates. With Eq. (9) then obliging $e(y) = e(|y|)$ to be a constant, we can then set $n(y) = 1/a(y)$, with Eq. (8) then recovering the solution of Eq. (11) provided

$$p_b = -\rho_b/3.$$  

(12)

As a check on this solution, we note that the Israel junction conditions which follow from Eq. (10) in our particular case, viz. [5]

$$(a'(y = 0^+) - a'(y = 0^-))/a(y = 0) = -\kappa_5^2\rho_b/3,$$

(13)

$$(n'(y = 0^+) - n'(y = 0^-))/n(y = 0) = \kappa_5^2(3p_b + 2\rho_b)/3,$$

(14)

are indeed satisfied by our obtained discontinuity. With $n^2(y) \to 1 + \kappa_5^2\rho_b|y|/3$ in the weak gravity limit, we see that we nicely recover the linear potential characteristic of a Newtonian gravity sheet (though, as will be clarified below, one actually not with the standard weak gravity coefficient), with gravity not at all being localized to the brane, and with the strong gravity limit even possessing a singularity at $|y| = 3/\kappa_5^2\rho_b$.

While we thus see that we can obtain the anticipated non-localized solution, we find that it is only obtainable for a very particular equation of state, one with a negative pressure. In order to understand this result we need to distinguish between the role that gravity plays in

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6With all of the metric coefficients being functions of $|y|$, the junction conditions require the terms linear in $|y|$ to be first order in $\kappa_5^2$, viz. $a(|y|) = \alpha(1 - \kappa_5^2\rho_b|y|/6) + O(|y|^2)$, $n(|y|) = \beta(1 + \kappa_5^2(3p_b + 2\rho_b)|y|/6) + O(|y|^2)$. The junction conditions thus generate a contribution to the bulk Riemann tensor (see Eq. (41) below) which only begins in order $\kappa_5^2$, something which is to be expected since the order $\kappa_5^2$ constant Newtonian gravitational acceleration associated with a metric with $n(y) = \beta(1 + \kappa_5^2\rho_b|y|/6)$ is removable by a coordinate transformation, with coordinate independent gravitational effects thus only beginning in order $\kappa_5^2$, with true gravity thus needing the $O(\kappa_5^2)$ terms of both of the $n(y)$ and $a(y)$ metric coefficients to be non-zero.

7To see that this is in fact a generic effect we note that for a flat 2-brane embedded in an empty 4 space, viz. one described by the metric $ds^2 = -n^2(z)dt^2 + a^2(z)(dx^2 + dy^2) + dz^2$, the 4-dimensional Einstein equations $G_{\mu\nu} = -\kappa_4^2T_{\mu\nu}$ take the form $G^{00} = -2\omega\alpha'' + \alpha''/a^2 = \kappa_4^2\rho_b\delta(z)$, $G^{2x} = -(a''n + a'n' + an)/an = -\kappa_4^2\rho_b\delta(z)$, $G^{2z} = -a'(a'n + 2an')/a^2 = 0$, with solution $a(z) = 1/n^2(z) = (1 - 3\kappa_4^2\rho_b|z|/8)^{2/3}$, constraint $p_b = -\rho_b/4$. Israel junction conditions of the form
an ordinary 4-dimensional world and the one that it appears to be playing in the embedded case. As regards first the conventional pure 4-dimensional situation with no embedding into a fifth dimension, we note that there the fluid equation of state is usually taken as a fixed, gravity independent input, and the gravity which it produces is then determined as output. Nonetheless, even in that case the $p_b/\rho_b$ ratio need not necessarily be positive. Thus even while the energy density and pressure of a high temperature ideal gas due to the kinematic motions of the gas particles are both positive, if the gas is cooled into a solid phase it then undergoes a phase transition, a long range order effect such as condensation into an ordered crystal lattice, an effect which can be associated with the negative pressure characteristic of vacuum breaking,\(^8\) with non-gravitational physics thus being capable of leading to fluids with negative pressure.\(^9\) Thus, if the fluid equation of state is to be taken as a fixed, gravity independent input in the embedded gravity case as well, we would have to conclude that unless the fluid actually possesses the needed equation of state, then no (static) embedding would in fact be possible.\(^10\)

However, in the brane-embedded case, it turns out that gravity can potentially play a different role, one in which it could be instrumental in actually fixing the fluid equation of state in the first place, with the equation of state then being output to the problem rather than input. In particular, the key difference between the embedded and non-embedded cases is that in the embedded case the matter sources are assumed to be confined to the brane, with no brane matter contribution to $T_{55}$ being permitted. As a consequence, the $G_{55}$ component of the empty bulk Einstein tensor has to vanish, a quite non-trivial requirement which has dynamical implications not present in a non-brane-embedded situation where the

\[ K_{\mu\nu}(y = 0^+) - K_{\mu\nu}(y = 0^-) = -\kappa_4^2(T_{\mu\nu} - q_{\mu\nu}T^\alpha_\alpha/2) \quad \text{(viz.} \quad (a'(y = 0^+) - a'(y = 0^-))/a(y = 0) = -\kappa_4^2\rho_b/2, \quad (n'(y = 0^+) - n'(y = 0^-))/n(y = 0) = \kappa_4^2(2p_b + \rho_b)/2), \quad \text{and a Riemann tensor which is again of order} \quad \kappa_4^4. \]

In passing we note also that our result confirms an old result of Vilenkin \cite{9} that no static solution is possible for 2-branes with equation of state $p_b = -\rho_b$. However, we now see that a static solution is possible when $p_b = -\rho_b/4$.

\(^8\)In such a case the harmonic phonon mode fluctuations in the lattice will still have positive pressure, it is just that they have nothing to do with the mechanism which put the atoms onto the lattice sites in the first place by minimizing the free energy. Rather they are only a perturbation around such a minimum, a minimum to which gravity however is sensitive.

\(^9\)The $p_b = -\rho_b/3$ equation of state required above, for instance, could be associated with an isotropic network of cosmic strings, with such a negative pressure fluid potentially leading to the cosmic acceleration (see e.g. \cite{10} for a recent review) associated with quintessence models \cite{11}.

\(^10\)In their original papers Randall and Sundrum noted the need to have a fixed relation between bulk and brane cosmological constants in models in which $n(y)$ was initially set equal to $a(y)$. We now see that restrictions on the structure of the energy-momentum tensor are of much broader generality, in principle involving the energy densities and pressures of all brane matter sources, restrictions which are intrinsic to all embeddings (even flat ones) and not just only to those associated with $AdS_5$. 

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fluid is otherwise free to flow in all available spatial directions. Such a vanishing is then a constraint imposed by the geometry, and even when there are no explicit bulk matter fields to apply stresses on the brane, nonetheless there is still a non-vanishing Riemann tensor in the bulk, to thus enable the bulk gravitational field to provide such stresses instead, with the bulk curvature and the brane pressure then potentially being able to dynamically adjust to each other to thereby fix the pressure on the brane and yield the brane equation of state as output.\textsuperscript{11} Thus rather than a given input fluid equation of state imposing an output geometry on gravity, gravity instead could impose an output equation of state on the fluid.

The notion that the presence of an extra dimension might have dynamical implications and that higher dimensional gravity might play a role in stabilizing lower dimensional systems is certainly a very interesting one which requires further study. To illustrate its capability it is instructive to recall Einstein’s attempt to construct a static 4-dimensional model of the universe. As is well-known, in his attempt to do so Einstein introduced a cosmological constant and was then able to find a non-trivial static universe solution provided the spatial 3-curvature $k$ of the universe was taken to be positive. In such a situation the ordinary 4-dimensional Einstein equations take the form $G_{0i}^0 = k = \kappa_4^2 \rho_b, G_{ij} = k \delta^j_i = -\kappa_4^2 p_b \delta^j_i$ to precisely impose the selfsame $p_b = -\rho_b/3$ equation of state which we obtained above while fixing $k = \kappa_4^2 \rho_b/3$.\textsuperscript{12} As we now see, in the 5-dimensional $k = 0$ model discussed above, the binding role played by positive $k$ in the 4-dimensional world is instead provided by the embedding, with the constraint condition $G_{55} = 0$ then providing a dynamics not otherwise present in the 4-dimensional system itself.\textsuperscript{13} In fact this phenomenon is actually a quite

\textsuperscript{11}Thus while the Israel junction condition $(n'(y = 0^+) - n'(y = 0^-))/n(y = 0) = \kappa_4^2 (2p_b + \rho_b)/2$ associated with the embedding of a flat 2-brane in an empty 4 space would actually yield the standard weak gravity Gauss’ Law $(n'(y = 0^+) - n'(y = 0^-))/n(y = 0) = \kappa_4^2 \rho_b/2$ if we were to ignore the pressure $p_b$ on the brane (the usual weak gravity assumption), we see that the consistency of the embedding requires a very different brane pressure, one of the same order of magnitude as $\rho_b$, to thus yield to a weak ($\kappa_4^2$ small) gravity limit whose potential has a different normalization than that associated with a standard pressureless weak Newtonian gravity sheet. (Since the standard Newtonian potential of a sheet is just a coordinate artifact, there is no reason for it to have to correspond to the non-relativistic limit of the true gravity associated with a fully covariantized uniform sheet.)

\textsuperscript{12}Einstein himself satisfied the relation $p_b = -\rho_b/3$ by taking $\rho_b = \rho_m + \lambda, p_b = -\lambda$ where $\rho_m$ is the energy density of ordinary matter. In this solution then ordinary matter was taken to have no pressure and the cosmological constant was tuned to be given as $\lambda = \rho_m/2$. Thus already in this now quite ancient model we see the need for constraints (either input or output) on the components of the 4-dimensional matter energy-momentum tensor. In passing we note that current observations [12–14] almost a century later are apparently requiring a similar such fine tuning between 4-dimensional matter and vacuum energy densities (for some remedies to this perplexing problem see e.g. [15]).

\textsuperscript{13}Negative pressure solutions to the cosmic acceleration problem could thus arise as a 4-dimensional reflection of a higher dimensional embedding.
general one. Specifically if the Einstein equations are assumed to hold in the bulk, it turns out [16,17] that the induced gravitational equations on the brane actually deviate from the standard 4-dimensional Einstein equations, with the additional terms that are found being explicit consequences of the embedding, viz. they do not represent new matter sources in the 4 space, but rather they arise though the constraints associated with the very existence of the embedding. In particular the authors of [16] noted that since the difference between the 4-dimensional Riemann tensor of the surface and the 5-dimensional Riemann tensor of the bulk can be completely characterized by a function quadratic in the extrinsic curvature tensor $K_{\mu\nu}$ of the 4-dimensional surface according to the Gauss equation

$$R^\alpha_{\beta\gamma\delta} = R^A_{BCD} q_A {^\alpha} B q^C {^\gamma} D - K^\alpha_{\gamma} K^\beta_{\delta} + K^\alpha_{\delta} K^\gamma_{\beta}, \quad (15)$$

use of the bulk Einstein equations, the Israel junction conditions at the surface of the brane and the assumption of a $y \to -y$ symmetry around the $y = 0$ brane then enable us to express the 4-dimensional Einstein tensor in terms of quantities on the brane which must necessarily be quadratic in the energy-momentum tensor of the brane. In particular, for generic metrics of the form $ds^2 = dy^2 + q_{\mu\nu} dx^\mu dx^\nu$ and a brane energy-momentum tensor of the specific form

$$T^A_B = -\Lambda_5 \delta^A_B, \quad T_{\mu\nu} = -\lambda q_{\mu\nu} + \tau_{\mu\nu} \quad (16)$$

the authors of [16] found that the 4-dimensional Einstein tensor on the brane is given by

$$G_{\mu\nu} = \Lambda_4 q_{\mu\nu} - 8\pi G_N \tau_{\mu\nu} - \kappa^4_5 \pi_{\mu\nu} - E_{\mu\nu} \quad (17)$$

where

$$G_N = \lambda \kappa^4_5 / 48\pi, \quad \Lambda_4 = \kappa^2_5 (\Lambda_5 + \kappa^2_5 \lambda^2 / 6) / 2, \quad E_{\mu\nu} = C^A_{BCD} n_A q^C q^B q^D \quad (18)$$

$$\pi_{\mu\nu} = -\tau_{\mu\alpha} \tau^\alpha_{\nu} / 4 + \tau^\alpha_{\mu} \tau_{\nu} / 12 + q_{\mu\nu} \tau_{\alpha\beta} \tau^{\alpha\beta} / 8 - q_{\mu\nu} (\tau^\alpha_{\alpha})^2 / 24. \quad (19)$$

Thus, even in the event that gravity gets to be localized to the brane, we see in general that on the brane we would expect gravity to depart from that given by just the standard 4-dimensional Einstein equations associated with a non-embedded 4-dimensional world.\(^{15}\) Thus measurements within a 4-dimensional world embedded in a higher dimensional bulk

\(^{14}\)This particular form was chosen so that one of the quadratic terms would then yield a term linear in $\tau_{\mu\nu}$.

\(^{15}\)For a perfect fluid $\tau_{\mu\nu} = (p_b + p_b) U_\mu U_\nu + p_b q_{\mu\nu}$ for instance, the additional $\pi_{\mu\nu}$ tensor takes the form $\pi_{\mu\nu} = [U_\mu U_\nu (2p_b^2 + 2p_b q_{\mu\nu}) + q_{\mu\nu} (p_b^2 + 2p_b q_{\mu\nu})] / 12$, and thus acts like an additional perfect fluid with pressure $P = (p_b^2 + 2p_b q_{\mu\nu}) / 12$ and energy density $R = p_b^2 / 12$ (so that if $p_b = -p_b / 3$, $P = +R / 3$).
would in principle be able to reveal the presence of the higher dimensional space even if the gravitational field is localized to the 4-dimensional world.\footnote{Noting the special role played by the brane cosmological constant $\lambda$ in establishing the Newton constant term in Eq. (17), we see that the effective Newton constant $G_N$ would vary (possibly even in sign as well as magnitude) in different epochs separated by phase transitions, with early universe cosmology then potentially no longer being controlled by the Newton constant measured in a low energy Cavendish experiment. It is thus of interest to note that it is precisely an epoch dependence to both the sign and magnitude of the effective gravitational coupling constant which has recently been identified \cite{15} as a possible solution to the cosmological constant problem. In passing, we additionally note that Eq. (17) has an interesting implication for the Schwarzschild problem. Specifically, even though the 4-dimensional $R_{\mu\nu} = 0$ vacuum Schwarzschild solution can \cite{18} explicitly be embedded into an $AdS_5$ bulk (a case where every term in Eq. (17) then vanishes), the same is not true for the source which would, in 4 dimensions, produce the Schwarzschild metric in the first place. Specifically, the authors of \cite{16} showed that when there is a spatially inhomogeneous matter distribution on the brane, Eq. (17) then prevents the exterior bulk geometry from being pure $AdS_5$, with the $AdS_5$ embedded $R_{\mu\nu} = 0$ geometry thus not being the solution exterior to the source in the embedded case.}

To study further implications of such embeddings we turn now to our second soluble model, namely a 3-brane with non-zero $k$ embedded in an empty bulk. In the event of non-zero $k$ the most general empty bulk solution to Eq. (7) is directly given as

$$a^2(y) = \alpha(1 - \kappa_5^2 \rho_b |y|/3 + k|y|^2/\alpha),$$

with Eq. (9) then leading to

$$n^2(y) = (-\kappa_5^2 \rho_b \alpha/3 + 2k|y|)^2/\alpha(1 - \kappa_5^2 \rho_b |y|/3 + k|y|^2/\alpha),$$

and with Eq. (8) then entailing the equation of state

$$p_b = -\rho_b/3 - 12k/\alpha \kappa_5^4 \rho_b.$$  

Thus we see that the consistency of the embedding (cf. the non-trivial vanishing of $G_{55}$) again imposes constraints on the brane equation of state, with the metric away from the brane now growing quadratically with distance. It is thus to the issue of whether or not there is to be a quenching of this metric when a bulk cosmological constant is introduced to which we now turn.

**III. EMBEDDING A BRANE IN A NON-EMPTY BULK**

In the event of there being a bulk cosmological constant $-\rho_B = p_B = P_T = -\Lambda_5$ the structure of the solutions to the 5-dimensional Einstein equations will depend on whether there is a spatial curvature $k$ on the brane. Thus on setting $k = 0$ first and taking $a(y)$ to be a function only of $|y|$ just as before, Eq. (7) is then found to lead to
\[
\frac{3}{2} \frac{d^2 f(|y|)}{d|y|^2} = -\kappa_5^2 \Lambda_5 f(|y|)
\]

\[
[3 \frac{df(|y|)}{d|y|} + \kappa_5^2 \rho_b f(|y|)]\delta(y) = 0.
\]

According to the first of these two equations (viz. a pure bulk Einstein equation which would hold even in the absence of any brane at \( y = 0 \)), the metric coefficient \( a(y) = f^{1/2}(y) \) will thus have an exponential dependence

\[
a^2(y) = \alpha e^{\nu|y|}
\]

on distance if \( \Lambda_5 \) is negative (viz. anti de Sitter), with the exponent \( \nu \) being constrained according to

\[
\nu^2 = -2\kappa_5^2 \Lambda_5 / 3.
\]

With Eq. (23) thus not being able to fix the sign of \( \nu \), we see that in and of itself a bulk cosmological constant could lead to either suppression or exponential growth away from the brane - as we recall from Eq. (1), the AdS\(_5\) metric is quadratic in \( R/z \), with the most general solution to Eq. (23) in fact being of the generic form

\[
a^2(y) = \alpha e^{\nu|y|} + \beta e^{-\nu|y|},
\]

a function which is unbounded no matter what the sign of \( \nu \), with having an AdS\(_5\) bulk in and of itself thus not being sufficient to guarantee brane-localization of gravity.\(^{17}\) Within the Einstein equations there appears to be no reason to exclude the unbounded term, and so we shall just exclude it by fiat as a \( |y| = \infty \) boundary condition. Thus dropping the \( \beta \) dependent term in Eq. (27) (a point we shall reexamine below), and retaining only the \( \alpha \) dependent one, we then find that Eq. (24), viz. the discontinuity condition at the brane, then fixes the sign of \( \nu \) according

\[
\nu = -\kappa_5^2 \rho_b / 3,
\]

with gravity now being localized to the brane when \( \rho_b \) is positive (for \( \rho_b \) negative no localization would be obtained), with Eq. (26) then yielding the compatibility condition

\[
\Lambda_5 + \kappa_5^2 \rho_b^2 / 6 = 0.
\]

\(^{17}\)In the original Randall-Sundrum study the brane geometry was taken to maximally 4-symmetric (viz. Minkowski), to thus oblige the metric coefficients \( a^2(y) \) and \( n^2(y) \) to be equal to each other, and thereby only allow solutions with a single exponential. However, once the brane geometry is lowered to maximally 3-symmetric, two exponential terms with opposite sign exponents are then allowed.
To complete the solution in this particular case we find that the insertion of Eq. (25) into Eq. (9) then leads to $e(y) = \beta e^{\nu|y|}$, so that the metric coefficient $n(y)$ is then found to take the requisite $AdS_5$ form

$$n^2(y) = \beta^2 e^{\nu|y|}/\alpha,$$

with Eq. (8) requiring the compatibility condition

$$p_b = -\rho_b$$

(31)
on the matter fields. Thus again we see that there are constraints on both the bulk and brane matter fields, with the exponential dependence associated with Eqs. (23) and (24) nicely quenching the linear metric dependence found in the empty bulk case just as desired, while precisely imposing on the brane the equation of state associated with a cosmological constant $\lambda$, with the condition $-\Lambda_5 = \kappa^2_5 \rho_b^2/6 \equiv \kappa^2_5 \lambda^2/6$ then entailing the vanishing of the net effective brane cosmological constant $\Lambda_4$ of the general Eq. (18), just as found in the original Randall Sundrum study.

It is also of interest to repeat this analysis for the more general solution given in Eq. (27). Its insertion into the Einstein equations yields

$$e(y) = \alpha e^{\nu|y|} - \beta e^{-\nu|y|},$$

(32)

$$3\nu(\alpha - \beta) = -(\alpha + \beta)\kappa^2_5 \rho_b,$$

(33)

$$6\nu(\alpha + \beta) = (\alpha - \beta)\kappa^2_5 (\rho_b + 3p_b).$$

(34)

Solubility thus requires $\rho_b/ (\rho_b + 3p_b) < 0$, while use of Eq. (26) leads to the equation of state

$$\Lambda_5 - \kappa^2_5 \rho_b (\rho_b + 3p_b)/12 = 0,$$

(35)

with it being this latter relation, rather than that of Eq. (29) which matches on continuously to the $p_b = -\rho_b/3$ equation of state obtained earlier in the $\Lambda_5 = 0$ case. The reason for this is due to the limiting process needed to extract the term linear in $|y|$ as needed for Eq. (11) from a function only containing exponentials, with it being only a linear combination of two exponentials with appropriately chosen singular coefficients ($\simeq (1 \pm 1/\nu)$) which can generate a non-vanishing linear term when the exponent $\nu = (-2\kappa^2_5 \Lambda_5/3)^{1/2}$ is allowed to go to zero. Hence, by not retaining the $\beta$ dependent term, we have then taken a $\Lambda_5$ dependent metric, viz. that of Eq. (25), whose $\Lambda_5 \to 0$ limit does not generate any term linear in $|y|$. It is thus our assumed $|y| = \infty$ boundary condition which eliminates the linear term, to thus then lead to a geometry which is exponentially suppressed as we go away from the brane. As we shall now show, however, in the presence of a non-zero spatial curvature, i.e. in the presence of a non-trivial topology, retaining only the exponentially damped $\alpha$ dependent term will not prove sufficient to suppress the geometry away from the brane.

---

18 Equation (18) thus explains why this condition is in fact quadratic in $\lambda$. 

11
In the \( k \neq 0 \) non-empty bulk case, Eq. (7) is this time found to lead to

\[
a^2(y) = \alpha e^{\nu|y|} + 3k/k_5^2\Lambda_5
\]

(36)

where

\[
\nu^2 = -2k_5^2\Lambda_5/3, \quad \nu = -k_5^2\rho_b(1 + 3k/\alpha k_5^2\Lambda_5)/3,
\]

(37)

with the net brane cosmological constant \( \Lambda_4 \) vanishing this time if the brane matter energy density and brane cosmological constant (defined via \( \rho_b = \rho_m + \lambda \)) are fine-tuned according to

\[
\lambda = -\rho_m(1 + \alpha k_5^2\Lambda_5/3k).
\]

(38)

In this \( k \neq 0 \) case Eq. (9) again leads to \( e(y) = \beta e^{\nu|y|} \), with the metric coefficient \( n(y) \) now being given by

\[
n^2(y) = \beta^2e^{2\nu|y|}/(\alpha e^{\nu|y|} + 3k/k_5^2\Lambda_5),
\]

(39)

and with Eq. (8) now requiring the compatibility condition

\[
p_b = -\rho_b(1 + 2k/\alpha k_5^2\Lambda_5),
\]

(40)

so that the brane matter pressure defined via \( p_b = p_m - \lambda \) is then related to the brane matter energy density according to \( p_m = -\rho_m/3 \). As we thus see, even though the \( AdS_5 \) exponential damping factor does make an appearance in the \( k \neq 0 \) case, nonetheless we find that \( a^2(y) \) is not asymptotically suppressed far away from the brane. Rather it tends to the non-vanishing value \( 3k/k_5^2\Lambda_5 \) if \( k \) is negative (even as \( n^2(y) \) is then being suppressed), while if \( k \) is positive \( n(y) \) actually becomes singular. This lack of suppression is also evidenced in the Riemann tensor, with its \( R_{1212} \) component for instance, viz.

\[
R_{1212} = f'^2/4f^2 - k/f,
\]

(41)

tending to \(-k_5^2\Lambda_5/3\) at large \( y \) (viz. twice the pure \( AdS_5 \) value), with the bulk embedding never being able to counteract the effect of the spatial curvature of the brane. A similar situation is also found for the Weyl tensor where

\[
W_{1212} = [-2ef'f'' + 3ef'^2 - 2efk - 3e'ff' + 2f'^2e'']/12ef^2
\]

(42)

asymptotes to the non-vanishing value \(-k_5^2\Lambda_5/6\). Thus, unlike pure \( AdS_5 \), the metric associated with Eqs. (36) and (39) is not conformal to flat, with the bulk cosmological constant

\[\text{---}\]

\[19\] For metrics of the form given in Eq. (4) all non-vanishing components of the Weyl tensor are kinematically proportional to \( W_{1212} \).

\[20\] As Eq. (42) also shows, even when \( e(y) = f(y) = 1 \), \( W_{1212} \) is equal to \(-k/6\) and still does not vanish. Thus even while a constant curvature 3 space embedded in an otherwise flat 4 space (viz. standard 4-dimensional Robertson-Walker) produces a metric which is conformal to flat, the same is not true of the same 3 space embedded in an otherwise flat 5 space, something a bulk cosmological constant is simply unable to alter. With only the \( k = 0 \) Robertson-Walker geometry thus being localizable by an \( AdS_5 \) embedding, it would be of interest to see whether geometries with non-zero \( k \) but with a negligibly small current era value of \( \Omega_k(t) = -ke^2/R^2(t) \) could still admit of a brane-localized current era gravity.
$\Lambda_5$ term not being able to completely quench the quadratic growth previously found for $a^2(y)$ in Eq. (20) in the $k \neq 0$, $\Lambda_5 = 0$ case.\footnote{While, as had already been noted above, the bulk geometry would not be pure $AdS_5$ in the event that the brane matter source was spatially inhomogeneous, we also see that even when the brane distribution is homogeneous, the bulk geometry may still not be pure $AdS_5$.} Thus, to conclude, we see that while embedding in a higher dimensional $\Lambda_5 < 0$ bulk might lead to a brane-localized geometry in certain specific cases, it would appear from study of our somewhat idealized static cosmological model that such embeddings may not always lead to exponential suppression in general.

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