Supersymmetric Index
In Four-Dimensional Gauge Theories

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This paper is devoted to a systematic discussion of the supersymmetric index $\text{Tr} (-1)^F$ for the minimal supersymmetric Yang-Mills theory – with any simple gauge group $G$ – primarily in four spacetime dimensions. The index has refinements that probe confinement and oblique confinement and the possible spontaneous breaking of chiral symmetry and of global symmetries, such as charge conjugation, that are derived from outer automorphisms of the gauge group. Predictions for the index and its refinements are obtained on the basis of standard hypotheses about the infrared behavior of gauge theories. The predictions are confirmed via microscopic calculations which involve a Born-Oppenheimer computation of the spectrum as well as mathematical formulas involving triples of commuting elements of $G$ and the Chern-Simons invariants of flat bundles on the three-torus.

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1. Introduction

A constraint on the dynamics of supersymmetric field theories is provided by the supersymmetric index $\text{Tr} \ (-1)^F$ [1]. It is defined as follows. One formulates an $n$-dimensional supersymmetric theory of interest on a manifold $T^{n-1} \times \mathbb{R}$, where $\mathbb{R}$ parametrizes the “time” direction and $T^{n-1}$ is an $(n-1)$-torus (endowed with a flat metric and with a spin structure that is invariant under supersymmetry, that is with periodic boundary conditions for fermions in all directions). If the original theory is such that the classical energy grows as one goes to infinity in field space in any direction, then the spectrum of the theory in a finite volume is discrete. Under such conditions, let $n_B$ and $n_F$ be the number of supersymmetric states of zero energy that are bosonic or fermionic, respectively. The supersymmetric index is defined to be $n_B - n_F$ and is usually written as $\text{Tr} \ (-1)^F$, where $(-1)^F$ is the operator that equals $+1$ or $-1$ for bosonic or fermionic states and the trace is taken in the space of zero energy states. Alternatively, as states of nonzero energy are paired (with equally many bosons and fermions), one can define the index as

$$\text{Tr} \ (-1)^F e^{-\beta H},$$

where $H$ is the Hamiltonian, $\beta$ is any positive real number, and the trace is now taken in the full quantum Hilbert space. This definition of the index shows that it can be computed as a path integral on an $n$-torus $T^n$, with a positive signature metric of circumference $\beta$ in the time direction (and periodic boundary conditions on fermions in the time direction – to reproduce the $(-1)^F$ in the trace – as well as in the space directions).

The index is invariant under any smooth deformations of a supersymmetric theory that leave fixed the behavior of the potential at infinity (so that supersymmetric vacua cannot move to or from infinity). The reason is that since the states of nonzero energy are paired, any change in $n_B$ – resulting from a state moving to or from zero energy – is accompanied by an equal change in $n_F$. Because of this, it is usually possible to effectively compute the index by perturbing to some sufficiently simple situation while preserving supersymmetry. A nonzero index implies that the ground state energy is exactly zero for any volume of the spatial torus $T^{n-1}$, and hence that the ground state energy per unit volume vanishes in the infinite volume limit. It follows that if the infinite volume theory

\footnote{This condition notably excludes theories that have a noncompact flat direction in the classical potential.}
has a stable ground state (which will be so if the potential grows at infinity) then that state is supersymmetric.

This way of thinking illuminates supersymmetric dynamics in many situations. But application to four-dimensional gauge theories was hampered for some years by the fact that, though one could readily obtain attractive results for $SU(n)$ and $Sp(n)$ gauge groups, the results for other groups appeared to clash with expectations based on chiral symmetry breaking. Ultimately (see the appendix to [2]), it became clear that the discrepancy arose from overlooking the fact that certain moduli spaces of triples of commuting elements of a Lie group are not connected.

The purpose of the present paper is to systematically develop the theory of the supersymmetric index for the minimal supersymmetric gauge theories in $2 + 1$ and $3 + 1$ dimensions. Actually, for the $(2 + 1)$-dimensional theories, we only consider the special case that the Chern-Simons coupling is $k = \pm \hbar/2$, where confinement and a mass gap are expected. The index in the $(2 + 1)$-dimensional theories with more general Chern-Simons coupling has been analyzed elsewhere [3].

In either $2 + 1$ or $3 + 1$ dimensions, we consider supersymmetric theories with the smallest possible number of supercharges (two or four in $2 + 1$ or $3 + 1$ dimensions) and with the minimal field content: only the gauge fields and their supersymmetric partners. We will compute the index for bundles of any given topological type on $T^2$ or $T^3$. Because Yang-Mills theory of a product group $G' \times G''$ is locally the product of the $G'$ theory and the $G''$ theory, the index for a semisimple $G$ can be inferred from the index for simple $G$ (provided that in the simple case one has results for all possible $G$-bundles). So we will assume that $G$ is simple.

The index has a number of important refinements (some of which were treated in [1]) that we will explain. It is possible, by letting $G$ be non-simply-connected, to include a discrete electric or magnetic flux and thereby probe the hypothesis of confinement. In four dimensions, it is possible to refine the index to take into account a discrete chiral symmetry group and to give evidence that the chiral symmetry is spontaneously broken. It is also possible, by allowing $G$ to be disconnected, to give evidence that discrete symmetries associated with outer automorphisms of $G$ (such as charge conjugation for $G = SU(n)$) are unbroken.

In sections 2 and 3, we explain what predictions about the supersymmetric index and its refinements follow from standard claims about gauge dynamics. In section 4, we compute the index and its refinements microscopically, following the strategy of [1] but (in
four dimensions) including the contributions of all components of the moduli space of flat connections. Full agreement is obtained in all cases; in $3 + 1$ dimensions, obtaining such agreement depends on a result counting the moduli spaces of commuting triples that was proposed in [2] and has subsequently been justified. For connected and simply-connected $G$, the moduli spaces of commuting triples have been analyzed in [4,5,6,7]. A generalization to non-simply-connected $G$, necessary for including the discrete fluxes, and an analysis of the Chern-Simons invariants of flat bundles on $T^3$, necessary for testing the claims about chiral symmetry breaking, have been made in [7]. In section 5, we give some details about moduli spaces of commuting triples.

2. Expectations In 2 + 1 Dimensions

2.1. Preliminaries

The minimal $(2+1)$-dimensional supersymmetric theory has a field content consisting of the gauge field $A$ of some compact connected simple Lie group $G$, along with a Majorana fermion $\lambda$ in the adjoint representation of the gauge group. The usual kinetic energy for these fields is

$$L = \frac{1}{g^2} \int d^3x \, \text{Tr} \left( \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} \lambda \Gamma \cdot D\lambda \right).$$

(2.1)

However, there is a crucial subtlety in $2+1$ dimensions: it is possible to add an additional Chern-Simons coupling while preserving supersymmetry. The additional interaction is

$$-\frac{ik}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A + \overline{\lambda} \lambda \right).$$

(2.2)

Here $k$ must, for topological reasons, obey a quantization condition. For $G$ simply-connected, $k$ must be congruent to $h/2 \mod 1$, where $h$ is the dual Coxeter number of $G$; if $G$ is not simply-connected, $k$ must be congruent to $h/2 \mod s$ where $s$ is an integer that depends on $G$. The $h/2$ term in the quantization law comes from an anomaly involving the fermions [8,3].

The supersymmetric index for general $k$ has been studied in [3]. Our intent here is to describe what properties of the index, and related invariants, can be deduced if we assume that in bulk the theory has a unique vacuum with a mass gap and confinement. As has been explained in [3], it is reasonable to believe that these properties hold precisely if $k = \pm h/2$. Hence in the present paper, when we consider the $(2+1)$-dimensional theory, we will always specialize to these values of $k$. 

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First we consider the theory with a gauge group $G$ that is simply-connected. Since we assume a mass gap, there is no Goldstone fermion, so that the unique vacuum has unbroken supersymmetry. One expects that an isolated vacuum with mass gap will contribute $+1$ to the supersymmetric index, so one expects $\text{Tr} (-1)^F = 1$. The logic in this statement is that one can compute $\text{Tr} (-1)^F$ by formulating the theory on a two-torus of very large radius $R$ with $1/R$ much smaller than the mass gap; then on the length scale $R$, the system is locked in its ground state, which contributes 1 to the index. The mass gap is important here; an isolated vacuum without a mass gap makes a contribution to the index that is not necessarily equal to 1 (or even $\pm 1$), as was seen in some examples in [1].

Now we consider the case that $G$ is not simply-connected. Its fundamental group $\pi_1(G)$ is necessarily a finite abelian group (cyclic except in the special case $G = SO(4n)/\mathbb{Z}_2$, for which $\pi_1(G) = \mathbb{Z}_2 \times \mathbb{Z}_2$). From a Hamiltonian point of view, in quantizing on a two-torus $T^2$, there are basically two changes that occur when $G$ is not simply-connected. First of all, a $G$-bundle over $T^2$ can be topologically non-trivial if $G$ is not simply-connected. In the present paper, it will be important that if $X$ is a two-manifold or a three-manifold, the possible $G$-bundles are classified by a “discrete magnetic flux,” a characteristic class $m$ of the $G$-bundle which takes values in

$$M = H^2(X, \pi_1(G)).$$

(On a manifold of dimension higher than three, a $G$-bundle has the characteristic class $m$ plus additional invariants such as instanton number.) All values of $m$ can occur. To gain as much information as possible, we do not want to sum over $m$; we want to compute the index as a function of $m$.

The second basic consequence of $G$ not being simply-connected is that the group of gauge transformations, that is the group of maps of $T^2$ to $G$ (or more generally the group of bundle automorphisms if the $G$-bundle is non-trivial) has different components. Restricted to a non-contractible loop in $T^2$, a gauge transformation determines an element of $\pi_1(G)$ which may or may not be trivial. By restricting it to cycles generating $\pi_1(T^2)$, a gauge

\[2\] There is actually a small subtlety here. In $2 + 1$ dimensions, after compactifying on a torus, there is no natural definition of the sign of the operator $(-1)^F$, and hence the contribution of a massive vacuum to the index might be $+1$ or $-1$. It was shown in [3] that with a natural convention, if the index is $+1$ for $k = h/2$, then it is $(-1)^r$ at $k = -h/2$, where $r$ is the rank of $G$. For our purposes, we work at, say, $k = h/2$ and define the sign of $(-1)^F$ so that the index is $+1$. 

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