No Black Hole Theorem in Three-Dimensional Gravity

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A common property of known black hole solutions in (2+1)-dimensional gravity is that they require a negative cosmological constant. In this letter, it is shown that a (2+1)-dimensional gravity theory which satisfies the dominant energy condition forbids the existence of a black hole to explain the above situation.

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The (2+1)-dimensional theory provides us with one of useful approaches to more complicated (3+1)-dimensional classical gravity or conceptual problems in quantum gravity [1]. At first sight, the (2+1)-dimensional gravity looks trivial. In particular, the vacuum Einstein equation implies that the space-time is locally flat, corresponding to the absence of the gravitational radiation (Weyl tensor) in three dimensions. However, the local distribution of matter fields has a global effect on the outer empty space; for instance, the gravitational field of a point particle is described by a conical space with its deficit angle corresponding to the mass of the particle [2], which causes the gravitational lens effect. One should also note that the triviality of local geometry does not necessarily imply the triviality of the theory itself; namely, the topological degrees of freedom plays an important role in the theory of gravitation [3,4]. The triviality of local geometry in the (2+1)-gravity theory holds even if the cosmological term is taken into account. The Einstein space is simply a space of constant curvature in three dimensions, so that educated relativists would not imagine that there is a black hole solution in this theory until in 1992 Bañados et al. show that there actually exists a black hole in the locally anti-de Sitter space [5,6]. This black hole space-time, called BTZ black hole, is obtained by identifying certain points of (the covering manifold of) the anti-de Sitter space. A different identification makes a space-time representing the BTZ black hole in a closed universe [9], multiple BTZ black holes [10] or a creation of the BTZ black hole [11]. The BTZ black hole is characterized by the mass, angular momentum and cosmological constant, and has almost all features of the Kerr-anti-de Sitter black hole in the conventional four-dimensional Einstein gravity. The BTZ black hole was shown to be also the solution of a low energy string theory [7,8].

Since the discovery of the BTZ black hole, a number of authors have attempted to find a black hole solution in various theories in (2+1)-dimensions. Black holes in topologically massive gravity [12] with the negative cosmological constant were found by Nutku [13]. In Einstein-Maxwell-A system, a static (non-rotating) charged black hole had been already noted in the original paper by Bañados et al. [5]. Clément [14] generated from the charged BTZ black hole a class of rotating charged black holes. Though rotating solutions in Einstein-Maxwell-A theory seem to have infinite total mass and angular momentum, these divergences may be cured by adding a Chern-Simons term to the action [14]. Black holes with a dilaton field have been discussed by many authors. In Brans-Dicke theory, Sa et al. found black hole solutions [16,17], and their properties were extensively studied for different Brans-Dicke parameters. Black holes in Einstein-Maxwell-dilaton-A theory were obtained by Chan and Mann in non-rotating [18] and rotating [19] cases. Other families were given by Koikawa et al. [20] and by Fernando [21]. Chen [22] also derived rotating black hole solutions in this theory by means of the duality transformation in the equivalent non-linear σ-model. Black holes coupled to a topological matter field [23], conformal scalar field [24], Yang-Mills field [25], Born-Infeld field [26] etc. were also discussed.

Thus, many black hole solutions have been known. Here, it might be interesting to note that all the black hole solutions listed above require a negative cosmological constant, otherwise a certain kind of energy conditions is violated. A typical example might be the BTZ black hole. As already mentioned, the BTZ black hole may be constructed by making identifications in the anti-de Sitter space. We may also consider a similar construction in the de Sitter space. In this case, a natural procedure might be identifying two geodesic circles in each Poincaré disk associated with the open chart of the de Sitter space. The resultant space-time represents an inflating universe rather than a black hole. The absence of black hole in this example might be due to the difference in the causal structure of conformal infinity [27].

The purpose of this letter is to give a reason for this situation. In particular, we will be able to answer the question: “Why the BTZ black hole requires a negative cosmological constant?” In the following, we consider the possibility of the existence of an apparent horizon in three-dimensional space-time with the procedure given by Hawking [28] in terms of the spin-coefficient formalism [29].

Let \((M,g)\) be a three-dimensional space-time and let
\[ \rho \] be a space-like hypersurface in \( \Sigma \). Suppose that \( \Sigma \) contains outer trapped surfaces, then will be an apparent horizon \( H \) which is defined to be the outer boundary of the trapped region in \( \Sigma \), where the notion “outer” is assumed to be well-defined as in the case of the asymptotically flat (or anti-de Sitter) space-time. The apparent horizon \( H \) will be a smooth closed curve in \( \Sigma \). Let \( m \) be a unit tangent vector of \( H \), and let \( n \) and \( n' \) be future directed out-going and in-going null vectors orthogonal to \( H \), respectively, such that \( g(n, n') = 1 \). The vectors \( n \) and \( n' \) are arranged such that \( n - n' \) lies in \( \Sigma \), which is always possible by means of the boost transformation \( n \mapsto a^2 n, n' \mapsto a^{-2} n' \) by some positive function \( a \). Let us consider a local deformation of \( H \) within \( \Sigma \) outside the trapped region generated by a vector field \( X = e^f (n - n') \) with some smooth function \( f \). Accordingly, the null triad \( \{ n, n', m \} \) is extended such that the normalization \( g(n, n') = -g(m, m) = 1 \), \( g(n, n) = g(n', n') = g(n, m) = g(n', m) = 0 \) are preserved and that \( m \) is tangent to each deformed \( H \). Then, since \( X \) and \( Y = e^m m \) form holonomic base vectors on \( \Sigma \) for some function \( h \), \( n \) and \( n' \) are propagated such that

\[
\delta f = \kappa - \tau + \beta = \kappa' - \tau' - \beta, \tag{1}
\]

where Ricci rotation coefficients

\[
\kappa = g(m, Dn), \quad \tau = g(m, D'n), \quad \beta = g(n', \delta n),
\]

\[
\kappa' = g(m, D'n'), \quad \tau' = g(m, D'n') \tag{2}
\]

and the differential operators

\[
D = \nabla_n, \quad D' = \nabla_{n'}, \quad \delta = \nabla_m \tag{3}
\]

are defined following the spin-coefficient formalism in four space-time dimensions [29]. The convergence of light rays emitted outward from each deformed \( H \) is measured by the quantity

\[
\rho = g(m, \delta n). \tag{4}
\]

In particular, \( \rho = 0 \) holds on \( H \) since \( H \) will be a marginally trapped surface. The change in \( \rho \) along \( X \) is derived from the following equations

\[
D\rho - \delta \kappa = (\epsilon + \rho)\rho - (2\beta + \tau + \tau')\kappa + \phi_{++}, \tag{5}
\]

\[
D'\rho - \delta \tau = -\epsilon \rho - \kappa \kappa' - \tau'' + \rho \delta + \phi_{+-} - \Pi, \tag{6}
\]

where

\[
\epsilon = g(n', Dn), \quad \epsilon' = g(n, D'n'),
\]

\[
\phi_{++} = \phi(n, n), \quad \phi_{+-} = \phi(n, n'), \quad \Pi = R/6 \tag{7}
\]

with the trace-free part of the Ricci tensor \( \phi = -\text{Ric} + (R/3)g \).

Subtracting Eq. (6) from Eq. (5), we obtain the equation

\[
e^{-f} \mathcal{L}_X \rho = \delta(\kappa - \tau) - (2\beta + \tau + \tau')\kappa + \kappa \kappa' + \tau^2 + \phi_{++} + \phi_{+-} + \Pi
\]

\[
= \delta(\delta f - \beta) + (\kappa - \tau)^2 + \phi_{++} + \phi_{+-} + \Pi \tag{8}
\]

on \( H \), where Eq. (1) has been used. Now suppose that there is a positive cosmological constant \( \Lambda > 0 \) and that the stress-energy tensor \( T \) satisfies the dominant energy condition: (i) \( T(W, W) \geq 0 \), and (ii) \( T(W) \) is non-space-like, for every time-like vector \( W \). Then, the Einstein equation \( \text{Ric} - (R/2)g + \Lambda g = -8\pi T \) leads to the inequalities

\[
\phi_{++} \geq 0, \quad \phi_{+-} + \Pi > 0. \tag{9}
\]

The term \( \delta(\delta f - \beta) \) in the last line of the Eq. (8) can be made zero by an appropriate choice of \( f \); in fact, parametrizing \( H \) by the proper length \( s \in [0, \text{Length}(H)] \), such \( f \) can be explicitly written as

\[
f = \int_s^0 \beta ds - \left( \frac{\delta}{\delta s} \frac{\beta ds}{ds} \right) s. \tag{10}
\]

Then, the last line of the Eq. (8) is positive definite, \( L_X \rho > 0 \). This implies that there is an outer trapped surface outside \( H \), which contradicts the assumption that \( H \) is the outer boundary of such surfaces. Hence, we obtain the following no black hole theorem:

**Theorem 1** Let \((M, g)\) be a three-dimensional space-time subject to the Einstein equation \( \text{Ric} - (R/2)g + \Lambda g = -8\pi T \) with \( \Lambda > 0 \). If the stress-energy tensor \( T \) satisfies the dominant energy condition, then \((M, g)\) contains no apparent horizons.

This explains why black hole solutions require a negative cosmological constant. Strictly speaking, we can only say that there is no non-degenerate apparent horizon \( (\rho = 0, L_X \rho \neq 0) \) in the case of \( \Lambda = 0 \), however, the presence of matter fields such as the dilaton or Maxwell field will exclude even degenerate horizons.

Thus, a black hole in \((2+1)\)-gravity requires negative energy such as a negative cosmological constant. This implies the breakdown of the predictability in certain three-dimensional theories. As in four space-time dimensions, we may consider the Oppenheimer-Snyder model of the gravitational collapse. The homogeneous disk of dust will collapse to a central point and a naked conical singularity will be left. This picture of gravitational collapses will remain unchanged unless the negative cosmological constant is added. Even in the case of the non-symmetric gravitational collapse of gauge fields or scalar fields, there will not form a black hole, so that when a singularity is formed, such a singularity will be naked.

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