New model for the neutrino mass matrix

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Abstract

I suggest a model based on a softly broken symmetry $L_e - L_\mu - L_\tau$ and on Babu’s mechanism for two-loops radiative generation of the neutrino masses. The model predicts that one of the physical neutrinos ($\nu_3$) is massless and that its component along the $\nu_e$ direction ($U_{e3}$) is zero. Moreover, if the soft-breaking term is assumed to be very small, then the vacuum oscillations of $\nu_e$ have almost maximal amplitude and solve the solar-neutrino problem. New scalars are predicted in the 10 TeV energy range, and a breakdown of $e-\mu-\tau$ universality should not be far from existing experimental bounds.

In a model without right-handed (singlet) neutrinos, the three weak-interaction-eigenstate neutrinos $\nu_e$, $\nu_\mu$, and $\nu_\tau$ may acquire $|\Delta I| = 1$ Majorana masses given by the following term in the Lagrangian:

$$L_{\text{mass}}^{(\nu)} = \frac{1}{2} \left( \begin{array}{ccc} \nu_e^T & \nu_\mu^T & \nu_\tau^T \end{array} \right) C^{-1} M \left( \begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \right) - \frac{1}{2} \left( \begin{array}{ccc} \nu_e^T & \nu_\mu^T & \nu_\tau^T \end{array} \right) C M^* \left( \begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \right).$$

(1)

Here, $C$ is the Dirac–Pauli charge-conjugation matrix and $M$ is a $3 \times 3$ symmetric mass matrix. One may diagonalize $M$ with help of a unitary matrix $U$ in the following way:

$$U^T M U = \text{diag}(m_1, m_2, m_3),$$

(2)

where $m_1$, $m_2$, and $m_3$ are real and non-negative. The physical neutrinos $\nu_1$, $\nu_2$, and $\nu_3$ are given by

$$\left( \begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \right) = U \left( \begin{array}{c} \nu_1 \\ \nu_2 \\ \nu_3 \end{array} \right).$$

(3)

Then,

$$L_{\text{mass}}^{(\nu)} = \frac{1}{2} \sum_{i=1}^{3} m_i \left( \nu_i^T C^{-1} \nu_i - \nu_i C \nu_i^T \right).$$

(4)

Experiment indicates that two linearly independent squared-mass differences among the three physical neutrinos differ by a few orders of magnitude. Indeed, $\Delta m^2_{\text{atm}}$ is of order $10^{-3}$ eV$^2$, while $\Delta m^2_{\odot}$ may be either of order $10^{-5}$ eV$^2$, in the case of the MSW solution for the solar-neutrino puzzle, or of order $10^{-10}$ eV$^2$, in the case of the vacuum-oscillations (“just so”) solution. It is customary to identify $\nu_3$ as the neutrino which has a mass much different from the masses of the other two, viz.,

$$|m_3^2 - m_1^2| = \Delta m^2_{\odot} \ll |m_3^2 - m_1^2| \approx |m_3^2 - m_2^2| \approx \Delta m^2_{\text{atm}}.$$  

(5)

Then, the negative result of CHOOZ’s search for $\nu_e$ oscillations [1] is interpreted as $|U_{e3}| \leq 0.217$, which is valid for $\Delta m^2_{\text{atm}} \geq 2 \times 10^{-3}$ eV$^2$. 

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It has been pointed out [2] that the assumption of an approximate lepton-number symmetry $\bar{L} = L_e - L_\mu - L_\tau$ (where $L_e$ is the electron number, $L_\mu$ is the muon number, and $L_\tau$ is the tau number) may constitute a good starting point for a model of the neutrino mass matrix. Indeed, if there are no $|\Delta \bar{L}| = 2$ mass terms then

$$
\mathcal{M} = \begin{pmatrix} 0 & rb & b \\ rb & 0 & 0 \\ b & 0 & 0 \end{pmatrix},
$$

where $b$ and $r$ may, without loss of generality, be taken to be real and positive. The mass matrix in Eq. (6) yields $m_3 = 0$, $m_1 = m_2 = b\sqrt{1+r^2}$, and

$$
U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ \frac{r}{\sqrt{2(1+r^2)}} & \frac{i}{\sqrt{2(1+r^2)}} & \frac{1}{\sqrt{1+r^2}} \\ \frac{-i}{\sqrt{2(1+r^2)}} & \frac{-r}{\sqrt{2(1+r^2)}} & -\frac{1}{\sqrt{1+r^2}} \end{pmatrix}.
$$

This is good for the following reasons:

1. The negative result of CHOOZ’s search for $\nu_e$ oscillations gets explained through $U_{e3} = 0$.

2. Since $4|U_{e1}U_{e2}|^2 = 1$, vacuum oscillations of $\nu_e$ with maximal amplitude would occur were $m_1 \neq m_2$, opening way for the “just so” solution of the solar-neutrino problem to apply.

3. It is intuitive to expect $r$ to be close to 1. Now, if $r = 1$ then $\nu_\mu - \nu_\tau$ mixing is maximal, and this explains the atmospheric-neutrino anomaly.

On the other hand, $\bar{L}$ must be broken, because $m_1 = m_2$ does not allow for oscillations between $\nu_1$ and $\nu_2$ and a solution of the solar-neutrino puzzle. A good choice, in order to avoid unpleasant majorons, would be to have $\bar{L}$ to be softly broken; this would moreover permit a natural explanation for $\Delta m^2 \ll \Delta m^2_{\text{atm}}$. This option has been suggested by Joshipura and Rindani [3]; however, in those authors’ models there is no predictive power for the form of the mixing matrix $U$, a fact which impairs the immediate interest and experimental testability of those models.

In this paper I put forward a simple model with softly broken $\bar{L}$ which maintains some predictive power. The model is based on Babu’s mechanism for two-loops radiative generation of the neutrino masses [4]. I remind that, in general, Babu’s mechanism leads to one neutrino remaining massless; however, whereas that general mechanism cannot predict the $\nu_e$, $\nu_\mu$, and $\nu_\tau$ components of the massless neutrino, the specific model that I shall put forward retains the exact-$\bar{L}$ prediction $U_{e3} = 0$. Moreover, in my model there is a rationale for the $\nu_e$ oscillations of maximal amplitude, and for the tiny mass difference $\Delta m^2_{\odot}$, which allow a “just so” explanation of the solar-neutrino deficit; that rationale is provided by the naturalness of the assumption that the term which breaks $\bar{L}$ softly is very small.

In my model I just introduce in the scalar sector, above and beyond the usual standard-model doublet $\phi = (\varphi^+ \varphi^0)^T$, one singly-charged singlet $f^+$ with $\bar{L} = 0$, together with two doubly-charged singlets $g^{2+}$ and $h^{2+}$, and their Hermitian conjugates. The difference between $g^{2+}$ and $h^{2+}$ lies in that the former field has $\bar{L} = 0$ whereas $h^{2+}$ has $\bar{L} = -2$. The Yukawa couplings of the leptons are $\bar{L}$-invariant and are given by

$$
\mathcal{L}^{(i)}_Y = -\frac{m_e}{v} \left( \varphi^+ \varphi^0 \right) e_R - \frac{m_\mu}{v} \left( \varphi^+ \varphi^0 \right) \mu_R - \frac{m_\tau}{v} \left( \varphi^+ \varphi^0 \right) \tau_R + f^+ \left[ f_\mu \left( e^T_L C^{-1} \mu L - e^T_L C^{-1} \nu_{\mu L} \right) + f_\tau \left( \nu^T_L C^{-1} \tau L - e^T_L C^{-1} \nu_{\tau L} \right) \right] + e^T_R C^{-1} \left[ g^{2+} \left( g_{\mu R} + g_{\tau R} \right) \mu_R + h^{2+} h e R \right] + H.c.,
$$

where $f_\mu$, $f_\tau$, $g_{\mu R}$, $g_{\tau R}$, and $h_e$ are complex coupling constants. Notice that, in the first line of Eq. (8), I have already taken, without loss of generality, the Yukawa couplings of $\phi$ to be flavor-diagonal; $v$ denotes the vacuum expectation value of $\varphi^0$.  

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The scalar potential $V$ has a trivial part, $V_{\text{trivial}}$, which is a quadratic polynomial in $\phi^4$, $f^4$, $g^2$-$g^2$, and $h^2$-$h^2$. Besides, $V$ includes two other terms, with complex coefficients $\lambda$ and $\epsilon$:

$$V = V_{\text{trivial}} + (\lambda f^2 - g^2 + \epsilon g^2 h^2 + \text{H.c.}).$$

(9)

The term with coefficient $\epsilon$ breaks $L$ softly. I make the following assumptions: this is the only $L$-breaking term in the theory, and $\epsilon$ is small. These assumptions are technically natural in the sense of ‘t Hooft [5].

From now on I shall assume, without loss of generality, $f_\mu$, $f_\tau$, $g_\epsilon$, $h_\epsilon$, $\lambda$, and $\epsilon$ to be real and positive. Only $g_\epsilon$ remains, in general, complex.

The neutrino mass term $\mathcal{M}_{\nu\mu}$ does not break $L$ and is generated at two-loops level by the Feynman diagram in Figure 1. A similar diagram generates $\mathcal{M}_{\nu \tau}$. In both cases, there is in the diagram an inner charged lepton which may be either $\mu$ or $\tau$. It is clear that the mass terms thus generated obey the relation

$$r \equiv \frac{\mathcal{M}_{\nu\mu}}{\mathcal{M}_{\nu \tau}} = \frac{f_\mu}{f_\tau}.$$  

(10)

Contrary to what happens in Zee’s model [6], this ratio of mass terms is not proportional to a ratio of squared charged-lepton masses [7]. As seen before, in order to obtain maximal $\nu_\mu-\nu_\tau$ mixing one would like to have $r \approx 1$. In the present model, this means that the coupling constants $f_\mu$ and $f_\tau$ should be approximately equal. In Zee’s model, on the other hand, one winds up with the rather unrealistic constraint $f_\mu/f_\tau \approx (m_\tau/m_\mu)^2$.

Let us check whether the diagram in Figure 1 is able to yield neutrino masses of the right order of magnitude. As we shall see later, we would like to obtain $|\mathcal{M}_{\nu \mu}| \approx |\mathcal{M}_{\nu \tau}| \approx \sqrt{\Delta m^2_{\text{atm}}} \sim 10^{-2} \text{–} 10^{-1} \text{eV}$. Now, from the diagram in Figure 1 with an inner $\tau$ one obtains

$$\mathcal{M}_{\nu \mu} = -2\lambda f_\mu f_\tau g_\epsilon m_\epsilon m_\tau \frac{I}{(16\pi^2)^2}.$$  

(11)

where

$$I = \frac{1}{\pi^4} \int \frac{d^4k}{k^2 - m_f^2} \frac{1}{k^2 - m_e^2} \int \frac{d^4q}{q^2 - m_f^2} \frac{1}{q^2 - m_e^2} \int \frac{1}{(k - q)^2 - m_\mu^2}$$  

(12)

$$= \frac{1}{2} \left( \frac{m_e^2 - m_f^2}{m_f^2 - m_e^2} \right) \int_0^\infty \ln \left( \frac{y + 1}{y + x_e} \right) \left[ p \ln \frac{y + x_g + 1 + p}{y + x_g + 1 - p} - p' \ln \frac{y + x_g + x_\tau + p'}{y + x_g + x_\tau - p'} + (1 - x_\tau) \ln x_g + (x_\tau - x_\mu - y) \ln x_\tau \right].$$  

(13)

Here, $x_e = m_e^2/m_f^2$, $x_\tau = m_\tau^2/m_f^2$, $x_g = m_\mu^2/m_f^2$, and

$$p = \sqrt{(y + x_g - 1)^2 + 4y},$$  

(14)

$$p' = \sqrt{(y + x_g - x_\tau)^2 + 4yx_\tau}.$$  

(15)

The integral in Eq. (13) is convergent and may be computed numerically. For $m_\epsilon$, $m_\tau \ll m_f$, and $m_\mu \approx m_f$, one finds $I$ to be of order $m_f^{-2}$.

In my estimate $\mathcal{M}_{\nu \mu}$ I shall therefore set $I \approx m_f^{-2}$. The bounds from $e-\mu-\tau$ universality in $\mu$ decay and in $\tau$ decay are $f_\mu/m_f \lesssim 10^{-4}$ GeV$^{-1}$ and $f_\tau/m_f \lesssim 10^{-4}$ GeV$^{-1}$ [4]; if one allows $f_\mu f_\tau/m_f^2$ to be as high as $10^{-8}$ GeV$^{-2}$, then one obtains

$$|\mathcal{M}_{\nu \mu}| \approx 10^{-15} \lambda g_\epsilon.$$  

(16)

It is reasonable to assume that the Yukawa coupling $g_\epsilon$ is of the same order of magnitude as the Yukawa couplings $f_\mu$ and $f_\tau$, and that the dimensionful scalar-potential coupling constant $\lambda$ is of the same order

1Notice that the possible $L$-breaking term $f^2 f^2 h^2$ has dimension higher than the one of $g^2 h^2$, and therefore the assumption of its absence is natural.

2It is not possible to use the approximations $m_\epsilon = m_\tau = 0$ because they lead to infrared divergences. This is not a problem, since those divergences are logarithmic and $\mathcal{M}_{\nu \mu}$ in Eq. (11) also includes a factor $m_\epsilon m_\tau$. 

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of magnitude as both \(m_\nu\) and \(m_\mu\). This leads to \(g_\tau / \lambda \sim f_\mu / m_\tau \sim 10^{-4}\) GeV\(^{-1}\). Fortunately the product \(\lambda g_\tau\) stays free. In order to obtain \(|M_{\mu\mu}| \sim 10^{-2}\) eV it is then sufficient to assume

\[
\lambda \approx m_\mu \approx m_\tau \sim 10^4\text{ GeV},
\]

\[
f_\mu \approx f_\tau \approx g_\tau \sim 1.
\]

Extra factors of order 1 may easily enhance \(|M_{\mu\mu}|\) and bring it up to the desired value 0.06 eV.

The assumption, made in Eq. (18), that the Yukawa couplings are of order 1, may seem unrealistic.\(^3\) However, there are no experimental indications against this possibility when the masses of \(f^+\) and of \(g^2\) are assumed to be as high as 10 TeV.\(^4\) For instance, \(g^2\) mediates the unobserved decay \(\tau^- \rightarrow \mu^- e^+ e^-\); however, by comparing that decay with the standard decay \(\tau^- \rightarrow \mu^- \nu_\mu \nu_\tau\), one easily reaches the conclusion that \(\text{BR}(\tau^- \rightarrow \mu^- e^+ e^-)\) should be at least one order of magnitude below the present experimental bound, when \(m_\mu \approx 10\) TeV and \(|g_\mu g_\tau| \approx 1\). A more complicated process is \(e^- e^+ \rightarrow \tau^+ \tau^-\), which is mediated by \(g^2\) exchange in the \(t\) channel. The amplitude \(A\) for this process is

\[
A = \frac{ie^2}{s} [\bar{v}(e)\gamma^\mu u(e)] [\bar{u}(\tau)\gamma_\mu v(\tau)] + \frac{ig^2}{3(s - m^2_\tau)} [\bar{v}(e)\gamma^\mu_5 u(e)] [\bar{u}(\tau)\gamma_\mu \gamma_5 v(\tau)]
- \frac{ig^2}{8(t - m^2_\mu)} [\bar{v}(e)\gamma^\mu (1 + \gamma_5) u(e)] [\bar{u}(\tau)\gamma_\mu (1 + \gamma_5) v(\tau)].
\]

I have used the convenient approximations \(m_e = m_\tau = 0\) and \(\sin^2 \theta_w = 1/4\) in writing down the standard-model amplitude, and a Fierz transformation in the non-standard contribution. If one defines \(j = 2m_\mu^2/s\), \(z = g^2_\tau / (2e^2)\), and \(l = 3(s - m^2_\tau) / s\), then one finds

\[
\frac{d\sigma}{d\cos \theta} \propto \frac{l^2 + 1}{l^2} (1 + \cos^2 \theta) + \frac{4}{l} \cos \theta + \frac{l + 1}{l} \frac{(1 + \cos \theta)^2}{1 + j + \cos \theta} + \frac{z^2}{(1 + j + \cos \theta)^2},
\]

where \(\theta\) is the angle between the momenta of \(e^-\) and of \(\tau^-\) in the center-of-momentum frame. From the differential cross section in Eq. (20) one easily checks that the deviations of both the total cross section and the forward–backward asymmetry from their standard-model predictions are completely negligible when \(m_\tau \sim 10\) TeV, even if \(g_\tau\) is as large as 1.

Except for \(M_{e\mu}\) and \(M_{e\tau}\), all other matrix elements of \(M\) break \(\bar{L}\) and, therefore, they will all be proportional to the \(\bar{L}\)-breaking parameter \(\epsilon\), which is assumed to be small. The matrix elements \(M_{\mu\mu}\), \(M_{\mu\tau}\), and \(M_{\tau\tau}\) arise at two loops from the diagram in Figure 2. In order to obtain a non-zero \(M_{e\mu}\) one must go to three loops and use for instance the diagram in Figure 3. In that diagram there are two inner charged leptons which can be either \(\mu\) or \(\tau\); therefore, there is a contribution to \(M_{e\tau}\) proportional to \(m^2_\tau\), and that matrix element should not be neglected in spite of it only arising at three-loops level.

The diagram in Figure 2 clearly leads to the following relation:

\[
M_{\mu\mu}, M_{\mu\tau}, M_{\tau\tau} = f^2_\mu (f_\mu f_\tau) : f^2_\tau = r^2 : r : 1.
\]

One thus obtains that in the present model

\[
M = \begin{pmatrix}
a & rb & b \\
rb & r^2 c & rc \\
b & rc & c
\end{pmatrix}.
\]

where \(a\), \(b\), and \(c\) are complex numbers with mass dimension, while \(r = f_\mu / f_\tau\) is a real dimensionless number which should in principle be of order 1. The masses \(a\) and \(c\) are suppressed relative to \(b\) by the soft-breaking parameter \(\epsilon\).

The mass matrix in Eq. (22) immediately leads to two predictions of this model: there is one massless neutrino \((\nu_3)\) and its component along the \(\nu_e\) direction, i.e., \(U_{e3}\), vanishes. Indeed, the diagonalizing

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\(^3\)Notice however that, in the standard model, the top-quark Yukawa coupling is also very close to 1.

\(^4\)Concerns about the breakdown of perturbativity are only justified for Yukawa couplings \(\gtrsim 4\pi\), i.e., of order 10 or more.
matrix $U$ reads

$$U = \begin{pmatrix} \cos \psi & -i \sin \psi & 0 \\ e^{i \alpha} r \sin \psi & e^{i \alpha} \sqrt{1 + r^2} & 1 \\ e^{i \alpha} \sqrt{1 + r^2} & e^{i \alpha} \frac{1}{\sqrt{1 + r^2}} & 0 \end{pmatrix} \cdot \text{diag} \left( e^{i \theta_1}, e^{i \theta_2}, 1 \right),$$

(23)

cf. Eq. (7). In the matrix of Eq. (23) $\alpha \equiv \text{arg} \left[ a b^* + b c^* (1 + r^2) \right]$ is a physically meaningless phase. The Majorana phases $\theta_1$ and $\theta_2$ are necessary in order to obtain real and positive $m_1$ and $m_2$. The sole physically observable phase is $2 (\theta_1 - \theta_2)$ [8]. The mixing angle $\psi$ is given by

$$\tan \psi = \sqrt{1 + \epsilon^2 + \epsilon},$$

(24)

where

$$\epsilon = \frac{|c|^2 (1 + r^2) - |a|^2}{2 \sqrt{1 + r^2} |ab^* + b c^* (1 + r^2)|}$$

(25)

is a parameter of order $\epsilon$, just as $a/b$ and $c/b$, and may therefore be assumed to be very small. Thus, $\psi$ is close to $45^\circ$. The amplitude of the vacuum oscillations of $\nu_e$ relevant for the solution of the solar-neutrino problem is $4 |U_{e1} U_{e2}|^2 = (1 + \epsilon^2)^{-1}$, i.e., almost maximal. Thus, the present model favors a “just so” solution of the solar-neutrino puzzle.

The soft-breaking parameter $\epsilon$ should be tiny. Indeed, one finds

$$\frac{\Delta m_{12}^2}{\Delta m_{31}^2} \approx 2 \frac{|ab^* + b c^* (1 + r^2)|}{|b|^2 \sqrt{1 + r^2}} \sim \epsilon;$$

(26)

as we want the “just so” solution for the solar-neutrino puzzle to apply, we must accept $\epsilon$ to be of order $10^{-7}$. Such a tiny soft breaking of $\hat{L}$ may eventually be explained by some new physics at a very high energy scale.

From the non-observation of neutrinoless double beta decay one derives the bound $|M_{ee}| \leq 0.2 \text{eV}$ [9]. This is not a problem to the present model. Indeed, as $m_3$ is predicted to vanish, $m_1$ and $m_2$ should both be very close to $\sqrt{\Delta m_{31}^2} \approx 0.06 \text{eV}$. Thus, in the approximation $\cos^2 \psi = \sin^2 \psi = 1/2$, one has

$$|M_{ee}| \approx (0.03 \text{eV}) \left| e^{2i(\theta_1 - \theta_2)} - 1 \right| < 0.2 \text{eV}.$$  

(27)

Moreover, the phase $2 (\theta_1 - \theta_2)$ is very close to zero—indeed, it vanishes in the limit of $\hat{L}$ conservation.

In conclusion, the model that I have presented in this paper makes the exact predictions $m_{31} = 0$ and $U_{e3} = 0$, while it naturally accommodates maximal amplitude $\nu_e$ oscillations and a tiny $\Delta m_{31}^2$. Maximal $\nu_\mu$–$\nu_\tau$ mixing follows from the reasonable assumption that two Yukawa couplings are almost equal. Neutrino masses are small because they are radiatively generated at two-loops level. Indeed, the fact that two neutrino masses are as large as 0.06 eV practically forces the new mass scale, at which the extra scalars lie, to be in the 10 TeV range; while deviations from $e-$$\mu$–$\tau$ universality in $\mu$ decay and in $\tau$ decay should be close at hand. The model requires some physical mechanism for generating a tiny soft breaking of $\hat{L}$.

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References

Figure captions

Figure 1: Two-loops Feynman diagram which generates $M_{e\mu}$.

Figure 2: Two-loops Feynman diagram which generates $M_{\mu\tau}$.

Figure 3: One of the three-loops Feynman diagrams which generate $M_{ee}$. 


Figure 1

Figure 2

Figure 3