Abstract

There is an ongoing effort to quantify entanglement of quantum pure states for systems with more than two subsystems. In this paper, we consider three general approaches to this problem for three-qubit states: choosing a basis which puts the state into a standard form, enumerating “local invariants” (functions of the state which are invariant under local unitary transformations), and using operational quantities such as the number of maximally entangled states which can be distilled. We build on earlier work to extend these three approaches and describe relationships among them.
I. INTRODUCTION

The importance of quantum entanglement, both as a resource for quantum information processing and as a ubiquitous feature of quantum systems, has become increasingly apparent over the last few years [1–3]. Recent developments in quantum information theory, in particular, have stimulated interest in the quantification and manipulation of entanglement.

A good deal of progress has been made as far as bipartite entanglement is concerned: indeed, for bipartite pure states an essentially complete theory now exists [3,4], though the situation for mixed states is less definite [5]. Asymptotically, there is a unique measure for entanglement of bipartite pure states, the entropy of entanglement; if $|\Psi_{AB}\rangle$ is a bipartite pure state shared between two spatially-separated experimenters (conventionally referred to as Alice and Bob), then the entropy of entanglement $S_E(|\Psi_{AB}\rangle)$ is

$$S_E(|\Psi_{AB}\rangle) = -\text{Tr}_A\{\rho_A \log_2 \rho_A\}, \quad \rho_A = \text{Tr}_B\{|\Psi_{AB}\rangle\langle\Psi_{AB}|\}.$$  \hspace{1cm} (1)

$S_E(|\Psi_{AB}\rangle)$ is both the number of maximally entangled pairs of qubits needed by Alice and Bob to prepare the given entangled state $|\Psi_{AB}\rangle$ and the number of maximally entangled pairs they can distill from it, using only local operations and classical communication, in the asymptotic limit where they have many copies of the state. Any two states with the same $S_E$ can thus be reliably interconverted, asymptotically, and a state can also be reliably converted to any state with lower $S_E$. If one desires a single number to characterize bipartite entanglement, $S_E$ is the natural choice.

If one is restricted to a single copy of the state the situation is more complicated [4]. In this case, one in general needs a complete set of local invariants, i.e., quantities invariant under local unitary transformations, in order to quantify entanglement. For a bipartite system the canonical choice of these invariants is the Schmidt coefficients. It is possible to find orthonormal bases $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ for systems A and B such that we can write the joint state of the system in the form

$$|\Psi_{AB}\rangle = \sum_i \sqrt{p_i} |i\rangle_A \otimes |i\rangle_B, \quad p_i > 0, \quad \sum_i p_i = 1.$$  \hspace{1cm} (2)

These parameters $\{p_i\}$ are uniquely defined by the state $|\Psi_{AB}\rangle$, and are equal to the eigenvalues of the reduced density matrix $\rho_A$ (or equivalently, of $\rho_B$). Nielsen’s theorem shows that a state $|\Psi_{AB}\rangle$ can be reliably converted to a state $|\Phi_{AB}\rangle$ using only local operations and classical communication if and only if the Schmidt coefficients of the two states satisfy a majorization criterion. If this criterion is not satisfied, the state conversion cannot be done reliably, but it may still be possible with some probability. These two descriptions—in terms of Schmidt coefficients for a single copy, or the entropy of entanglement asymptotically—are linked by the equation

$$S_E(|\Psi_{AB}\rangle) = -\sum_i p_i \log_2 p_i.$$  \hspace{1cm} (3)

These two results give an essentially complete picture of entanglement in the case of bipartite systems in pure states. For tripartite or multipartite states, however, not nearly so much progress has been made. Techniques for quantifying bipartite entanglement cannot be straightforwardly extrapolated to the tripartite case [6]. There is in general no tripartite
equivalent of the Schmidt decomposition; hence, there is no obvious generalization either of $S_E$ or of Nielsen’s theorem.

One thing which does generalize is the idea of independent quantities invariant under local unitary transformations [7–11]. The minimum number of such quantities can be estimated using parameter counting arguments as we show in the next section. For a pure state of the simplest possible tripartite system, consisting of three qubits, five parameters are needed. These five parameters are sufficient to determine a three-qubit pure state up to local unitary transformations. Considerably more parameters are required for tripartite states of higher dimensionality. However, just as a single number suffices to characterize bipartite entanglement asymptotically, this multiplicity of locally-invariant parameters may simplify in the limit of many identical systems [12].

One way of determining these parameters is to put the state into a standard form, by changing the bases of the local subsystems (or equivalently by performing a local unitary transformation) [13,14,11,15]. If this standard form is uniquely determined, then any two states which are equivalent under local unitary transformations will have the same standard form, and any which are not will not. The coefficients of this standard form then represent one possible set of invariant parameters. This can be quite helpful in calculations—rather than assume a state is arbitrary, one can assume it to be in one or another standard form. Unfortunately, the coefficients of a standard form rarely have any obvious intrinsic physical significance. We examine several of these proposed forms in section II.

In this paper we consider several possible choices for a set of five locally-invariant parameters characterizing a pure state of three qubits. (There are in principle an infinite number of ways of choosing this set). We will focus on one of these choices in particular, which is defined by choosing a basis for each qubit from the Schmidt decomposition of that qubit with the other two bits considered as a single system. We discuss the physical interpretation of the five parameters yielded by this “Schmidt form,” each of which has a fairly straightforward meaning, also in section II.

The Schmidt form has proven analytically useful in demonstrating the reliability of a recently discovered technique for distilling maximally entangled (GHZ) three-qubit states [14]; we include this proof in section III.

In section IV we look at proposed polynomial quantities which are invariant under local unitary transformations, especially the “residual tangle” of Coffman, Kundu and Wootters [9]. This “residual tangle” can be seen as a measure of the “GHZ-ness” of a state, and we see how well it correlates with the yields of GHZ states from the aforementioned distillation protocol. We also look at the dependence of this yield on other parameters, and relate this distillation protocol to recent work by Carteret and Sudbery characterizing three-qubit states with “atypical” entanglement properties. We see how these atypical states are also singled out by the distillation protocol.

Our conclusions are summarized in section V.
II. STANDARD FORMS FOR THREE-QUBIT STATES

A. Two qubit states and the Schmidt decomposition

Two qubits can always be represented in their Schmidt decomposition (2); that is, for any state \( |\psi\rangle \) we can find bases \( \{ |0\rangle, |1\rangle \}_{a,b} \) for each qubit such that

\[
|\psi\rangle = \sqrt{p} |00\rangle + \sqrt{1-p} |11\rangle.
\]

The positive real values \( p \) and \( 1-p \) are eigenvalues of the reduced density matrices for each of the two qubits, while the basis vectors \( |0\rangle \) and \( |1\rangle \) are the corresponding eigenvectors. Without loss of generality, we adopt the convention that \( p \geq 1/2 \).

No representation similar to (4) exists for more than two subsystems, in general [6]. But it is possible to make a Schmidt decomposition between each subsystem and the others, and use that to define a Schmidt basis for each subsystem. For instance, we can make a Schmidt decomposition between qubit 1 and qubits 2 and 3, writing the three-qubit state in the form

\[
|\psi\rangle = \sqrt{p} |0\rangle |\psi_0\rangle_{23} + \sqrt{1-p} |1\rangle |\psi_1\rangle_{23}.
\]

Choosing the Schmidt basis for qubit 1 guarantees that the correlated states of qubits 2 and 3 must be orthogonal: \( \langle \psi_0 | \psi_1 \rangle = 0 \).

A generic pure state of three qubits lies in an eight-dimensional Hilbert space, and requires eight complex amplitudes or sixteen real parameters. One of these can be fixed by normalizing the state. The set of local unitary operations on three qubits is a group of dimension ten (including overall phase). Thus, we would expect at least five parameters to be necessary to describe properties of the state which are invariant under local unitary transformations—entanglement properties.

While investigation has shown that five parameters are indeed sufficient to determine the entanglement properties of a three-qubit pure state, it is far from obvious how to choose these parameters. Many parametrizations are possible, but some may not be defined for all states, requiring different coordinate “patches” to cover different regions of Hilbert space.

B. The Linden-Popescu-Schlienz Standard Form

One simple parametrization that has been proposed [13] is the Linden-Popescu-Schlienz (LPS) standard form. One begins by putting qubit 1 in its Schmidt basis, giving a state of form (5). One can then choose one of the two correlated states, say \( |\psi_0\rangle_{23} \), and find its corresponding Schmidt basis. The resulting state for the three qubits has the form

\[
|\psi\rangle = \sqrt{p} |0\rangle \left( a |00\rangle + \sqrt{1-a^2} |11\rangle \right) \\
+ \sqrt{1-p} |1\rangle \left( \gamma \left( \sqrt{1-a^2} |00\rangle - a |11\rangle \right) + f |01\rangle + g |10\rangle \right),
\]

where \( p, a \) and \( f \) are real positive numbers, \( g \) is complex, and \( \gamma = (1 - f^2 - |g|^2)^{1/2} \). Together these give five independent real parameters. (Note that \( f \) is made real by choosing the phases of the basis vectors appropriately. One could equally well use a different convention, for instance making \( g \) real and \( f \) complex.)
C. The Griffiths-Niu representation

Just as in the case of the LPS form, we begin by putting qubit 1 in its Schmidt basis, giving the form (5). The two state vectors $|\psi_0\rangle_{23}$ and $|\psi_1\rangle_{23}$ span a two-dimensional subspace of the Hilbert space for qubits 2 and 3. Niu and Griffiths showed [16] that any such two-dimensional subspace can be given basis vectors of the form

$$|\chi_0\rangle = \sqrt{q} |00\rangle_{23} + \sqrt{1-q} |11\rangle_{23},$$
$$|\chi_1\rangle = \sqrt{r} |01\rangle_{23} + \sqrt{1-r} |10\rangle_{23},$$

by some choice of a product basis for the 4-D Hilbert space of the two bits, where $q$ and $r$ are real numbers between 0 and 1. By choosing an appropriate phase for the product basis vectors we can write

$$|\psi_0\rangle_{23} = a |\chi_0\rangle + b |\chi_1\rangle,$$
$$|\psi_1\rangle_{23} = -b^* |\chi_0\rangle + a |\chi_1\rangle,$$

where $a$ is real and $a^2 + |b|^2 = 1$. The choice of $|\chi_0,1\rangle$ is then unique, and the three-qubit state is

$$|\psi\rangle = \sqrt{p} |0\rangle \left( a\sqrt{q} |00\rangle + a\sqrt{1-q} |11\rangle + b\sqrt{r} |01\rangle + b\sqrt{1-r} |10\rangle \right)$$
$$+ \sqrt{1-p} |1\rangle \left( -b^* \sqrt{q} |00\rangle - b^* \sqrt{1-q} |11\rangle + a\sqrt{r} |01\rangle + a\sqrt{1-r} |10\rangle \right),$$

which is described by five real parameters: $p$, $q$, $r$, $a$, and the phase of $b$. This form has an obvious resemblance to the LPS form, but treats the $|0\rangle$ and $|1\rangle$ terms more symmetrically; however, there is still a lack of symmetry under interchange of the bits.

D. The minimal representation

Another interesting representation has been proposed by Acin et al. [15], who have shown that all three-qubit states can be written in the form

$$|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\phi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle$$

by a suitable choice of basis, where the $\lambda_i$ are all real and positive and $\phi$ is a phase between 0 and $\pi$. With only five terms, this is a minimal description, and in that sense a generalization of the bipartite Schmidt decomposition (which is also minimal). However, the $\lambda_i$ and $\phi$ have no obvious physical interpretation, and this representation is once again asymmetric under interchange of the bits.

E. The Schmidt representation

Suppose we choose to represent a pure state for three qubits in their Schmidt bases. The state has the form
\[ |\psi\rangle = a|000\rangle + b|001\rangle + c|010\rangle + d|011\rangle + e|100\rangle + f|101\rangle + g|110\rangle + h|111\rangle, \]

(11)

which looks just like a generic three-qubit state with 16 parameters. However, using each of the three qubits in turn we can write \( |\psi\rangle \) in a form similar to (5), with the pairs of states \( \{|\psi_0\rangle_{23}, |\psi_1\rangle_{23}\}, \{|\psi_0\rangle_{12}, |\psi_1\rangle_{12}\}, \) and \( \{|\psi_0\rangle_{13}, |\psi_1\rangle_{13}\} \) orthogonal. These conditions impose restrictions on the possible values of the coefficients in (11).

By redefining the relative phases of the basis vectors

\[ |0\rangle_j, |1\rangle_j \rightarrow \exp(i\phi_j)|0\rangle, \exp(i\theta_j)|1\rangle, \]

(12)

we can choose to make four of the coefficients real. A convenient choice is to make \( a, d, f, g \) real, while \( b, c, e, h \) remain complex. The state must also be normalized, which imposes the condition

\[ a^2 + |b|^2 + |c|^2 + d^2 + |e|^2 + f^2 + g^2 + |h|^2 = 1. \]

(13)

This leaves 11 undetermined parameters.

We can now express the larger eigenvalues \( p_{A,B,C} \) of the reduced density matrices \( \rho_{A,B,C} \) in terms of the coefficients:

\[
\begin{align*}
    p_A &= a^2 + |b|^2 + |c|^2 + d^2, \\
    p_B &= a^2 + |b|^2 + |e|^2 + f^2, \\
    p_C &= a^2 + |c|^2 + |e|^2 + g^2,
\end{align*}
\]

(14)

(the smaller eigenvalues obviously being \( 1 - p_{A,B,C} \)). Finally, the states correlated with basis vectors \( |0\rangle_j \) and \( |1\rangle_j \) must be orthogonal to each other. This gives three more equations:

\[
\begin{align*}
    ae^* + bf + cg + dh^* &= 0, \\
    ac^* + bd + eg + fh^* &= 0, \\
    ab^* + cd + ef + gh^* &= 0.
\end{align*}
\]

(15)

Because these equations are complex, they are equivalent to six real equations.

Combining these restrictions, we now have fourteen equations in sixteen unknowns. Thus, in addition to the eigenvalues \( p_{A,B,C} \) we would expect there to be two more free parameters. Can we identify reasonable candidates for these parameters? It turns out that natural choices are the two probabilities \( a^2 \) and \( |h|^2 \). These parameters are symmetric under interchanges of the three qubits, and have a fairly simple physical interpretation: they are the probabilities of all three qubits giving the same result when measured in their Schmidt bases. What is more, the coefficients of the other state vectors can all be calculated in terms of the five probabilities \( a^2, |h|^2 \), and \( p_{A,B,C} \).

The expressions for the norms of the coefficients are relatively simple:

\[
\begin{align*}
    |b|^2 &= \frac{(2p_C - 1)|h|^2 - (p_A + p_B - 1)(2a^2 - p_A - p_B - p_C + 1)}{2p_A + 2p_B + 2p_C - 3}, \\
    |c|^2 &= \frac{(2p_B - 1)|h|^2 - (p_A + p_C - 1)(2a^2 - p_A - p_B - p_C + 1)}{2p_A + 2p_B + 2p_C - 3}.
\end{align*}
\]
\(|e|^2 = \frac{(2p_A - 1)|h|^2 - (p_B + p_C - 1)(2a^2 - p_A - p_B - p_C + 1)}{2p_A + 2p_B + 2p_C - 3}\)

\(d^2 = \frac{(2p_A - 1)a^2 - (p_B + p_C - 1)(2|h|^2 + p_A + p_B + p_C - 2)}{2p_A + 2p_B + 2p_C - 3}\)

\(f^2 = \frac{(2p_B - 1)a^2 - (p_A + p_C - 1)(2|h|^2 + p_A + p_B + p_C - 2)}{2p_A + 2p_B + 2p_C - 3}\)

\(g^2 = \frac{(2p_C - 1)a^2 - (p_A + p_B - 1)(2|h|^2 + p_A + p_B + p_C - 2)}{2p_A + 2p_B + 2p_C - 3}\). \hfill (16)

The phases of \(b, c, e\) are more complicated. If we define the variables \(\phi_{b,c,e}\) by \(b = |b|\exp(i\phi_b), c = |c|\exp(i\phi_c),\) and \(e = |e|\exp(i\phi_e)\), the constraint equations (13,14,15) imply after a bit of algebra that

\[
\cos(\phi_b) = (Q_1/|b|)(-2adf + g(a^2 + d^2 + f^2 - g^2)),
\]

\[
\cos(\phi_c) = (Q_1/|c|)(-2adg + f(a^2 + d^2 - f^2 + g^2)),
\]

\[
\cos(\phi_e) = (Q_1/|e|)(-2afg + d(a^2 - d^2 + f^2 + g^2)),
\]

\[
\cos(\phi_h) = (Q_2/|h|)(-2dfg + a(-a^2 + d^2 + f^2 + g^2)),
\]

\[
\sin(\phi_b) = (Q_2/|b|)(2adf + g(a^2 + d^2 + f^2 - g^2)),
\]

\[
\sin(\phi_c) = (Q_2/|c|)(2adg + f(a^2 + d^2 - f^2 + g^2)),
\]

\[
\sin(\phi_e) = (Q_2/|e|)(2afg + d(a^2 - d^2 + f^2 + g^2)),
\]

\[
\sin(\phi_h) = (Q_2/|h|)(2dfg + a(-a^2 + d^2 + f^2 + g^2)),
\]

where \(Q_1\) and \(Q_2\) are two constants. We can solve for the values of \(Q_1\) and \(Q_2\) by using the identity \(\sin^2(\phi) + \cos^2(\phi) = 1\), which gives us

\[Q_1^2 = \left(\frac{|c|^2(2adf + g(a^2 + d^2 + f^2 - g^2))^2}{8adf g(2p_A - 1)(2p_B - 1)(2p_C - 1)(p_B - p_C)} - \frac{|b|^2(2adg + f(a^2 + d^2 - f^2 + g^2))^2}{8adfg(2p_A - 1)(2p_B - 1)(2p_C - 1)(p_B - p_C)}\right) \times \left(\frac{2p_A + 2p_B + 2p_C - 3}{2a^2 + 2|h|^2 + p_A + p_B + p_C - 2}\right)^4, \hfill (18)\]

\[Q_2^2 = \frac{|b|^2 - Q_1^2(-2adf + g(a^2 + d^2 + f^2 - g^2))^2}{(2adf + g(a^2 + d^2 + f^2 - g^2))^2}. \hfill (19)\]

Of course, the earlier expressions must be substituted for \(|b|, |c|, d, |e|, f, g\) in terms of \(a^2, |h|^2\) and \(p_{A,B,C}\), which results in formulae of great complexity but no intrinsic difficulty. The expressions for \(Q_1^2\) and \(Q_2^2\) are not unique; different expressions can be derived, but they must of course all be equal. These equations determine \(Q_1\) and \(Q_2\) only up to a sign.

**F. Interpretation of the Schmidt form parameters**

In the Schmidt form for three-qubit pure states, each of the five parameters has a reasonably straightforward physical interpretation. The three parameters \(p_A, p_B, p_C\) are the larger
(i.e., $p > 1/2$) eigenvalues of the reduced density operators for each of the three qubits, and correspond to the probabilities of obtaining the more likely of the two possible outcomes (which by convention we label $|0\rangle$) when we measure each of the qubits in its Schmidt basis. These parameters are closely related to the minimum absolutely selective information for each qubit, which is given by the entropy function

$$\min S_i = -(p_i \log_2 p_i + (1 - p_i) \log_2(1 - p_i)).$$

(20)

This quantity is the minimum amount of fundamentally unpredictable classical information generated by carrying out a measurement on qubit $i$, given a free choice of measurement basis [14]. By using the Schmidt form to choose measurement bases we can simultaneously minimize the absolutely selective information for all three qubits.

Given a supply of copies of the state $|\psi\rangle$, the probabilities $p_A, p_B, p_C$ can each be determined by local measurements on one of the three qubits. These local probabilities (and functions of them) are the only locally invariant quantities that can be measured in this way; however we choose to select the remaining two parameters, they can only be determined by simultaneous measurements on more than one qubit. (Note that these don’t necessarily have to be joint measurements; local coincidence measurements will also work.) The existence of two other parameters beyond the locally determinable ones gives rise to the phenomenon of locally indistinguishable states that are incommensurate [8]. Two states with the same values of $p_A, p_B, p_C$ will be locally indistinguishable, but if they differ in their values for one or both of the other two parameters, then they cannot be interconverted in either direction by local operations and classical communication, even if nonunitary operations (e.g., measurements) are permitted. This is in contrast to the bipartite case, for which two states with identical local density operators are always interconvertible locally; indeed in the bipartite case this interconversion is achievable by local unitary operations alone. Sets of incommensurate states have potential cryptographic applications.

For the remaining two parameters in the Schmidt form, we generally use $a^2$ and $|h|^2$ as being the most symmetric choice. Both $a^2$ and $|h|^2$ have a fairly straightforward physical interpretation; they are the probabilities to obtain the same outcome for all three qubits, either 000 or 111, when we measure them in their Schmidt bases.

The parameters $p_A, p_B, p_C$ range from 1/2 to 1 (since they are defined to be the larger eigenvalues of their corresponding local density matrices). Similarly, $a^2$ ranges from 0 to 1, and $|h|^2$ from 0 to 1/2 (where this asymmetry arises from the convention that $|0\rangle_{A,B,C}$ is the eigenstate corresponding to the larger eigenvalue $p_{A,B,C}$). However, this does not mean that these parameters can take arbitrary values within these ranges. Some choices of parameter values correspond to no physical state, and give nonsensical values for (16) and (17).

In particular, the local probabilities must obey the triangle inequalities

$$p_A(1 - p_A) + p_B(1 - p_B) \geq p_C(1 - p_C),$$

$$p_B(1 - p_B) + p_C(1 - p_C) \geq p_A(1 - p_A),$$

$$p_C(1 - p_C) + p_A(1 - p_A) \geq p_B(1 - p_B);$$

(21)

these imply, for instance, that if $p_A = 1$ then $p_B = p_C$. The restrictions on $a^2$ and $|h|^2$ are more complicated, but some idea can be gotten from the graphs in figures 1a–d. Here we see that the possible values for $a^2$ and $|h|^2$ depend on both the local probabilities and each other.
III. PROOF OF DISTILLABILITY

The Schmidt form can help to provide analytical insight when addressing specific problems. For example, the efficacy of a recently proposed tripartite distillation protocol [14] can be demonstrated with the help of the Schmidt form.

Consider a state of three qubits in an arbitrary product basis, which can be written in the form (11). We can straightforwardly calculate the quantity

\[ p_A(1 - p_A) = |af - be|^2 + |ag - ce|^2 + |ah - de|^2 \]
\[ + |bg - cf|^2 + |bh - df|^2 + |ch - dg|^2, \]  

(22)

This expression is a polynomial in the coefficients and their complex conjugates, and is correct in any basis. If the state is in the Schmidt form, this expression simplifies to

\[ p_A(1 - p_A) = (a^2 + |b|^2 + |c|^2 + d^2)(|e|^2 + f^2 + g^2 + |h|^2). \]  

(23)

Let us now assume that we have written the state in Schmidt form, such that the states \{|0\}, \{|1\}\} for each qubit \(j\) are eigenstates of the local density matrix with eigenvalues \(p_j\) and \(1 - p_j\), respectively. Suppose we now perform a weak measurement on each of the three qubits. First, allow each qubit to interact with a separate ancilla bit initially in state \(|0\rangle_{\text{anc}}\), such that

\[ |0\rangle \otimes |0\rangle_{\text{anc}} \rightarrow \sqrt{1 - \epsilon} |0\rangle \otimes |0\rangle_{\text{anc}} + \sqrt{\epsilon} |0\rangle \otimes |1\rangle_{\text{anc}}, \]
\[ |1\rangle \otimes |0\rangle_{\text{anc}} \rightarrow |1\rangle \otimes |0\rangle_{\text{anc}}, \]  

(24)

where \(\epsilon \ll 1\). Then measure the three ancilla bits. With a probability of \(\epsilon(p_A + p_B + p_C)\) one will find one or more of the ancilla bits in state \(|1\rangle_{\text{anc}}\), in which case the procedure has failed. Otherwise, this step has succeeded and the three qubits are now in a new state with slightly different coefficients \(a', b', \ldots, h'\). The changes in the coefficients are

\[ \Delta a = -(\epsilon/2)(3 - p_A - p_B - p_C)a, \]
\[ \Delta b = -(\epsilon/2)(2 - p_A - p_B - p_C)b, \]
\[ \Delta c = -(\epsilon/2)(2 - p_A - p_B - p_C)c, \]
\[ \Delta d = -(\epsilon/2)(1 - p_A - p_B - p_C)d, \]
\[ \Delta e = -(\epsilon/2)(2 - p_A - p_B - p_C)e, \]
\[ \Delta f = -(\epsilon/2)(1 - p_A - p_B - p_C)f, \]
\[ \Delta g = -(\epsilon/2)(1 - p_A - p_B - p_C)g, \]
\[ \Delta h = (\epsilon/2)(p_A + p^2 + p_C)h. \]  

(25)

This very simple form results because the state is in Schmidt form. After this procedure the bases for the three bits will generally no longer be the correct Schmidt basis (though it will be close to it), so the expression (23) cannot be used; but (22) is always correct. Thus we get a change in \(p_A(1 - p_A)\)
\[
\Delta[p_A(1 - p_A)] = -\epsilon(4 - 2(p_A + p_B + p_C))(|af - be|^2 + |ag - ce|^2)
- \epsilon(3 - 2(p_A + p_B + p_C))(|ah - de|^2 + |bg - cf|^2)
- \epsilon(2 - 2(p_A + p_B + p_C))(|bh - df|^2 + |ch - dg|^2)
= -\epsilon(3 - 2(p_A + p_B + p_C))p_A(1 - p_A)
- (\epsilon/2)(|af - be|^2 + |ag - ce|^2 - |bh - df|^2 - |ch - dg|^2)
\]

By making use of equations (14) and (15), this expression simplifies to

\[
\Delta[p_A(1 - p_A)] = \epsilon\left[(2(p_A + p_B + p_C) - 3)p_A(1 - p_A)
+ p_A(a^2 - |e|^2 + |h|^2 - d^2) + d^2 - a^2\right],
\]

which using (16) further simplifies to

\[
\Delta[p_A(1 - p_A)] = \frac{\epsilon(2p_A - 1)}{2p_A + 2p_B + 2p_C - 3}\left[2(a^2 + |h|^2)(p_B + p_C - 1)
- (2p_A - 1)(p_A + p_B + p_C - 1)(p_A + p_B + p_C - 2)\right].
\]

The prefactor to (28) is strictly positive, as is the first term inside the brackets. The second
term is positive if \(p_A + p_B + p_C < 2\); any state that satisfies this criterion will evolve towards
the GHZ state and have a nonzero yield.

For \(p_A + p_B + p_C \geq 2\), the sign of (28) depends on the relative sizes of the first and
second terms inside the brackets. A brief examination of the equations (16) shows that for
\(p_A + p_B + p_C \geq 2\) the quantity \(a^2\) must obey the inequality

\[2a^2(p_B + p_C - 1) \geq (2p_A + p_B + p_C - 2)(p_A + p_B + p_C - 2),\]

which when plugged into the expression for \(\Delta[p_A(1 - p_A)]\) yields the inequality

\[\Delta[p_A(1 - p_A)] \geq \epsilon(2p_A - 1)(1 - p_A)(p_A + p_B + p_C - 2) \geq 0.\]

Because of the symmetry of the protocol, \(p_B(1 - p_B)\) and \(p_C(1 - p_C)\) must also increase. So one
step of this protocol must move the state towards the GHZ with nonvanishing probability,
and will (in general) produce a nonzero yield of GHZ triplets.

There are three circumstances in which this result can fail. First, no product state can
ever be distilled to a GHZ by this method. At least one of \(p_A, p_B, p_C\) must equal 1 in this
case, which causes the rate (28) corresponding to it to vanish. This is not immediately
obvious from the form of (28), but it is easily checked using (16) and (21)—if \(p_A = 1\), then
\(p_B = p_C = a^2\), and (28) is equal to zero.

Second, there are states with \(p_A + p_B + p_C = 2\) for which \(a^2 = |h|^2 = 0\), again making
(28) vanish. These are a subset of the triple states described below, which are equivalent to
states of the form (33); these states also minimize the residual tangle. Finally, it is possible
for a state with \(p_A + p_B + p_C > 2\) to evolve to one of these triple states. The class of triple
states includes states with \(p_A + p_B + p_C > 2\) which do exactly that, and other states can
also do so, though this set is of lower dimension.
IV. POLYNOMIAL INVARIANTS AND DISTILLABILITY

A. Polynomial invariants

A somewhat different approach to the parametrization problem, rather than putting the state in a standard form, is to look for \emph{locally invariant quantities}—functions of the state which are invariant under local unitary transformations [7,10,11]. (Of course, the parameters of a standard form are exactly such functions, making that approach a special case.) A potentially infinite number of such invariants exists, but it suffices to find five independent quantities, preferably with clear physical interpretations.

Much of the effort in this respect has focussed on polynomial invariants (that is, polynomial in the state coefficients and their complex conjugates), which are analytically tractable. An obvious such invariant is simply the norm of the state, usually taken to be 1. Somewhat more interesting are the quantities

\[ \text{Tr}\{\rho_i - \rho_i^2\} = 2p_i(1-p_i), \]  

where \( \rho_i \) is the reduced density matrix of subsystem \( i \). This quantity is an invertible function of the \( p_i \) used above in the Schmidt form, or of the von Neumann entropy \( S_i \), but unlike those two quantities is a polynomial in the state coefficients.

Another proposed invariant quantity is the \emph{residual tangle} of Coffman et al. [9]. This can be written

\[ \tau_{ABC} = 2(\lambda_{ij}^1 \lambda_{ij}^{AB} + \lambda_{ij}^1 \lambda_{ij}^{AC}), \]  

where \( \lambda_{ij}^1 \) and \( \lambda_{ij}^2 \) are the (positive) eigenvalues of the matrix \( \sqrt{\rho_{ij}\tilde{\rho}_{ij}} \). Here \( \rho_{ij} \) is the density operator for the two-party \( ij \) system, and \( \tilde{\rho}_{ij} \) is the “spin-flipped” density operator: \( \tilde{\rho}_{ij} = (\sigma_y \otimes \sigma_y)\rho_{ij}^*(\sigma_y \otimes \sigma_y) \). It has been suggested that the residual tangle is a measure of the irreducible three-way (“GHZ-type”) entanglement of a tripartite state, beyond any two-party (“EPR-type”) entanglement that may be contained in such a state. As such, it is of particular interest in discussing distillability below. While not itself a polynomial quantity, its square \( \tau_{ABC}^2 \) is.

There is a particular set of states for which the residual tangle vanishes [9]. We have previously described states in this set as “triple” states [14], because they are equivalent under local unitary transformations to states with just three components:

\[ |\psi_{tr}\rangle = b|001\rangle + c|010\rangle + e|100\rangle. \]

Carteret and Sudbery [10] refer to these as “beechnut” or “tetrahedral” states, and they clearly include all product states as a limit. In terms of the Schmidt parameters, these states have \( p_A + p_B + p_C \geq 2 \). Interestingly, as well as having zero residual tangle, states of this type have vanishing primary yield for the tripartite distillation protocol described in section III and in [14]. This means that it is impossible by using protocols of this type to distill a GHZ triplet directly from a single copy of a triple state. These states can yield GHZ states only by \emph{secondary distillation}, which involves first distilling pairs of subsystems from the starting state into EPR pairs, and then using these EPR pairs in combination to prepare GHZ triplets.
This correspondence between residual tangle and primary distillation yield for the triple state case suggests that there may be a more general connection between these quantities. (This will be explored graphically below.) Furthermore, it suggests that the operational criterion of primary distillability may be directly related to fundamental three-way entanglement.

Together with (31), $\tau^{2}_{ABC}$ gives four independent quantities. Sudbery [11] suggests for a fifth Kempe’s invariant [8],

$$I_5 = \sum_{ijklmnopq} t_{ijk} t_{ilm}^{*} t_{nlo} t_{pjo}^{*} t_{pqm} t_{nqk}^{*},$$

where

$$|\psi\rangle = \sum_{ijk} t_{ijk} |ijk\rangle.$$  

(34)

This quantity is independent of the others, though its physical significance is not entirely clear. Other quantities have also been proposed [17], but these five are sufficient to determine states which are equivalent under local unitary transformations.

B. Distillability and ‘Special’ states

The very interesting results of Carteret and Sudbery [10] demonstrate the existence of classes of ‘special’ states for three qubits, which behave nongenerically under local unitary transformations. A possible interpretation is that these ‘special’ classes represent states with unusual entanglement properties. This suggests in turn that these states may have unusual distillability properties. This is clearly true for the triple states described above, which are one such class. We evaluate this possibility in terms of the primary yield achievable by the distillation protocols described above. This yield is the percentage of an ensemble of systems in the given starting state that arrives in the GHZ state after the protocol has been carried out; equivalently, it is the probability of successfully transforming a single copy of the system into a GHZ triplet. (We do not include secondary distillation.)

Carteret and Sudbery’s result concerns the stabilizer of a three-qubit state: the set of all local unitary transformations which leave that state unchanged. For a generic state, this stabilizer has dimension zero, i.e., it includes only discrete members. Certain exceptional states, however, have a stabilizer with nonzero dimension.

Interestingly, the steps of the GHZ distillation technique commute with local unitary transformations. Because of this, the distillation procedure preserves the stabilizer of the initial state, and hence must take ‘special’ states to other ‘special’ states of the same type. This gives a different way of understanding why triple states have zero primary yield. Because all triple states have $p_A + p_B + p_C \geq 2$, they cannot include the GHZ state (with $p_A + p_B + p_C = 3/2$) as a limit. The protocol always takes triple states to triple states; once on the set of triple states, it is impossible to move off of it, so the yield of GHZ triplets must be zero.

The set of product states (or “bystander states” in the terminology of Carteret and Sudbery) is also ‘special’ in this sense, and also cannot be distilled. In this case, there is no
tripartite entanglement present to be concentrated. All the other ‘special’ classes include the GHZ as a limit, and therefore are distillable. Let us examine them briefly:

**Generalized GHZ states.** These states can be written in Schmidt form

\[ |\psi\rangle = a|000\rangle + h|111\rangle. \] (36)

They have \( p_A = p_B = p_C = a^2 \), residual tangle \( \tau = 4a^2h^2 \), and primary yield \( Y = 1 - \sqrt{1 - \tau} = (2/3)(3 - p_A - p_B - p_C) \). Unlike most tripartite states, the generalized GHZs can be distilled by asymptotic distillation techniques developed for bipartite pure states; the asymptotic yield is \(-a^2 \log_2 a^2 - h^2 \log_2 h^2 \geq Y\).

**Slice states.** In Schmidt form these are

\[ |\psi\rangle = a|000\rangle + d|011\rangle - e|100\rangle + h|111\rangle, \quad ae = dh, \] (37)

and similar states derived by permuting the order of the bits. These states are simultaneously in both Schmidt and LPS standard form. They have \( p_B = p_C = a^2 + e^2 \), \( p_A = a^2 + d^2 \), \( \tau = 4(ah + de)^2 = 4a^2(h + e^2/h)^2 \). The yield is difficult to evaluate analytically, but numerical evidence shows no unusual behavior under distillation. However, there is a subclass of these states which do behave nongenerically, the maximal slice states.

**Maximal slice** or **Slice-ridge states** are of form (37) with \( a^2 + e^2 = 1/2 \); for these states \( Y = 2(1 - p_A) = 2(2 - p_A - p_B - p_C) = 1 - \sqrt{1 - \tau} \) and \( p_B \) and \( p_C \) are both equal to 1/2. The expression for the primary yield in terms of the residual tangle is identical to that for the GHZ-type states, while in terms of \( p_A + p_B + p_C \) it is identical to the generalized triple states below.

**Generalized triple states.** These are not special states in the sense of Carteret and Sudbery; however, though still discrete, their stabilizers are larger than most states, and this class is also preserved by the primary distillation procedure. These states are written

\[ |\psi\rangle = b|001\rangle + c|010\rangle + e|100\rangle + h|111\rangle. \] (38)

We are mainly interested here in the symmetric state \( b = c = e \); for this case \( p_A = p_B = p_C = 2b^2 \) and \( Y = 2(2 - p_A - p_B - p_C) \); the residual tangle is \( \tau = 16b^3\sqrt{1 - 3b^2} = \sqrt{(4 - Y)^3Y/27} \). They are interesting because they are the least distillable states with \( p_A + p_B + p_C < 2 \).

We can put these results together by looking at the relationship between the primary yield \( Y \), \( \tau_{ABC} \), and \( p_A + p_B + p_C \). We have plotted these quantities for a large sample of randomly generated states in figures 2 and 3, with the families of ‘special’ states indicated. We see that most of these states are indeed special as far as distillation is concerned: they form the boundaries of the plotted regions. The quantity \( \tau_{ABC} \) does seem to be closely related to distillability, though this relationship is not exact, at least for our technique; for a given value of \( \tau_{ABC} \) states with a range of \( Y \) values exist, but this range is not very wide. This range is bounded at the top by the generalized GHZ and maximal slice states, and at the bottom by the symmetric generalized triple state. All true triple states have \( Y = \tau_{ABC} = 0 \). There is also a relationship between \( p_A + p_B + p_C \) and \( Y \), though again for a given \( p_A + p_B + p_C \) there is a range of \( Y \) values. This range too is bounded above by the generalized GHZs, and below by the triple states, generalized triples and maximal slice states. (The maximal slice states have the curious distinction of being highly distillable by
one measure and highly undistillable by another; for a given $\tau_{ABC}$ they have the minimal value of $p_A + p_B + p_C$.

We have looked for a relationship between primary yield and Kempe’s invariant polynomial discussed in section IVA above; there seems little relationship between them, but it is possible that this invariant has some other, as yet unknown, physical interpretation.

V. CONCLUSIONS

We have examined tripartite entanglement from both an analytical and an operational point of view. In the bipartite case, which is well understood and to which we have turned for clues, the analytical and operational aspects of entanglement are very closely related; asymptotic distillation yields and preparation resources for a given state are given by one parameter, while the entanglement properties of a single copy are given by the locally invariant parameters, the Schmidt coefficients. We have looked for similar connections in the three-qubit case. Here at least five locally invariant parameters are required, as opposed to just one in the two-qubit case, which inevitably leads to a much greater complexity in the tripartite description. We have examined several ways of choosing these five parameters, looking for a clear physical interpretation, mathematical tractability, and usefulness is solving particular problems. One representation in particular, the “Schmidt form,” seems especially promising in all three respects, though it is still quite complex. We have also looked at some proposed polynomial invariants, especially the residual tangle. In particular, we have looked for connections between these parameters and yields in distilling GHZ triplets, showing that ‘special’ classes of states within Carteret and Sudbery’s recently introduced scheme extremize the distillation yield for particular values of the residual tangle and $p_A + p_B + p_C$.

Although a certain amount amount of progress towards understanding tripartite entanglement has been made, at least for qubits, many important questions remain unanswered. For example, the number of states in the asymptotic minimum reversible entanglement generating set (MREGS) for three-qubit states, and for tripartite states in general, is still unknown. No asymptotically reversible (or optimal but irreversible) distillation technique for GHZ states is known. The search for solutions to these and related problems is ongoing.

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REFERENCES

Figure 1. These figures illustrate the restrictions on the allowed ranges for the Schmidt invariants $a^2$ and $|h|^2$ as a function of $p_{\text{sum}} = p_A + p_B + p_C$ and each other. For a randomly-generated sample of states we plot a) $a^2$ vs. $p_{\text{sum}}$, b) $|h|^2$ vs. $p_{\text{sum}}$, c) $a^2 + |h|^2$ vs. $p_{\text{sum}}$, and d) $a^2 - |h|^2$ vs. $p_{\text{sum}}$. Some of the boundaries of these regions are occupied by the ‘special states’ of section IV.

Figure 2. Here we plot the primary yield of GHZ triplets from the infinitesimal distillation algorithm of section III vs. $p_{\text{sum}} = p_A + p_B + p_C$ for various ‘special states’ as well as a random sample of generic states. We see that all states lie between two linear boundaries; the generalized GHZ states lie on the upper boundary, while the maximal slice and generalized Triple states lie on the lower boundary, and the triple states are the zero-yield states between $p_{\text{sum}} = 2$ and $p_{\text{sum}} = 3$. The upper linear boundary corresponds to the yield of Bernstein and Bennett’s Procrustean method of EPR distillation in the bipartite case.

Figure 3. Here we plot the primary yield of GHZ triplets from the infinitesimal distillation algorithm of section III vs. the square of the residual tangle $\tau_{ABC}^2$ for various ‘special states’ as well as a random sample of generic states. We see that all states lie between two curved boundaries; the generalized GHZ and maximal slice states lie on the upper boundary, while the generalized triple states lie on the lower boundary. The triple states all have both $\tau_{ABC}$ and the yield equal to zero. Interestingly, the maximal slice states appear to be high-yield states when plotted against $\tau_{ABC}$, but low-yield when plotted against $p_{\text{sum}} = p_A + p_B + p_C$; for a given value of $\tau_{ABC}$ these states minimize $p_{\text{sum}}$. 

16
Figure 1a.
Figure 1b.

\[ p_A + p_B + p_C \]
Figure 1c.
Figure 1d.

\[ p_A + p_B + p_C \]

\[ a^2, h^2 \]
Primary yield of GHZ triplets

Figure 2.
Generalized GHZ and slice-ridge states
Generalized triple states
Random states

Primary yield of GHZ triplets

Figure 3.