Superactivation of Bound Entanglement

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We show that, in a multi-party setting, two non-distillable (bound-entangled) states tensored together can make a distillable state. This is an example of true superadditivity of distillable entanglement. We also show that unlockable bound-entangled states cannot be asymptotically separable, providing the first proof that some states are truly bound-entangled in the sense of being both non-distillable and non-separable asymptotically.

The joint state of more than one quantum system cannot always be thought of as a separate state of each system, nor even as a correlated mixture of separate states of each system [1], a situation known as quantum entanglement. Entanglement leads to the most counterintuitive effects in quantum mechanics, including the disturbing idea due to Bell that quantum mechanics is incompatible with local hidden variable theories [2]. Even today new quantum oddities with their basis in entanglement are being found, and the study of entanglement is at the heart of quantum information theory.

A state belonging to parties A, B, C, etc. is said to be inseparable if it cannot be written in separable form

\[ \rho = \sum_i p_i \rho_i^A \otimes \rho_i^B \otimes \rho_i^C \ldots \]  

for any positive probabilities \( p_i \) summing to one and set of density matrices \( \rho_1^A, \rho_1^B, \rho_1^C, \ldots \), where, for example, \( \rho_i^A \) operates on the Hilbert space belonging to party A and need not be the same as \( \rho_i^B \). It is known that many inseparable quantum mixed states may be distilled into pure entanglement, while separable states cannot [3,4]. More recently it has been shown that some mixed states which are entangled in the sense of being inseparable nevertheless cannot be distilled into any pure entanglement [5,6]. Such states are known as bound-entangled states.

In the bipartite case, bound entanglement may sometimes be useful in a kind of quasi-distillation process known as activating the bound entanglement [7] in which a finite number of free-entangled mixed states are distilled with the help of a large number of bound-entangled states. This is not a true distillation of the bound entanglement in that no more pure entanglement is produced than the distillable entanglement of the free-entangled mixed states, the distillable entanglement being defined as the pure entanglement distillable per state from an infinite number of copies of a state.

In the case of more than two parties the bound entanglement can be more truly activated by the presence of free entanglement. In examples given by Cirac, Tarra Rap and Dür [8–10], and in the equivalent formulation of unlockable bound-entangled states [11], when several parties share certain bound entangled states, and when some subset of the parties get to share pure entanglement then some pure entanglement may be distilled between parties where it would be impossible to obtain any without having shared the bound-entangled state. This is a kind of superadditivity of distillable entanglement, though in the known cases no more entanglement is distilled than the pure entanglement that was shared, rather it is in a different place.

In this letter we present an effect we call superactivation of bound entanglement. It is “super” in the sense of being superadditivity of distillable entanglement, but without the restrictions of either of the earlier types of activation of bound entanglement. In superactivation two entangled mixed states \( \rho, \rho' \) are combined to yield more pure entanglement than the sum of what a set of parties could distill from either \( \rho \) or \( \rho' \) on their own, even if many copies of \( \rho \) or \( \rho' \) are shared. In particular, both states in our example are bound-entangled states from which no pure entanglement can be distilled.

We will use the usual notation for the maximally entangled states of two qubits (the Bell states):

\[ |\Psi^\pm\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle), \quad |\Phi^\pm\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle) \]  

For convenience we adopt the following notation as well:

\[ \Psi = \{ \Psi^-, \Psi^+, \Phi^+, \Phi^- \} \text{ with elements } \Psi_i, \text{ and } \sigma = \{ \mathbf{1}, 2, (1 \ 0, 0 \ -1), (0 \ -1, 1 \ 0), (0 \ 1, 1 \ 0) \} \text{ with elements } \sigma_i. \]  

In the text, we shall refer to a Bell state as any one of the four states (3) and to an EPR state as the standard singlet state \( |\Psi^-\rangle \). The Bell states \( |\Psi_i\rangle \) may be related to the standard EPR state \( |\Psi^-\rangle \) by the following identities, up to an overall phase which is unimportant here:

\[ |\Psi^-\rangle = \mathbf{1} \otimes \sigma_i |\Psi_i\rangle = \sigma_i \otimes \mathbf{1} |\Psi_i\rangle \]  

\[ |\Psi_i\rangle = \mathbf{1} \otimes \sigma_i |\Psi^-\rangle = \sigma_i \otimes \mathbf{1} |\Psi^-\rangle \]  

We will also need the simple lemma that if a state \( |\psi\rangle \) is teleported [12] from Alice to Bob using an incorrect one of the Bell states \( |\Psi_i\rangle \) rather than \( |\Psi^-\rangle \) as normally required.
by the protocol, then the result of the teleportation will be \( \sigma_i |\psi\rangle \), again up to an overall phase. This is easily seen by using (6) to write the incorrect Bell state as \( |\Psi^-\rangle \) with \( \sigma_i \) operating on Bob’s part of the \( |\Psi^-\rangle \). If the rotation which is the final step in teleportation could be squeezed in before the \( \sigma_i \) the proof would be complete, but instead it follows the \( \sigma_i \). However, the rotations used in teleportation are also the \( \sigma \) matrices, and all the \( \sigma_i, \sigma_j \) either commute or anticommute \( (\sigma_i \sigma_j = \pm \sigma_j \sigma_i) \) and so their order can be freely interchanged up to a phase. □

In [11] a four-party bound entangled state was presented:

\[
\rho_{ABCD} = \frac{1}{4} \sum_{i=0}^{3} |\Psi_i\rangle^{AB} \langle \Psi_i|^{CD} \langle \Psi_i|
\]

In other words, \( A \) and \( B \) share one of the four Bell states, but don’t know which one, and \( C \) and \( D \) share the same Bell state, also not knowing which one.

This state has several properties:

- **Symmetry under interchange of parties:** \( \rho_{ABCD} = \rho_{ABDC} = \rho_{ADBC} = \rho_{ACBD} \), etc. This may be verified by writing out the state as a \( 16 \otimes 16 \) matrix and interchanging indices. A more enlightening way is to use our lemma and think of the state in terms of teleportation. First, we note that some of the symmetries are obvious, for example interchanging \( A \) and \( B \) because Bell states are themselves symmetric under interchange. So the only symmetry we need to consider is the interchange of \( B \) with \( C \) and the rest can be constructed trivially.

Consider the state in its original form, with \( A \) and \( B \) sharing an unknown Bell state and \( C \) and \( D \) sharing the same one. Now consider \( A \) and \( C \) getting together and performing a Bell measurement and obtaining the result \( |\Psi_j\rangle \), which we can think of as \( A \) and \( C \) doing the first step required to teleport \( A \)’s particle to \( D \) using the unknown Bell state shared by \( C \) and \( D \). The result \( |\Psi_j\rangle \) is random since \( A \) and \( C \) had halves of completely separate unknown Bell states. The state being teleported is half of a Bell state given by Eq. (6) \( \sigma_i \otimes 1_2 |\Psi^-\rangle \) as is the state used in the teleportation. So, by our lemma, if the teleportation were completed an extra \( \sigma_i \) would be introduced, and the two \( \sigma_i \)’s would cancel being self-inverse. Thus, \( B \) and \( D \) would share a standard \( |\Psi^-\rangle \). But if the \( \sigma_i \) needed to complete teleportation is not performed, this means that \( B \) and \( D \) share the Bell state \( \sigma_j^{-1} \otimes 1_2 |\Psi^-\rangle = \sigma_j \otimes 1_2 |\Psi^-\rangle = |\Psi_j\rangle \), which is the result obtained by \( A \) and \( C \). So \( AC \) and \( BD \) share identical random Bell states, which was the original form of the density matrix, but with \( A \) and \( C \) interchanged.

- **Non-distillability:** When all four parties remain separated and cannot perform joint quantum operations, then they cannot distill any pure entanglement by local operations and classical communication (LOCC), even if they share many states, each having density matrix \( \rho_{ABCD} \). This comes from the fact every party is separated from every other across a separable cut. This is easy to see since the state (7) is separable across the \( AB : CD \) cut by construction and the state has the symmetry property.

- **Unlockability:** The entanglement of the state can be unlocked. If \( A \) and \( B \) come together and perform a joint quantum measurement, they can determine which of the four Bell states they have (the four Bell states form an orthogonal basis) and tell \( C \) and \( D \) the outcome. Since \( C \) and \( D \) then know which Bell state they have, they can convert it into the standard \( |\Psi^-\rangle \) state using local operations by Eq. (5). Because of the symmetry property any two parties can join together to help the other two (get a \( |\Psi^-\rangle \)). Note that the unlockability property implies the state must not be fully separable, or no entanglement could be distilled between separated parties, even when some of the parties come together.

Because the state is both non-distillable and entangled, it is by definition a bound-entangled state [5,6].

Now we consider the mixed state

\[
M = \rho_{ACBD} \otimes \rho_{ABCE}
\]

where the \( \rho_{ACBD} \) and \( \rho_{ABCE} \) are the states of Eq. (7) but with the qubits assigned to different parties. Technically \( \rho_{ACBD} \) could be written as \( \rho_{ABCD} \) due to the symmetry property but it will be useful to have it explicitly written in the form where it is an unknown Bell state shared between \( A \) and \( C \) and the same state shared by \( B \) and \( D \). The state \( M \) is illustrated in Figure 1a. \( M \) is the tensor product of two density matrices, neither of which is independently distillable. We now show how to distill a \( |\Psi^-\rangle \) between \( D \) and \( E \).

First, party \( A \) teleports her half of the unknown Bell state she shares with \( B \) (the solid arrow connecting \( A \) and \( B \) in Figure 1a) to \( C \) using the unknown Bell state she shares with \( C \) (the dashed arrow connecting \( A \) and \( C \) in the figure). This results in the situation of Figure 1b, where now \( C \) shares an unknown Bell state with \( B \), her half of which has additionally picked up the unknown rotation \( \sigma_j \) from having been teleported with an incorrect Bell state \( |\Psi_i\rangle \). The Bell state connecting \( A \) and \( C \) is gone in the figure, since it has been expended performing the teleportation. Then \( B \) teleports his half of that state to \( D \) using the unknown Bell state (again \( |\Psi_i\rangle \) that they share, resulting in the situation of Figure 1c, where now
$C$ and $D$ share the unknown Bell state originally shared by $A$ and $B$, both halves of which having been rotated by $\sigma_i$. It is important to note here that because of the structure of $\rho^{ABCBD}$ this is the same $\sigma_i$. Now, using Eq. (6) and the fact that $\sigma_i^2$ is the identity (once again except for a phase), we can see that the $\sigma_i$'s cancel and we are left with the state $\rho^{CDCE}$. This is the same form as the four-party unlockable state (Eq. (7)) but with one party sharing two of the qubits, and it is therefore distillable into a pure EPR pair between $D$ and $E$.

$M$ cannot be distilled into EPR pairs between any of the other parties. This is because if we give the five parties the additional power of having $D$ and $E$ in the same room, then $M$ is just two copies of $\rho^{ABCD}$ which are known not to be distillable (by definition if $\rho$ is not distillable, then neither is $\rho^{\otimes N}$). To construct a state out of tensor products of bound-entangled states that is distillable into any kind of pure entanglement, it is sufficient to symmetrize $M$, i.e.

$$M_{\text{symmetric}} = \rho^{ABCD} \otimes \rho^{ABCE} \otimes \rho^{ABDE} \otimes \rho^{ACDE} \otimes \rho^{BCDE}. \quad (9)$$

Then the distillation protocol just described can be used to obtain an EPR pair between any two of the parties, and using more copies of $M_{\text{symmetric}}$ one can obtain EPR pairs between all pairs of parties. Once this is accomplished any arbitrary multi-party entangled state can be constructed by one party creating it in his lab and teleporting the pieces as needed to the others.

It has been an open question whether bound entangled states are actually entangled at all in an asymptotic sense. A state $\rho$ is said to be asymptotically separable (cf. [13]) if for any positive $\epsilon$ there exists a number of copies $N$, a number $m$ sublinear in $N$ of EPR pairs shared in some way among the parties, and an LOCC method of constructing from those EPR pairs a state $\rho'$ such that $F(\rho, \rho') > 1 - \epsilon$ for some sensible definition of the fidelity $F$ between two density matrices (the definition of fidelity for pure states $F(\psi, \phi) = |\langle \psi | \phi \rangle|^2$ cannot be directly applied to density matrices). One definition with nice properties is given in [14], but the main property we need from any definition is that two states of fidelity $F > 1 - \epsilon$ will with probability greater than $1 - C\epsilon$ behave the same under any quantum mechanical test, where $C$ is a constant. Because of the linearity of quantum mechanics, it is not hard to create such fidelity measures: The following simple norm will suffice: $F(\rho, \rho') = 1 - 1/d^2 \sum_{ij} |\rho_{ij} - \rho'_{ij}|$ if $\rho$ and $\rho'$ are written as matrices in $d \times d$. This fidelity is only close to one when every entry is the same in the two matrices $\rho$ and $\rho'$ and the results on any quantum test only depend on the entries in a state's density matrix, two states of high fidelity with necessarily behave nearly identically.

Because $M$ (Eq. (8)) is distillable, it cannot be that the original state $\rho^{ABCD}$ is asymptotically separable. If it were, then many copies $N$ of $\rho^{ABCD}$ and $\rho^{ABCE}$ could be created arbitrarily precisely using a number of EPR pairs sublinear $N$. These could be used to create $N$ copies of $M$ which could then be distilled into $N$ pure EPR pairs between $D$ and $E$. These $DE$ EPR pairs would, to arbitrarily high probability, pass any test that pure EPR pairs would pass. Thus, an amount of entanglement sublinear in $N$ would have been converted into $N$ EPR pairs by LOCC, which is impossible [4]. A further discussion on multipartite separability and distillability will be presented in [13].

In fact, all unlockable bound-entangled states are asymptotically inseparable. This is because when some subset $S$ of the parties possessing such a state come together in the same lab the state becomes distillable. If the state were asymptotically separable then it could be made arbitrarily precisely with asymptotically no entanglement even when parties in $S$ are actually together in the same lab (it cannot hurt for them to be together as they can conveniently ignore this fact as they carry out whatever procedure results in the creation of the state). But then they can distill a finite amount of arbitrarily pure entanglement per state from the sublinear amount of entanglement they started with with, which is impossible. It is worth noting then that the unlockable bound-entangled states are the first states shown to be true bound-entangled states in the sense of both being non-distillable and being non-separable asymptotically.

It is clearly a necessary condition for superactivation that at least one of the states involved must not be asymptotically separable. It is by no means a sufficient one, however, since the states $\rho^{ABCD}$ and $\rho^{EFGH}$ are each not asymptotically separable but $\rho^{ABCD} \otimes \rho^{EFGH}$ is not distillable as the two pieces are on disconnected
sets of parties.

In the individual states $\rho_{ABCD}$ and $\rho_{ABCE}$, every party is separated from every other party by at least one separable cut. In order for the combined state $M$ to be distillable into a $D-E$ EPR pair, and for $M_{\text{symmetric}}$ to be distillable into EPR pairs between any pair of parties, it is necessary that the parties who get EPR pairs no longer be separated by any separable cut, as is indeed the case by construction for these states [15]. Using this observation, Dür has reported a whole family of superactivated states [16] based on the unlockable bound-entangled states of [8–10]. It is possible, though not known, that all states for which a party is separated from some other party by no asymptotically separable cuts are distillable into an EPR pair between those two parties. This is known to be true for pure states, as was shown in [13]. The simple mixed state case of bipartite bound-entangled states [5,6] which are entangled but not distillable are not necessarily counterexamples, as those states might turn out to be asymptotically separable.

A possible route to constructing a counterexample is to modify the state $\rho_{ABCD}$ (7) slightly,

$$\rho'_{ABCD} = \frac{1}{4} \sum_{i=0}^{3} \sum_{j=0}^{3} c_{ij} |\Psi_i\rangle \langle \Psi_j| \otimes \mathbf{1}_{CD}$$

$$+ \frac{1}{4} \sum_{i=0}^{3} \sum_{j=0}^{3} c_{ij} \mathbf{1}_{AB} |\Psi_i\rangle \langle \Psi_j| \otimes |\Psi_i\rangle \langle \Psi_j|$$

(10)

where $c_{ij} = f$ for $i = j$ and $c_{ij} = (1-f)/3$ for $i \neq j$. For $f = 1$ this is the same as $\rho_{ABCD}$ and for $f < 1$ it is similar, except that the correlation between the Bell states shared by $AB$ and $CD$ is no longer perfect. This new state is still an unlockable bound-entangled state for $f > 1/2$. The separability across various cuts is the same as before, and if $A$ and $B$ jointly measure their Bell state and tell $C$ and $D$, the state of $CD$ will be a Werner state of fidelity $f$. Such states are distillable if and only if $f > 1/2$ [3,4]. Now, if the state $M_f = \rho_{ABCD}^{\prime} \otimes \rho_{ABCE}^{\prime}$ ($f > 1/2$) is constructed in parallel to Eq. (8), $D$ and $E$ will not be separated by any separable cut, but if the distillation procedure we have outlined above is attempted it will fail for some values of $f$. The cancellation of $\sigma$ matrices as the $AB$ state is teleported to $C$ and $D$ will be imperfect due to the lack of total correlation between the $AC$ and $BD$ Bell states. This imperfect cancellation, combined with the initial lack of perfect correlation between the $AB$ and $DE$ Bell states, will result in the final state of $CDE$ being $\rho_{CDE}^{\prime}$ with $f' = f^2 + (1-f)^2/3$. For $f < (1 + \sqrt{3})/4$ the resulting $f'$ will be less than 1/2 and the final state will not be distillable. This is not to say that no method can distill a $DE$ EPR pair from $M_f$, $1/2 < f < (1 + \sqrt{3})/4$, merely that our particular procedure which works for $f = 1$ will now fail, leaving the question open.

[15] It is easy to check that for every bipartite cut separating $D$ and $E$ for $M$ or any pair of parties for $M_{\text{symmetric}}$, one side of the cut will have exactly three of the four parts of at least one of the unlockable bound entangled states making up the total state. Thus, rather than being separable, the state is actually distillable across all such cuts.