Confining strings, Wilson loops and extra dimensions

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Abstract

We study solutions of the one-loop $\beta$-functions of the critical bosonic string theory in the framework of the Renormalization Group (RG) approach to string theory, considering explicitly the effects of the 21 extra dimensions. In the RG approach the 26-dimensional manifold is given in terms of $M^4 \times R^1 \times H_{21}$. In calculating the Wilson loops, as it is well known for this kind of confining geometry, two phenomena appear: confinement and over-confinement. There is a critical minimal surface below of which it leads to confinement only. The role of the extra dimensions is understood in terms of a dimensionless scale $l$ provided by them. Therefore the effective string tension in the area law, the length of the Wilson loops, as well as, the size of the critical minimal surface depend on this scale. When this confining geometry is used to study a field-theory $\beta$-function with an infrared attractive point (as the Novikov-Shifman-Vainshtein-Zakharov $\beta$-function) the range of the couplings where the field theory is confining depends on that scale. We have explicitly calculated the $l$-dependence of that range.

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In a previous paper [1] we studied the holographic renormalization group flow associated to the Yang-Mills theory where the coupling constant diverges in the infrared limit\(^2\) and, \(\beta\)-functions with an infrared attractive point, as the Novikov-Shifman-Vainshtein-Zakharov (NVSZ) one [3]. It has been considered in the framework of the Renormalization Group approach to the string theory developed by Álvarez and Gómez [2]. In [1] it has been calculated the Wilson loops and analyzed the features of the metric which is a solution of the vanishing string theory sigma model \(\beta\)-function equations, at leading order in \(\alpha'\). The metric studied shows confinement and also over-confinement, which depend on the size of the minimal surfaces used to calculate the Wilson loops. In fact there is a critical minimal surface between those regimes. In the case of the NSVZ \(\beta\)-function, as it was shown in [4], there exist two branches: one behaves asymptotically free in the UV limit and the other one shows a strongly coupled regime in that limit. There is also an attractive infrared point which means that the RG flow of the theory goes from the UV limit to the IR limit. In the branch of the strong coupling an interesting result emerges [1], around the attractive infrared point, there is a range of couplings where the theory only shows confinement. From those calculations an open question arises. That is about the meaning of the extra 21-dimensional manifold which is included in order to solve the vanishing one-loop \(\beta\)-function equations of the string theory sigma model in the critical dimension\(^3\). Notice that this solution also satisfies the equations of motion of 5-dimensional gravity coupled to the dilaton field. In this sense there is a connection between the RG approach [2] and the confining strings [5]\(^4\), which has been pointed out by Álvarez and Gómez [2].

In this paper we study the above mentioned question. In order to understand the role played by those extra coordinates we will calculate the Wilson loops using a metric which is a solution of the closed bosonic string theory sigma model \(\beta\)-function equations, by considering that the 21-extra dimensions correspond to a hyper-plane \(\mathcal{H}\). We consider two situations, one is when the Wilson loop is drawn in such a way that the quark and the anti-quark are separated both in the Minkowski spacetime by \(L\), and also in the hyper-plane \(\mathcal{H}\) by a distance \(|\Delta \vec{y}|\). This point will be clarified latter. The other situation, when

\(^2\)This has been early studied in reference [2].

\(^3\)In the case of superstrings similar analysis can be done by means an extra 5-dimensional manifold.

\(^4\)See also [6] where a Liouville-string approach to confinement in four dimensional gauge theories is presented.
the quark-antiquark pair lies at the same point of the 21-dimensional hyper-plane \( \mathcal{H} \) has already discussed in reference [2] (and in our context also in [1]). We will use it as a check of consistency in the limit when \( |\Delta \vec{y}| \to 0 \), in the previous case. The expected result is that the critical size of the minimal surfaces depends on a scale \( l \) which is a function of the separation \( |\Delta \vec{y}| \) on the 21-dimensional hyper-plane. This has an interesting consequence when it is applied to study the NSVZ \( \beta \)-function in the infrared limit (on the branch of strong coupling). The range of couplings in which the theory is only confining reduces as far as \( |\Delta \vec{y}| \) increases, but as we shall see this reduction is bounded. It implies that the effect due to those extra coordinates leads to a reduction of the range of pure confinement of the theory.

The first concrete realization of a connection between gravity and field theories has been made by Maldacena [7], showing the duality between type IIB supergravity in \( \text{AdS}_5 \times S^5 \) and the \( \mathcal{N} = 4 \) super Yang-Mills theory at the boundary. This AdS/CFT duality was developed in [7, 8, 9, 10]. In order to extend the duality to non-conformal field theories it has been proposed several possibilities [11]. On the other hand, Polyakov proposed a string representation of Yang-Mills theories based on the zig-zag invariance of pure Yang-Mills Wilson loops [5, 12, 13, 14, 15]. Here we are going to deal with the RG approach to the strings where the idea is to model the renormalization group equations of gauge theories from the string theory \( \beta \)-functions. It implies to associate to the couplings of the gauge field theories some background fields of a closed string theory, so that the geometry dictates the properties of the gauge theory.

Following reference [2] we solve the vanishing one-loop \( \beta \)-function equations of the closed bosonic string theory in the critical dimension. Using the Liouville ansatz [2, 5] those equations are trivially satisfied by any dilaton. In the metric we identify 4 of the 26 coordinates as the Euclidean (\( \Sigma \)) (or Minkowski (\( M^4 \))) spacetime, where the gauge theory lives. In this prescription it is assumed the following relation between the string coupling \( g_s \), the Yang-Mills coupling \( g_{YM} \) and the dilaton: \( g_s = e^{\Phi} = g_{YM}^2 \). The second part of this is related with the soft dilaton theorem [16, 17].

Let us consider the following metric

\[
d s^2 = e^{2\Phi}(\pm dt^2 + dx_i dx_i) + 4l_c^2 e^{4\Phi}(d\Phi)^2 + d\vec{y}_{21}^2 ,
\]
where $x_i, i = 1, 2, 3$, and $l_c$ is an arbitrary scale of dimension of length related to the running-energy scale of the field theory considered\(^5\), let us call it $\mu$. Here $\Phi = \Phi(\mu)$ is the dilaton field. As we said before we consider a 21-dimensional hyper-plane $\mathcal{H}$. The metric of Eq.(1) is a solution of the vanishing one-loop $\beta$-functions of the bosonic strings [18] at the leading order in $\alpha'$, which for the critical dimension become

$$R_{\mu\nu} + 2 \nabla_\mu \nabla_\nu \Phi = 0 \ ,$$

and

$$\nabla^2 \Phi - 2 (\nabla \Phi)^2 = 0 \ ,$$

where $\nabla_\mu \Phi = \partial_\mu \Phi$ and $\nabla_\mu \nabla_\nu \Phi = \partial_\mu \partial_\nu \Phi - \Gamma^a_{\mu
u} \partial_a \Phi$. We set the antisymmetric three-form and the tachyon to zero. Using the ansatz $\rho = e^{2\Phi}$ the metric of Eq.(1) becomes

$$ds^2 = \rho \ (\pm dt^2 + dx_i dx_i) + 4 \rho^2 (d\rho)^2 + d\vec{y}_{21}^2 \ ,$$

where $\rho = \infty$ corresponds to the horizon and $\rho = 0$ is the singularity. The region near that singularity deserves more study since the present description breaks down around it.

Figure 1: Schematic representation of the prescription of calculating the Wilson loops in terms of $e^\Phi$-coordinates.

\(^5\)Please notice that in that follows we will use three different scales: the dimensionless $l$ related to the extra dimensions on $\mathcal{H}$, the above mentioned $l_c$ (which in [2] is related to the inverse of $\Lambda_{QCD}$), and also $l_s$ related to the string constant $\alpha' = l_s^2$.  

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By computing the expectation value of the Wilson loops it is possible to obtain the energy of the separation of the quark-antiquark pair. Let us consider a Wilson loop $C$, drawn in a 4-dimensional Euclidean (Minkowski) space-time $\Sigma$, placed at $e^{\Phi}$, (see figure 1). As it is usual we consider the limit when the time $T$ is large and use the relation between the energy of the static configuration $E$ and the action $S$, ($E = S/T$). We follow the Maldacena’s proposal [19] for the expectation value of the Wilson loop $<W(C)>$. This expectation value behaves like the exponential of the world-sheet area of a fundamental string describing the closed curve $C$ on $\Sigma$ (the curve $C$ also lies on $\mathcal{H}$ when $|\Delta \vec{y}| \neq 0$).

In figure 2 we show a schematic representation of the quark-antiquark separation in the hyper-plane $\mathcal{H}$. The figure on the left shows the case when there is a separation in $x$-coordinates ($L$) which lies on $\Sigma$, but not on the hyper-plane $\mathcal{H}$ (here indicated by the vertical $y$-axis). When the quark and the antiquark have different positions at the hyper-plane, let us say the points $\vec{y}_a$ and $\vec{y}_b$, the distance $|\Delta \vec{y}|$ naturally arises. It is indicated in the figure on the right.

We start from the Nambu-Goto action

$$S_{NG} = \frac{1}{2\pi l_s^2} \int d\sigma \, d\tau \sqrt{\det G_{MN} \partial_\alpha X^M \partial_\beta X^N},$$

where $G_{MN}$ is the metric given by Eq.(1), and $X^M$ is a generic coordinate on the 26-dimensional space-time. Since we are interested in a static configuration we can take
\( \tau = t \) and \( \sigma = x \). Therefore the metric becomes

\[
ds^2 = \pm e^{2\Phi} d\tau^2 + (e^{2\Phi} + 4l_c^2 e^{4\Phi} \Phi_\sigma^2 + (\partial_\sigma \vec{y})^2) \ d\sigma^2, \tag{6}
\]

where \( \Phi_\sigma = \frac{\partial \Phi}{\partial \sigma} \), while \( \partial_\sigma \vec{y} = \frac{\partial \vec{y}}{\partial \sigma} \). In addition let us remember that \([\tau] = [\sigma] = [l_c] = [l_s] = [y]\) have dimension of length. In the static configuration the action is

\[
S_{NG} = \frac{T}{2\pi l_s^2} \int_0^L d\sigma \ e^\Phi \sqrt{e^{2\Phi} + 4e^{4\Phi} l_c^2 \Phi_\sigma^2 + (\partial_\sigma \vec{y})^2}. \tag{7}
\]

We consider configurations as it is shown in figure 1, where \( e^{\Phi I} \) is the limit when \( \Phi \to -\infty \) (represented by a dot at the origin). Since the scalar curvature \( R \) goes like \( e^{-4\Phi} \), it implies that at this point the metric has a naked singularity. As we said before in that limit the present description breaks down. Since \( e^\Phi = g_s = g_{YM}^2 \), it is possible to choice the position of the 4-dimensional hyper-plane \( \Sigma \) at any point of the \( \mu \) coordinate, which corresponds to choosing a particular value of the coupling constant, here denoted by \( e^{\Phi c} \). The string world-sheet can fluctuate orthogonally to the hyper-plane \( \Sigma \), but the minimal surfaces are those corresponding to fluctuations in the direction of \( e^{\Phi I} \). As we pointed out in [1], in terms of the gauge theory language this means that semi-classically the RG flow goes from the strongly coupled regime to the weakly coupled one. However, if there is an attractive infrared point, it flows from the ultraviolet limit to the infrared one. We will consider the vertical axis as the \( \vec{x} \)-direction and study a Wilson-loop of size \( L \), as it is shown in figure 1.

In the symmetric configuration \( \vec{x} = 0 \) corresponds to the minimum \( e^{\Phi_0} \), that is when one takes \( \Phi_0 = \Phi(\mu_0) \). For such a minimum \( \partial_\sigma \vec{y} \) takes the value \( \partial_\sigma \vec{y}_0 \). Classically the following conditions are straightforwardly obtained by solving the Euler-Lagrange equations for the action of Eq.(7)

\[
\frac{e^{3\Phi}}{\sqrt{e^{2\Phi} + 4e^{4\Phi} l_c^2 \Phi_\sigma^2 + (\partial_\sigma \vec{y})^2}} = e^{2\Phi_0} \sqrt{1 - l^2}, \tag{8}
\]

and

\[
\frac{e^{\Phi - \Phi_0} \left| \partial_\sigma \vec{y} \right|}{\sqrt{e^{2\Phi} + 4e^{4\Phi} l_c^2 \Phi_\sigma^2 + (\partial_\sigma \vec{y})^2}} = \frac{\left| \partial_\sigma \vec{y}_0 \right|}{\sqrt{e^{2\Phi_0} + (\partial_\sigma \vec{y}_0)^2}} = l. \tag{9}
\]

Those are obtained since the Lagrangian in the above action is independent of \( \vec{x} \) and \( \vec{y} \). In this way Eq.(8) comes from the energy conservation and Eq.(9) comes from momentum conservation in the hyper-plane \( \mathcal{H} \). From these expressions one gets the following relation

\[
\left| \partial_\sigma \vec{y}_0 \right|^2 = \frac{l^2 e^{2\Phi_0}}{1 - l^2}, \tag{10}
\]
where \( l \) is related to the distance between the quark-antiquark pair in the hyper-plane \( \mathcal{H} \). In fact \( l \) is the \( \sin \alpha \), where \( \alpha \) is the angle between the quark and the antiquark as it is indicated in figure 2. Notice that from Eq.(9), for \( |\partial_\sigma \vec{y}_0| \to \infty \) we have \( l \to 1 \), while \( |\partial_\sigma \vec{y}_0| < e^{\Phi_0} \) corresponds to \( l \to 0 \), which means that the distance on the hyper-plane \( \mathcal{H} \) has been taken to be very small. Taking into account the previous conditions of Eqs.(8) and (9), we are able to invert those expressions in order to obtain

\[
d\sigma = 2e^{\Phi_0} l_c \sqrt{1 - l^2} \frac{dv}{\sqrt{(v^2 - 1)(v^2 - l^2 + 1)}} ,
\]

(11)

where we use the new variable \( v = \frac{e^\phi}{e^{\Phi_0}} \), while

\[
d\gamma = 2e^{2\Phi_0} l_c \sqrt{1 - l^2} \frac{v^2}{\sqrt{(v^2 - 1)(v^2 - l^2 + 1)}} dv .
\]

(12)

Notice that for \( l = 0 \) the distance \( |\Delta \vec{y}| \) trivially vanishes and Eq.(11) reduces to the case considered in [1]. Integrating Eq.(11) one obtains the length of the Wilson loop

\[
\frac{L}{2} = 2l_c e^{\Phi_0} \sqrt{1 - l^2} \int_{1}^{e^{\Phi_c}/e^{\Phi_0}} \frac{1}{\sqrt{(v^2 - 1)(v^2 - l^2 + 1)}} dv
\]

\[
= 2l_c e^{\Phi_0} \sqrt{1 - l^2} \frac{F(\arccos(\frac{e^{\Phi_0}}{e^{\Phi_c}}), \sqrt{\frac{1-l^2}{2-l^2}})}{\sqrt{2-l^2}} ,
\]

(13)

which in the limit of \( l \to 0 \) reduces to the case studied in [1]. On the other hand, the distance on \( \mathcal{H} \) results

\[
\frac{\Delta y}{2} = 2e^{2\Phi_0} l_c l \int_{1}^{e^{\Phi_c}/e^{\Phi_0}} \frac{v^2}{\sqrt{(v^2 - 1)(v^2 - l^2 + 1)}} dv
\]

\[
= 2e^{2\Phi_0} l_c l \frac{F(\arccos(\frac{e^{\Phi_0}}{e^{\Phi_c}}), \sqrt{\frac{1-l^2}{2-l^2}})}{\sqrt{2-l^2}}
\]

\[
- 2e^{2\Phi_0} l_c l \sqrt{2-l^2} E(\arccos(\frac{e^{\Phi_0}}{e^{\Phi_c}}), \sqrt{\frac{1-l^2}{2-l^2}})
\]

\[
+ 2l_c l e^{2\Phi_0 - \Phi_c} \sqrt{(e^{2(\Phi_c - \Phi_0)} + 1 - l^2)(e^{2(\Phi_c - \Phi_0)} - 1)} ,
\]

(14)

which trivially reduces to 0 as far as \( l \to 0 \), as it is expected from [1]. Here \( F \) and \( E \) are elliptic integrals of the first and second kind [20], respectively. From Eqs.(13) and (14) we see that as far as \( l \) approaches to one, the separation of the pair on \( \Sigma \) decreases, while the separation in \( \mathcal{H} \) increases. We will come back on this point latter. Since \( L \) is function
of $e^{\Phi_c}$ and $e^{\Phi_0}$, we can plot $L/e^{\Phi_c}$ as a function of $e^{\Phi_0}/e^{\Phi_c}$. There is a maximum between the two regions (I and II) [1, 2, 21]. The region I, i.e. below the maximum, corresponds to large world-sheets.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure3.png}
\caption{Derivative of the scaled Wilson-loop size $L/e^{\Phi_c}$ (in units of $l_c$) in terms of the dimensionless variable $e^{\Phi_0}/e^{\Phi_c}$. The zero of that derivative (maximum of $L/e^{\Phi_c}$) moves to the right as far as $l \to 1$. The zeros of the solid lines correspond to the extreme values of $l = 0$ (left, lower solid straight line) and $1$ (right, upper solid straight line), respectively.}
\end{figure}

In figure 3 we plot the derivative of $L/e^{\Phi_c}$ with respect to $e^{\Phi_0}/e^{\Phi_c}$ as a function of $e^{\Phi_0}/e^{\Phi_c}$. We show three curves, the lower one (solid line) corresponds to $l = 0$. The maximum of $L/e^{\Phi_c}$ occurs at $e^{\Phi_0}/e^{\Phi_c} \approx 0.62$ and it is $L/e^{\Phi_c}|_{\text{Maximum}} = 0.42 l_c$. It is precisely what happens when one reduces the problem to consider the quark and the antiquark at the same point in $\mathcal{H}$ [1, 2]. On the other hand we also plot the cases when $l = 0.7$ (showed by the dashed line) and $l = 1 - 10^{-15}$ (this is the upper solid line). As far as $l \to 1$ the size of both the critical minimal surface and the Wilson loop $L$ decreases. For instance when we take $l = 1 - 10^{-15}$, we are considering the limit of large separation on the 21-dimensional hyper-plane (see for instance figure 4). In this situation $e^{\Phi_0}/e^{\Phi_c}$ is approximately 0.652. Therefore the effect of the extra dimensional manifold $\mathcal{H}$ is reflected on the size of the minimal surfaces and the corresponding Wilson loop sizes. By using the RG approach to the strings [2], if one considers a $\beta$-function with a pole in the infrared [1, 4], there is a range of couplings $(g_{YM}^{\Lambda}, 0.62^{-1/2} g_{YM}^{\Lambda} (\equiv 1.27 g_{YM}^{\Lambda}))$ for $l = 0$, and $(g_{YM}^{\Lambda}, 0.65^{-1/2} g_{YM}^{\Lambda} (\equiv 1.24 g_{YM}^{\Lambda}))$ for $l = 1$, where only confinement takes place. Here $\mu = \Lambda$ is the infrared attractive point.
In figure 4 we plot $|\Delta \vec{y}|/e^{2\Phi_c}$ (in units of $l_c$) as a function of $e^{\Phi_0}/e^{\Phi_c}$ and $l$. For $l = 0$, $|\Delta \vec{y}|/e^{2\Phi_c} = 0$. For finite values of $l$ the maximum moves up to $e^{\Phi_0}/e^{\Phi_c} = 0.652$, which corresponds to the limit for $l = 1$.

![Figure 4](image)

Figure 4: Plot of $|\Delta \vec{y}|/e^{2\Phi_c}$ (in units of $l_c$) as a function of the dimensionless variable $e^{\Phi_0}/e^{\Phi_c}$ (horizontal axis) and $l$ (transverse horizontal axis).

Let us calculate the energy for the static configuration.

\[
E = 2 e^{3\Phi_0} \frac{l_c}{\pi l_s^2} \left(1 + l^2\right) \frac{F \left(\arccos \left(\frac{e^{\Phi_0}}{e^{\Phi_c}}\right), \sqrt{\frac{1-l^2}{2-l^2}}\right)}{3\sqrt{2-l^2}} - 4 e^{3\Phi_0} \frac{l_c}{\pi l_s^2} l^2 (2-l^2) \frac{E \left(\arccos \left(\frac{e^{\Phi_0}}{e^{\Phi_c}}\right), \sqrt{\frac{1-l^2}{2-l^2}}\right)}{3\sqrt{2-l^2}} + \frac{2 l_c}{3\pi l_s^2} e^{4\Phi_0-\Phi_c} (2l^2 + e^{2(\Phi_c-\Phi_0)}) \sqrt{(e^{2(\Phi_c-\Phi_0)} - l^2 + 1)(e^{2(\Phi_c-\Phi_0)} - 1)}. \quad (15)
\]

Expanding Eqs.(13) and (15) in series of powers of $e^{\Phi_0}/e^{\Phi_c}$ we get

\[
\frac{L}{l_c} = 4 e^{\Phi_0} \sqrt{1-l^2} \frac{K \left(\sqrt{\frac{1-l^2}{2-l^2}}\right)}{\sqrt{2-l^2}} - 4 \sqrt{1-l^2} e^{2\Phi_0-\Phi_c} + O(e^{\Phi_0-3\Phi_c}) , \quad (16)
\]

and

\[
\frac{\pi l_s^2 E}{2l_c} = \frac{e^{3\Phi_c}}{3} + \frac{l^2}{2} e^{\Phi_c+2\Phi_0} + \left(-2 l^2 \sqrt{2-l^2} E \left(\frac{1-l^2}{2-l^2}\right) + \left(1 + l^2\right) \frac{K \left(\sqrt{\frac{1-l^2}{2-l^2}}\right)}{\sqrt{2-l^2}}\right) \frac{e^{3\Phi_0}}{3} + O(e^{4\Phi_0-\Phi_c}). \quad (17)
\]
Here $K(t) = F(\frac{\pi}{2}, t)$ and $E(t) = E(\frac{\pi}{2}, t)$ are the complete elliptic integrals of the first and second kind, respectively. Replacing $L$ in terms of $e^{\Phi_0}$ in Eq.(17) we get

$$E = \frac{1}{96\pi l_s^2 l_c^2} \left( \sqrt{\frac{2 - l^2}{1 - l^2}} \right)^3 \left( -2 l^2 (2 - l^2) E \left( \sqrt{\frac{1 - l^2}{2 - l^2}} \right) + (1 + l^2) K \left( \sqrt{\frac{1 - l^2}{2 - l^2}} \right) \right) L^3 + \mathcal{O}(e^{3\Phi_c})$$

which leads to over-confinement. In the strongly coupled limit ($e^{\Phi_c} \to \infty$) the above expression becomes divergent. However the divergent part is independent of the size of the Wilson loop. In the context of AdS/CFT duality this kind of divergence was regularized by a mass renormalization [19] or by considering the Legendre transform of the minimal area [22]. In our case observing that zero-size Wilson loops diverge we will drop the $L$ independent divergent term and measure the energy with respect to the zero-size Wilson loops. In this limit $L$ becomes

$$L_\infty = 4 l_c e^{\Phi_0} \sqrt{\frac{1 - l^2}{2 - l^2}} K \left( \sqrt{\frac{1 - l^2}{2 - l^2}} \right) ,$$

which is exactly the same as in [1] when one takes $l = 0$, that is

$$L_\infty^0 = \sqrt{\pi} l_c e^{\Phi_0} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}.$$  

The energy for that configuration is

$$E_\infty = 2 e^{3\Phi_0} \frac{l_c}{\pi l_s^2} (1 + l^2) \frac{K \left( \sqrt{\frac{1 - l^2}{2 - l^2}} \right)}{3 \sqrt{2 - l^2}} - 4 e^{3\Phi_0} \frac{l_c}{\pi l_s^2} l^2 (2 - l^2) \frac{E \left( \sqrt{\frac{1 - l^2}{2 - l^2}} \right)}{3 \sqrt{2 - l^2}} .$$

In the limit $l = 0$ it reduces to

$$E_\infty = \frac{2 l_c e^{3\Phi_0}}{3 \sqrt{2} \pi l_s^2} K \left( \frac{1}{\sqrt{2}} \right) = \frac{l_c e^{3\Phi_0}}{6 \sqrt{\pi} l_s^2} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} ,$$

which is, as it is expected, proportional to $L^3$.

A quite different situation arises when the region II is analyzed. In this region $e^{\Phi_0}$ is close to $e^{\Phi_c}$. In such a case we have to do the integration between 1 and $1 + \epsilon$, so that we have

$$L_\epsilon = 2 l_c e^{\Phi_0} \sqrt{\frac{1 - l^2}{2 - l^2}} \left( \sqrt{2} \epsilon^{1/2} - \frac{5}{6} \frac{\sqrt{2}}{\sqrt{\epsilon}} e^{3/2} + \mathcal{O}(\epsilon^{\frac{3}{2}}) \right) ,$$

and

$$E_\epsilon = \frac{2 \sqrt{2} l_c}{\pi l_s^2} e^{3\Phi_0} \frac{1}{\sqrt{2 - l^2}} \sqrt{\epsilon} + \mathcal{O}(\epsilon^{\frac{3}{2}}) .$$

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At order $\sqrt{\epsilon}$ it gives

$$E_c = \frac{e^{2\Phi}}{2\pi l_s^2} \frac{1}{\sqrt{1 - l^2}} L_c ,$$

indicating confinement at region II (area law). The above expression reduces to the case of [1] when $l = 0$. In the present context the extra dimensions increase the value of the effective tension by a factor $\frac{1}{\sqrt{1 - l^2}}$, which is expected since it takes into account the separation of the quark-antiquark pair on $\mathcal{H}$. The separation $|\Delta \vec{y}|/e^{2\Phi}$ in the hyper-plane $\mathcal{H}$ also behaves linearly as a function of $L$ and it is an increasing function of $l$ as it is shown in figure 4. For every value of $l$ the maxima of $|\Delta \vec{y}|/e^{2\Phi}$ and $L/e^{\Phi}$ take place at the same point, i.e. $e^{\Phi_0}/e^{\Phi_c}$, as it should be expected.

In figure 5 we show the shape of critical minimal surfaces for different values of $l$. The hyper-plane $\Sigma$ is placed at $e^{\Phi_c}$. The solid line represents the critical minimal surface corresponding to $l = 0$. The dotted lines represent critical minimal surfaces for $0 < l < 1$. Given some value of $l$ there is a critical minimal surface below of which the theory shows confinement only. Above of that there is over-confinement.

![Figure 5: Schematic shape of the critical minimal surfaces used in calculating the Wilson-loops.](image)

In the particular case of the NSVZ $\beta$-function, it has a pole in the infrared, it means that given a value of the coupling, i.e. $g_{YM}^2 = e^{\Phi_c}$, the size of the critical minimal surface
is upper bounded. It implies that the range of couplings where the theory has only confinement decreases as \( l \) increases from \( g_{YM} = 1.27 \, g_{YM}^4 \) to \( g_{YM} = 1.24 \, g_{YM}^4 \).

There are effects due to the flat extra dimensions on the separation \( L \) and also on the string tension. As far as the parameter \( l \) increases, the length of the Wilson loop \( L \) (measured on the Minkowski spacetime) decreases. On the other hand the separation of the quark-antiquark pair increases in the hyper-plane \( \mathcal{H} \). The string tension also increases by a factor \( \frac{1}{\sqrt{1-l^2}} = (\cos \alpha)^{-1} \), as it is expected.

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