A Novel Spin-Statistics Theorem in $(2 + 1)d$ Chern-Simons Gravity

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It has been known for some time that topological geons in quantum gravity may lead to a complete violation of the canonical spin-statistics relation: there may be no connection between spin and statistics for a pair of geons. We present an algebraic description of quantum gravity in $(2+1)d$ based on the first order formalism of general relativity and show that, although the usual spin-statistics theorem is not valid, statistics is completely determined by spin. Hence, a new spin-statistics theorem can be formulated.

On a space-time $M$ of the form $M = \Sigma \times \mathbb{R}$, the topology of a spatial slice is well-captured by soliton-like excitations of $\Sigma$ called topological geons [1–3]. In this letter we will be concerned with connected $(2+1)d$ orientable manifolds $M$. In this case, the topology of the orientable surface $\Sigma$ is determined by the number of handles of $\Sigma$, with each handle corresponding to a topological geon. Geons can have particle-like features. For example, one can assign to them properties such as spin and statistics. However, unlike ordinary particles they may violate the spin-statistics relation [2,3].

We first examine the meaning of spin and statistics before describing our approach to this problem. Suppose we have a configuration space $Q$ describing a pair of identical geons. One such configuration can be visualized as two handles on the plane. Now, the quantization of two geons on the plane is not unique. One has to choose some hermitian vector bundle $B_k$ over $Q$ whose square-integrable sections serve to define the domains of appropriate observables [1–3]. The index $k$ labels inequivalent quantizations. The space of these sections is the quantum Hilbert space $\mathcal{H}_k$ of the two-geon system. Physical operations can be implemented as operators on $\mathcal{H}_k$. If we perform a $2\pi$-rotation of one of the geons, described by an operator $C_{2\pi}$, then its eigenstate will change by a phase $e^{i2\pi S}$, where $S$ is the spin. Just like particles in $(2+1)d$, geons can carry fractional spin, i.e., $S$ can be any real number [3,4]. Similarly, if we exchange the position of the two geons, the wave function will change by the action of an operator $R$, the “statistics operator”. The standard spin-statistics relation would tell $R = C_{2\pi}$ on one of the geons. Note that there is no a priori reason for this relation to hold. Now one can ask if it is true for each quantization procedure parametrized by $k$. The authors of [2] and [3] show that some quantizations violate the spin-statistics theorem, but leave open the question of which are the ones that do not. Furthermore, as emphasized in [4], the list of quantum theories derived in [3] is completely based on kinematic considerations, only the diffeomorphism constraint being imposed. Imposing the latter would further restrict the states, and in this sense some of the values of $k$ may not be dynamically allowed.

In this letter we show that, at least for $(2 + 1)d$ gravity in the first order formalism, there is a generalization of the standard spin-statistics connection relating $R$ and $C_{2\pi}$. In the quantization scheme given in [3], one considers the mapping class group $M_2$ and finds a vector bundle $B_k$ for each unitary irreducible representation of $M_2$ [5]. Then, one sees no relation between $R$ and $C_{2\pi}$ for a generic $k$. Instead we will look at $M_2$ as part of a larger algebra $A$ of operators describing the quantum theory of geons. It contains the group algebra of $M_2$. Furthermore, the first order formalism naturally takes into account the dynamical constraints. The possible quantizations are given by unitary irreducible representations $\Pi_r$ of $A$, where the index $r$ parameterizes inequivalent quantizations. We show that there is a large class of quantizations (representations) $\Pi_r$ such that statistics is totally determined by spin according to the formula

$$\Pi_r(R) = e^{i(2\pi S - \theta[r])} \mathbb{I},$$

(1)

on state vectors of spin $S$. Here the extra phase $\theta[r]$ is completely fixed by the choice of the representation $\Pi_r$.

Topological censorship theorems [6] restrict the kinds of geon solutions one can have, at least in the metric framework, but it is still possible to construct explicit geon solutions in $(2+1)d$ gravity [7]. These are in general geodesically incomplete, but have a Cauchy surface $\Sigma$ with flat initial data, which is all we need in this paper.

In the first order formalism, the fundamental variables a triad are $e^a = e^a_\mu dx^\mu$ and an $SO(2,1)$ connection one-form $A^a = \frac{i}{2}e^{abc}\omega_{bc}dx^a$. The Einstein-Hilbert action takes the form

$$S = \int_M e^a \wedge F_a,$$

(2)

where $F_a = da_a + \frac{i}{2}e^{abc}A^b \wedge A^c$ is the usual curvature for the connection $A$. In our convention, Lorentz space-time indices are represented by Greek letters, and spatial...
with a marked point \( = 1 \). The constraints imply \([4]\) that the space of flat connections, i.e., the set of equivalence classes of flat connections on \( \Sigma \) under gauge transformations. It turns out that \( \pi_1(\Sigma) \to SO(2,1) \).

Let us choose a reference point \( p_0 \in \Sigma \), and for each closed curve \( \gamma \) based on \( p_0 \) compute the holonomy \( W(\gamma) = P e^{\int_\gamma A} \). This quantity is invariant under gauge transformations that are identity at \( p_0 \). Since \( A \) is flat, \( W(\gamma) \) is invariant under small deformations of \( \gamma \). Therefore the formulation is on the space \( Q \) under gauge changes. It turns out that \( \pi_1(\Sigma) \to SO(2,1) \).

Let \( Q \) be the set of all such maps. We recall that \( W(\gamma) \) changes to \( gW(\gamma)g^{-1}, \ g \in SO(2,1) \), under gauge transformations that are not identity (and equal \( g \) at \( p_0 \). For closed (i.e., compact and boundaryless) surfaces one must make an identification \( W \sim gWg^{-1} \) to get the moduli space of flat connections. In other words, \( Q = \tilde{Q}/SO(2,1) \).

In our case, \( \Sigma \) is a compactified two-dimensional surface with a marked point \( p_\infty \), the “point at infinity”, which is our basepoint. Gauge transformations which are not trivial at \( p_\infty \), taking a value \( g \) (say) at \( p_\infty \), change \( W \) to \( gWg^{-1} \) as before, but these are no longer equivalent. We call this action of \( SO(2,1) \) by conjugation the gauge action. The group \( Diff^\infty(\Sigma) \) of orientation-preserving spatial diffeomorphisms (diffeos) which are trivial at infinity acts on the holonomies \( W \) by changing the curve \( \gamma \). Its subgroup \( Diff^\infty_0(\Sigma) \subset Diff^\infty(\Sigma) \), connected to the identity (the group of small diffeos) cannot change the homotopy class of \( \gamma \). Therefore the formulation is already invariant by small diffeos, and the physical configuration space is \( Q \). Large diffeos, on the other hand, act nontrivially on the holonomies. So, we can work with the quotient group \( M_{\Sigma} = Diff^\infty_0(\Sigma)/Diff^\infty_0(\Sigma) \), known as the mapping class group. In particular, the elements \( C_{2\pi} \) and \( R \) are large diffeos \([1–3]\). For the sake of simplicity, we will denote the elements of \( Diff^\infty(\Sigma) \) and its classes in \( M_{\Sigma} \) by the same letters. An important fact is that elements of \( M_{\Sigma} \) commute with the gauge action.

We now describe the algebra \( A \) used for quantization. We consider only the minimum needed to investigate the spin-statistics connection. First, we comment on its general structure. Its first component consists of the operators of “position” type on the space \( Q \) and corresponds to the commutative algebra \( F(Q) \) of continuous functions of compact support \( f : \tilde{Q} \to \mathbb{C} \). The gauge action of \( SO(2,1) \) on \( Q \) induces an action on functions. Instead of \( SO(2,1) \), we take its group algebra \( \mathcal{G} \). Finally, we also include the algebra \( \mathcal{U} \) of (suitable) remaining observables acting on \( F(Q) \). In other words, \( A \) has the structure

\[
A = (\mathcal{U} \otimes \mathcal{G}) \times F(Q) \tag{3}
\]

We choose the algebra \( \mathcal{U} \) to be the group algebra of \( M_{\Sigma} \).

Let us give an explicit presentation of \( A^{(1)} \), the algebra \( A \) for a single geon. We choose the generators of \( \pi_1(\Sigma) \) to be the homotopy classes of the loops \( \gamma_1, \gamma_2 \) of Fig.1. Each flat connection provides us with a pair of holonomies \( (a,b) = (W(\gamma_1),W(\gamma_2)) \). Since there are no relations among the generators of \( \pi_1(\Sigma) \), any pair of values \( (a,b) \) can occur. Therefore \( Q \) is \( SO(2,1) \times SO(2,1) \).

Instead of working with \( F(Q) \) directly, we work with one of its representations. Note that the Haar measure on \( SO(2,1) \) induces a measure on \( Q \). Using this measure we may define an inner product on \( F(Q) \) in the obvious way. The completion of \( F(Q) \) in this norm is a Hilbert space \( \mathcal{H}_0 \) carrying what we call the defining representation of \( F(Q) \). A function \( f \in F(Q) \) acts on \( \varphi \in \mathcal{H}_0 \) as a multiplication operator:

\[
(f \varphi)(a,b) = f(a,b)\varphi(a,b) \tag{4}
\]

With \( g \in SO(2,1) \), let \( \delta_g \) denote the generators of the group algebra \( \mathcal{G} \). These \( \delta_g \) ’s are gauge transformations, and act by conjugating holonomies:

\[
(\delta_g \varphi)(a,b) = \varphi(g^{-1}ag,g^{-1}bg) \tag{5}
\]

The mapping class group of \( \Sigma \) has two generators \( A \) and \( B \), which correspond to Dehn twists along the loops. Their effect on loops \( \gamma_1, \gamma_2 \) is given by

\[
(A \varphi)(a,b) = \varphi(ab^{-1}), \\
(B \varphi)(a,b) = \varphi(ab^{-1}, b) \tag{6}
\]

Fig. 1: The figure shows \( \Sigma \) for a single geon (opposite sides of the rectangle are to be identified) and loops \( \gamma_i \) (\( 1 \leq i \leq 3 \)). The homotopy classes \( [\gamma_1] \) and \( [\gamma_2] \) generate the fundamental group, while \( [\gamma_3] \) is not independent of \( [\gamma_1] \) and \( [\gamma_2] \).
The generators of $A^{(1)}$ are functions $f \in \mathcal{F}(\tilde{Q})$, diffeos $A,B$ of the mapping class group and gauge transformations $\delta_g$.

The mapping class group includes $C_{2\pi}$ [1–3,8]. Its action on the defining representation is

$$(C_{2\pi}\varphi)(a,b) = \varphi(aca^{-1}, cbc^{-1})$$

where $c := aba^{-1}b^{-1}$. One can verify that $C_{2\pi} = (AB^{-1}A)^4$.

These operators can be encoded in what is called a transformation group algebra [9]. Let $G$ be a group with a left-invariant measure acting on a space $X$. The transformation group algebra is just the set of continuous functions $\mathcal{F}(G \times X)$, with compact support and with the product

$$(F_1F_2)(g,x) = \int \! F_1(z,x)F_2(z^{-1}g,z^{-1}x)dz.$$  (8)

Here $x \rightarrow z^{-1}x$ is the group action on $X$, $z^{-1}g$ is the group product of $z^{-1}$ and $g$, and $dz$ is the left-invariant measure on $G$. The irreducible representations of a transformation group algebra have been worked out in [9]. In our case, $X = Q$ and $G = SO(2,1) \times M_\Sigma$, where $G$ can be made into a topological group by giving $M_\Sigma$ the discrete topology. The measure on $SO(2,1)$ is the Haar measure and the measure on $M_\Sigma$ is given by $\sum_{m \in M_\Sigma} f(m)$ for any function $f$ on $M_\Sigma$ with appropriate convergence properties. The measure on $G$ is then the product measure. Finally, $A^{(1)} = \mathcal{F}(SO(2,1) \times M_\Sigma \times Q)$, where we use the bijection

$$\mathcal{O}(G) \otimes \mathcal{F}(X) \leftrightarrow \mathcal{F}(G \times X)$$

by interpreting $\delta_g \otimes f$ as the distribution

$$\delta_g \otimes f : (h,x) \mapsto \delta_g(h)f(x)$$

(10) on $G \times X$, $\delta_g$ being the $\delta$-function supported at $g$.

Let $Y = \tilde{Q}/G$ be the set of orbits of $G$ in $\tilde{Q}$, one such orbit being $O_\omega$. Let us choose one representative $(a_\omega,b_\omega) \in \tilde{Q}$ for each orbit $O_\omega$, and write $O_\omega = [(a_\omega,b_\omega)]$. We define the stabilizer group $N_\omega \subset G$ as the set of elements $(g,\lambda)$ of $G$ such that $(g,\lambda) \cdot (a_\omega,b_\omega) = (a_\omega,b_\omega)$, where the $G$ action has been denoted by a dot. Let $\alpha$ be a unitary irreducible representation of $N_\omega$ on some Hilbert space $V_\alpha$. Now consider the space of square-integrable functions $\phi : G \rightarrow V_\alpha$ such that $\phi(hg,\lambda) = \alpha(g^{-1},\lambda^{-1})\phi(h,\xi)$ for all $(g,\lambda) \in N_\omega$ and $(h,\xi) \in G$. They are called equivariant functions. The set of these functions can be completed into a Hilbert space $L^2(G,V_\alpha)$ [9]. The irreducible unitary *-representations $\Pi_{(\omega,\alpha)}$ of $\mathcal{F}(G \times \tilde{Q})$ can be realized on the Hilbert spaces $\mathcal{H}_{(\omega,\alpha)} = L^2(G,V_\alpha)$ and, up to unitary equivalence, labeled by $r = (\omega,\alpha)$. This label is a quantum number characterizing a single geon. The action of the operators $F = \Pi_r(F)$, $F \in A^{(1)}$ on a vector $\phi^r \in \mathcal{H}_r$ is given by

$$(\tilde{F}\phi)^r(h,\xi) = \int_{SO(2,1) \times M_\Sigma} F((h,\xi) \cdot (a_\omega,b_\omega),(g,\lambda)) \times \phi^r(g^{-1}h,\lambda^{-1}\xi)dz,$$  (11)

for any $h \in SO(2,1)$ and $\xi \in M_\Sigma$. We find, in particular, that

$$\begin{align*}
\delta_{h'}(\phi^r)(h,\xi) &= \phi^r(h^{-1}h',\xi) \\
\tilde{A}\phi^r(h,\xi) &= \phi^r(h,A^{-1}\xi) \\
\tilde{B}\phi^r(h,\xi) &= \phi^r(h,B^{-1}\xi) \\
\tilde{f}\phi^r(h,\xi) &= f(h\xi\tilde{q}_y)\phi^r(h,\xi).
\end{align*}$$

(12)

Now, let $\Sigma$ be an orientable surface of genus two with a marked point, representing the point at infinity. It supports a system of two geons. Their algebra $A^{(2)}$ can be presented in the defining representation space $\mathcal{H}_0 \otimes \mathcal{H}_0$ of $A^{(1)} \otimes A^{(1)}$. It is generated by elements of $A^{(1)} \otimes A^{(1)}$ plus the elements of the mapping class group that mix up the geons, with the proviso that we retain only “diagonal” elements of the form $\delta_\omega \otimes \delta_\gamma$ from the gauge transformations. There are only two independent generators of $M_\Sigma$ involving both geons. One of them, the diffeo $R$ that exchanges the position of the geons, has already been discussed in connection with the spin-statistics relation. The other one is the so-called handle slide $H$. Unlike the exchange $\mathcal{R}$, the handle slide $H$ has no analogue for particles. Its existence comes from the fact that a geon is an extended object. As the name indicates, it corresponds to the operation of sliding an end of one of the handles through the other handle.

Our description of a pair of geons should be given by an algebra $A^{(2)}$ which also includes $H$. But since $H$ does not enter directly in the spin-statistics relation, we will not include it in $A^{(2)}$.

Although $A^{(1)}$ is not a Hopf algebra, there is an element $R \in A^{(1)} \otimes A^{(1)}$ that plays the role of an $R$-matrix. In other words, we can write $R = \sigma R$ where $\sigma : \mathcal{H}_0 \otimes \mathcal{H}_0 \rightarrow \mathcal{H}_0 \otimes \mathcal{H}_0$ is the flip automorphism $\sigma(f_1 \otimes f_2) = f_2 \otimes f_1$. The $R$-matrix turns out to be

$$R = \int \int \! da \, db \, P_{(ab)} \otimes \delta_{a^{-1}b^{-1}}$$

(13)

where $P_{(ab)}(\tilde{q},h,\xi) = \delta(\tilde{q},(a,b)) \delta(h,e)\delta(\xi,e)$, the $\delta$’s being $\delta$-functions. The existence of the $R$-matrix is essential to establish the connection between spin and statistics. It relates a diffeo performed on a pair of objects with operators acting on each object individually.

Each geon carries a representation $\mathcal{H}_r$ labeled by quantum numbers $r = (\omega,\alpha)$. However, we only need to consider eigenstates of $\tilde{C}_{2\pi} := \Pi^*(C_{2\pi})$ with spin $S$. Let $\{ \phi_i^{S}\} \subset \mathcal{H}_r$ be a basis for the eigenspace of spin $S$ in $\mathcal{H}_r$ for some fixed $r$. Two geons are said to be identical if they carry the same quantum numbers $r$ and $S$. We consider identical geons, fix an element $(a_\omega,b_\omega)$ in the corresponding class $\omega$ and denote the net flux $\alpha_\omega b_\omega a_\omega^{-1}b_\omega^{-1}$ by $c_\omega$. Consider the characteristic function $F_c$ which at $(a,b)$ is
If $ab^{-1}h^{-1} = c$ and zero otherwise. It is clear that a generic vector $\phi^r_{i,S}$ is not an eigenstate of $\hat{P}_c$. A simple computation shows that $\epsilon^{r}_{i,S}$ is an eigenstate of $\hat{P}_c$ if and only if it has support only on points $(h, \xi)$ such that $hc_{\omega}h^{-1} = c_{\omega}$.

The quantum state for two identical geons is a linear combination of vectors of the form $\phi^r_{i,S} \otimes \phi^r_{j,S}$. It is enough to show the spin-statistics connection (1) for such decomposable vectors. We must act with the operator $\hat{R} = (\Pi_r \otimes \Pi_r)(\hat{R})$ on these vectors. By using eq. (11), we easily see that

$$P(a,b)\phi^r_{i,S}(h,\xi) = \delta((a,b),(h,\xi) \cdot (a_\omega,b_\omega))\phi^r_{i,S}(h,\xi)$$  \hspace{1cm} (14)

for every $(h, \xi) \in SO(2,1) \times M_\Sigma$. Also,

$$\hat{\delta}_{c_{\omega}^{-1}} \phi^r_{j,S}(h,\xi) = \phi^r_{j,S}(ch,\xi),$$  \hspace{1cm} (15)

where we have put $c = aba^{-1}b^{-1}$. Using (13) and the flip automorphism we conclude that

$$\hat{R}\phi^r_{i,S}(h_1,\xi_1) \otimes \phi^r_{j,S}(h_2,\xi_2) = \hat{\delta}_{c_{\omega}^{-1}h_1^{-1}} \phi^r_{j,S}(h_1,\xi_1) \otimes \phi^r_{i,S}(h_2,\xi_2).$$  \hspace{1cm} (16)

At this point we make the assumption that $\phi^r_{i,j}$ are eigenstates of the net flux $\hat{P}_c$, explaining its physical meaning later. So we can set $h_2c_{\omega}h_1^{-1} = c_{\omega}$. But we have

$$\hat{\delta}_{c_{\omega}^{-1}} \phi^r_{j,S}(h_1,\xi_1) = e^{i2\pi S} \hat{\delta}_{c_{\omega}^{-1}C_{2\pi}^{-1}} \phi^r_{j,S}(h_1,\xi_1) = \phi^r_{j,S}(c_{\omega}h_1,C_{2\pi}\xi).$$  \hspace{1cm} (17)

Note that $\phi^r_{j,S}(c_{\omega}h_1,C_{2\pi}\xi) = \phi^r_{j,S}(h_1c_{\omega},\xi C_{2\pi})$ because of the above assumption, and because $c_{\omega}$ commutes with $h_1$ and $C_{2\pi}$ commutes with every element of $M_\Sigma$. On the other hand, $(c_{\omega},C_{2\pi}) \in N_\omega$ and hence we can use the equivariance property of $\phi^r_{j,S}$ to rewrite the r.h.s. of the last equality in (17) as

$$\phi^r_{j,S}(c_{\omega}h_1,C_{2\pi}\xi) = \alpha(c_{\omega}^{-1},C_{2\pi}^{-1}) \phi^r_{j,S}(h_1,\xi).$$

Now, every $\delta_{\theta}$ commuting with $a_\omega$ and $b_\omega$ commutes also with $c_{\omega}$, while $C_{2\pi}$ is in the center of $M_\Sigma$. Therefore, $(c_{\omega},C_{2\pi})$ is in the center of $N_\omega$, and by Schur’s lemma we conclude that $\hat{\delta}_{c_{\omega}^{-1}C_{2\pi}^{-1}}$ is equal to a phase, say $e^{-i\theta(r)}$. Eq. (1) then follows:

$$\hat{R}\phi^r_{i,S} \otimes \phi^r_{j,S} = e^{i(2\pi S - \theta(r))} \phi^r_{j,S} \otimes \phi^r_{i,S}. \hspace{1cm} (18)$$

We were able to establish a connection between spin and statistics for all eigenstates of the net flux $P_c$. The other vectors in the representation space of $r$ are not physically allowed as a consequence of a superselection rule. Indeed, the total flux of a geon commutes with all elements of the algebra except the gauge transformations at infinity. Therefore the total flux can be regarded as a superselected charge. The total flux of a pair of geons is similarly superselected. For example, the exchange diffeo changes the net flux of individual geons but not the total flux. As one geon is sent to infinity, the exchange becomes non-local and the net flux of the remaining geon becomes itself superselected. In other words, each physically realizable pure single geon state gives an eigenstate of $P_c$.

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