Anomaly Matching in Gauge Theories at Finite Matter Density

Stephen D.H. Hsu*
Department of Physics,
University of Oregon, Eugene OR 97403-5203

Francesco Sannino†
Department of Physics,
Yale University, New Haven, CT 06520

Myck Schwetz‡
Department of Physics,
Boston University, Boston, MA 02215, USA

June, 2000

Abstract

We investigate the application of ’t Hooft’s anomaly matching conditions to gauge theories at finite matter density. We show that the matching conditions constrain the low-energy quasiparticle spectrum associated with possible realizations of global symmetries.

*hsu@duende.uoregon.edu
†francesco.sannino@yale.edu
‡ms@bu.edu
1 Introduction

Quark matter at high density provides an interesting laboratory for the controlled study of non-perturbative physics [1]. Novel phenomena such as color superconductivity and color-flavor locking have been shown to result from physics near the quark Fermi surface. Rigorous results can be obtained in the limit of asymptotic density. Less, however, is known about what happens at intermediate densities, where the effective QCD coupling is still large. The intermediate density region is more likely to be realized in neutron star cores or laboratory experiments.

The 't Hooft anomaly matching conditions [2] constrain the realizations of chiral symmetry in the low energy phase of a gauge theory. In this paper we investigate whether these conditions, orginally derived at zero density, can be used to constrain the behavior of matter at non-zero density. There are several obstacles to this generalization. A basic observation is that Lorentz symmetry, which is broken at non-zero matter density, plays an important role in the original derivation of these matching conditions [2, 3, 4], and in the physics of the anomaly. In gauge theories at zero density, unbroken global chiral symmetries imply the existence of massless spin-1/2 states. As elaborated in [3, 4, 5], these degrees of freedom are responsible for the IR singularities associated with the anomaly. It is not at all obvious that the same considerations apply to the quasiparticles comprising the low-energy spectrum at finite density. Yet, as noted by Sannino [6], the various phases of color superconductivity do indeed satisfy the anomaly matching conditions.

't Hooft’s original argument for his anomaly matching conditions involves the use of “spectator” fermions to cancel any anomalies which result from the gauging of flavor symmetries. In the low energy limit of the theory, one is left only with the massless states (composite or elementary) and the spectator fermions. If this long-wavelength theory is to be consistent, the anomaly generated by the spectators must be cancelled by that of the other massless states. This argument can be repeated in the case of finite density, with gapless quasiparticles playing the role of massless fermions in the low-energy effective theory. However, due to the existence of a Fermi surface there is a large degeneracy of zero energy states. Naively, one might expect an enhancement of the anomaly prefactor by an amount proportional to the area of the Fermi surface, which scales like $\mu^2$ (the square of the chemical potential). Clearly, we need to understand more precisely how quasiparticles contribute to the anomaly.

We will show that the anomaly matching conditions (AMCs) continue to apply at finite density, and constrain the possible quasiparticle spectrum [6]. In essence, the singularity structure of the three-current correlator implied by the anomaly must be reproduced by the
effects of quasiparticles in the finite density theory, thereby constraining their flavor quantum numbers. Applying the Landau-Cutkowsky rules, we see that only quasiparticles satisfying special kinematic conditions contribute to the singularity – there is no degeneracy factor due to the Fermi surface.

This letter is organized as follows. In section 2 we discuss the computation of the anomaly at finite density and show that it is unaffected by the presence of a chemical potential. Further, the anomaly continues to imply the existence of IR singularities, even at non-zero density. In section 3 we use the Landau-Cutkowsky rules to show that these singularities (in the absence of spontaneous symmetry breaking) require the existence of gapless quasiparticles in the low energy spectrum.

2 Anomaly at Finite Density

In this section we show that the anomaly is unaffected by the presence of a chemical potential, as are the implications for singularities of the three current correlator. Heuristically, density effects are IR in nature whereas the anomaly can be computed from the UV behavior of the theory. Points (A) and (B) below follow simply from this observation, although (C) does not.

A. The standard derivation of the anomaly from the point of view of UV divergencies involves the careful treatment of the variation of the fermionic partition function [7]

\[ Z = \int dM(\psi, \bar{\psi}) \exp \left[ i \int d^4x \bar{\psi} \left( i \slashed{D} \right) \psi \right], \quad (1) \]

under the infinitesimal change of variables

\[ \psi \rightarrow (1 + i\alpha(x)\gamma^5) \psi, \]
\[ \bar{\psi} \rightarrow \bar{\psi}(1 + i\alpha(x)\gamma^5). \quad (2) \]

Although the action in the exponent of the integrand is invariant under (2), the fermionic measure \( dM \) is not: \( dM \rightarrow J^{-2} dM \). The Jacobian determinant \( J \) is evaluated using gauge invariant regularization in the basis of the eigenstates of Dirac operator \( \slashed{D} \), i.e. \( \psi(x) = \sum_n a_n \phi_n(x), \bar{\psi}(x) = \sum_n \bar{a}_n \bar{\phi}_n(x) \) and \( dM = \Pi da_m d\bar{a}_m \).

Note, that if the chemical potential is not zero, one may still define the measure in the basis of the eigenstates of Dirac operator with zero chemical potential. All further consideration follow the same path and produce the familiar result

\[ J = \exp \left[ -i \int d^4x \alpha(x) \left( \frac{e^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}(x) \right) \right] \quad (3) \]
This implies that the anomaly equation for the axial current will take the same form at non-zero fermion density:

\[ \partial_\mu j_5^\mu = - \frac{e^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \]  \hspace{1cm} (4)

**B.** It is known that the anomalies arise via quantum corrections, (i.e. renormalization). In particular axial anomalies are associated, in a diagrammatic language, with triangular diagrams.

The time-honored three point function in electrodynamics is: (the generalization to non abelian theories is straightforward)

\[ T_{\mu\nu\lambda}(k_1, k_2, q) = i \int d^4x_1 d^4x_2 \langle 0 | T(V_\mu(x_1) V_\nu(x_2) A_\lambda(0)) | 0 \rangle e^{ik_1 \cdot x_1 + ik_2 \cdot x_2} , \]  \hspace{1cm} (5)

with \( q = k_1 + k_2 \) and \( V \) and \( A \) vector and axial currents respectively. Since the anomaly is independent of a fermion mass we set it to be zero. The classical axial Ward identity \( q_\lambda T_{\mu\nu\lambda} = 0 \) is modified according to (for a more detailed discussion see [8] before and after formulae (6.31-32)):

\[ q_\lambda T_{\mu\nu\lambda} = \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} , \]  \hspace{1cm} (6)

with

\[ \Delta^{(1)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \frac{i}{p - \mu \gamma_0} \frac{\gamma_5 \gamma_\nu}{p - \vec{k}_1} \frac{i}{\gamma_\mu} - \frac{i}{p - \vec{k}_2} \frac{\gamma_5 \gamma_\nu}{p - \vec{q}} \frac{i}{\gamma_\mu} \right] , \]  \hspace{1cm} (7)

and \( \Delta^{(2)}_{\mu\nu} \) is obtained by interchanging \( \mu \rightarrow \nu \) as well as \( k_1 \rightarrow k_2 \). Note that if we could shift the integration variable \( p \) to \( p + k_2 \) in the second term of (7) the \( \Delta^{(1)}_{\mu\nu} \) term would vanish identically. Since the integrals are linearly divergent, it can be shown that (at zero density) a translation of the integration variable produces extra finite terms ruining the classical Ward identity.

It is natural to ask what happens to the triangle anomaly at finite matter density. In principle, the contributions from triangle diagrams could depend on the chemical potential \( \mu \): \( \Delta^{(i)}_{\mu\nu}(\mu) \).

We compute the difference \( \Delta^{(i)}_{\mu\nu}(\mu) - \Delta^{(i)}_{\mu\nu}(0) \) parameterizing a possible deviation in the anomaly calculations. We explicitly evaluate the finite density effects for the \( i = 1 \) case (the effects for the \( i = 2 \) are identical):

\[ \delta \Delta^{(1)}_{\mu\nu}(\mu) = \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \frac{i}{p - \mu \gamma_0} \frac{\gamma_5 \gamma_\nu}{p - \vec{k}_1} \frac{i}{\gamma_\mu} - \frac{i}{p - \vec{k}_2} \frac{\gamma_5 \gamma_\nu}{p - \vec{q}} \frac{i}{\gamma_\mu} - \frac{i}{p - \vec{k}_2} \frac{\gamma_5 \gamma_\nu}{p - \vec{q}} \frac{i}{\gamma_\mu} \right] . \]  \hspace{1cm} (8)

By applying a momentum shift (\( p \) to \( p - u \)), with the shift vector \( u^\tau = (\mu, 0, 0, 0) \), we expect a possible non null contribution to be encoded in the following integral (see [8]):
\[
\delta \Delta_{\mu\nu}^{(1)}(\mu) = u^\tau \int \frac{d^4 p}{(2\pi)^4} \frac{\partial}{\partial p^\tau} f(p, k_1, k_2),
\]
and
\[
f(p, k_1, k_2) = -i 4 \epsilon_{\rho\sigma\mu\nu} \left[ \frac{p^\rho (p - k_1)^\sigma}{p^2 (p - k_1)^2} - \frac{(p - k_2)^\rho (p - q)^\sigma}{(p - k_2)^2 (p - q)^2} \right].
\]

Applying Gauss’s theorem for the case of four-dimensional Minkowski space, we have:
\[
\delta \Delta_{\mu\nu}^{(1)}(\mu) = u^\tau \int \frac{d^4 p}{(2\pi)^4} \frac{\partial}{\partial p^\tau} f(p, k_1, k_2) = u^\tau \frac{2i\pi^2}{(2\pi)^4} \lim_{p \to \infty} p^2 p_\tau f(p, k_1, k_2)
= \frac{\mu^\tau}{2\pi^2} \epsilon_{\rho\sigma\mu\nu} \lim_{p \to \infty} p_\tau \left( -\frac{p^\rho k_1^\sigma}{p^2} + \frac{p^\rho q^\sigma + p^\sigma k_2^\rho}{p^2} \right)
= \frac{\epsilon_{\rho\sigma\mu\nu}}{8\pi^2} \left( u^\rho k_2^\sigma + u^\sigma k_2^\rho \right) = 0,
\]
and, in the last step, we used \( \lim_{p \to \infty} p_\tau p^\rho = \frac{g^{\tau\rho}}{4} \).

Thus, for any finite density we have no extra contribution to the axial anomaly with the respect to the zero density one.

Adler and Bardeen [9] showed, at zero density, that the axial anomaly coefficient is one loop exact. We expect this theorem to hold at finite density, although we will not attempt to provide a proof. Heuristically we can say that the introduction of a density term does not affect the superficial degree of divergence of the higher-order triangle diagrams with respect to the zero density case. Since the latter have a lower degree of divergence with the respect to the simple triangle diagram we have no momentum-routing ambiguity and hence vanishing contribution to the anomaly.

C. At zero density the anomaly implies an IR singularity in the three-point function (5) involving vector and axial currents [3]. The form of \( T_{\mu\nu\lambda}^{abc} \) is strongly restricted by Lorentz, Bose and permutation symmetries (for simplicity, one works in the domain \( k_1^2 = k_2^2 = q^2 = -Q^2 \)):
\[
T_{\mu\nu\lambda}^{abc}(k_1, k_2, q) = s^{abc} F(Q^2) \left[ \epsilon_{\mu\alpha\lambda} k_1^\alpha q^\beta k_2^\lambda + \epsilon_{\nu\lambda\alpha} q^\alpha k_2^\beta k_1^\lambda + \epsilon_{\lambda\mu\alpha} k_2^\alpha q^\beta k_1^\mu \right] + \ldots,
\]
where \( s^{abc} \) is symmetric under permutation of current’s indices. In general (12) may contain tensor-like structures (denoted here by \( \ldots \)) which unlike pseudo-tensorial ones do not contribute to the anomaly.

At non-zero matter density \( T_{\mu\nu\lambda} \) may in principle contain more terms, due to the presence of a constant Lorentz four-vector \( u = (\mu, 0) \). These terms will, however, be still restricted by the rotational \( SO(3) \), Bose and permutation symmetries. Due to the presence of a new
independent 4-vector $u$ the $F$ functions, a priori, depend on any scalar combination of $Q$ and $u$ (i.e. $Q^2$, $u^2 = \mu^2$ and $Q\alpha u^\alpha = \mu Q_0$).

$$T^{abc}_{\mu\nu\lambda}(k_1, k_2, q) = s^{abc} \left\{ F_1(Q^2, u^2, Q \cdot u) \left[ \epsilon_{\mu\nu\alpha\beta} k_1^\alpha q^\beta k_2^\lambda + \epsilon_{\nu\lambda\alpha\beta} q^\alpha k_2^\beta k_1^\mu + \epsilon_{\lambda\mu\alpha\beta} k_2^\alpha k_1^\beta q^\nu \right] + F_2(Q^2, u^2, Q \cdot u) \left[ (k_1 \rightarrow u) + (k_2 \rightarrow u) + (q \rightarrow u) \right] + F_3(Q^2, u^2, Q \cdot u) \epsilon_{\mu\nu\lambda\alpha} u^\alpha \right\} + \ldots ,$$  

(13)

The second and third terms are at variance with the form of the usual anomaly term. Their existence could, in principle, alter the singularity structure of the theory. Indeed, we can also use gauge invariance to limit the allowable terms. It is straightforward to see that, at the operator level, the second and third term do not correspond to any gauge-invariant pseudo-tensor operator built in terms of $\epsilon_{\alpha\beta\mu\nu}$, $\partial_\mu$, $F_{\alpha\beta}$ and $u^\mu$. This is true not only for the Abelian case but in general for a non-Abelian theory as well. However in the latter case, due to gauge invariance, we should use the covariant derivative rather than the ordinary partial derivative.

Hence at finite density we have:

$$T^{abc}_{\mu\nu\lambda}(k_1, k_2, q) = s^{abc} F_1(Q^2, u^2, Q \cdot u) \left[ \epsilon_{\mu\nu\alpha\beta} k_1^\alpha q^\beta k_2^\lambda + \epsilon_{\nu\lambda\alpha\beta} q^\alpha k_2^\beta k_1^\mu + \epsilon_{\lambda\mu\alpha\beta} k_2^\alpha k_1^\beta q^\nu \right].$$  

(14)

Now, as demonstrated in A and B by explicitly evaluating the anomaly, there is no matter density effect. The anomaly equation can be formally written as (see [3])

$$-Q^2 F_1 \left( Q^2, u^2, Q \cdot u \right) = \text{Const.} ,$$  

(15)

for any $u$. As for the zero density case [3] $F_1$ possesses a simple pole in $Q^2$ with known residue. Thus at the same time it gives us the leading asymptotic form of $F_1$ for large $Q^2$ and the leading singularity for small $Q^2$ (associated with infrared physics) at any finite density. In order to better clarify how the infrared singularities emerge at finite density we will investigate, as done for the zero density case [3], the singularity properties of scattering amplitudes.

### 3 Quasiparticles and Singularities

Since the anomaly equation (4) is unmodified by finite density effects, we can apply 't Hooft's spectator fermion construction to conclude that quasiparticles must generate the same anomalies as the fundamental fermions. In this section we examine in more detail how this occurs. We are particularly interested in why the Fermi surface degeneracy does not alter the result, and why massive excitations do not contribute to the anomaly.
As discussed above, the anomaly at finite density still implies a $1/Q^2$ pole in the function $F_1(Q^2)$. We can use the Landau-Cutkowsky rules [10], to identify the intermediate states which are capable of providing this singularity. We first prove that the singularity structure is not modified by matter density effects. Then we investigate the kinematics associated with the singularity to gain a better understanding of why this is the case.

Using the Landau-Cutkowsky rules [10], the discontinuity in the triangle graph is given by:

$$\text{Disc } T_{\mu\nu\lambda} \propto \int \frac{dp_0 d^3p}{(2\pi)^4} (p + k_1 - u)_{\rho} (p + k_1 + k_2 - u)_{\sigma} (p - u)_{\tau} \Theta \left( p^0 - \mu \right) \delta \left[ (p - u)^2 \right] \Theta \left( p^0 + k_1^0 - \mu \right) \delta \left[ (p + k_1 - u)^2 \right] \Theta \left( p^0 + k_1^0 + k_2^0 - \mu \right) \delta \left[ (p + k_1 + k_2 - u)^2 \right],$$

(16)
times the trace factor $\text{Tr} \left[ \gamma_\gamma \gamma^\mu \gamma^\rho \gamma^\sigma \gamma^\tau \right]$. Here we used the result [10] that the leading singularity in the physical region is given by the graph(s) in which each internal particle is on-shell and propagates forward in time. In other words, we replaced each internal propagator by $2\pi i \Theta(l_0 - \mu) \delta \left[ (l_0 - \mu)^2 - \vec{l}^2 \right]$, where $l$ is the particle momentum.

We can now perform the integral in $p_0$ by using the first $\delta$ function, which requires that $p_0 - \mu = |\vec{p}|$. We obtain

$$\text{Disc } T_{\mu\nu\lambda} \propto \int \frac{d^3p}{2|\vec{p}|(2\pi)^4} (\vec{p} + k_1)_{\rho} (\vec{p} + k_1 + k_2)_{\sigma} (\vec{p})_{\tau} \Theta \left( |\vec{p}| + k_1^0 \right) \delta \left[ (\vec{p} + k_1)^2 \right] \Theta \left( |\vec{p}| + k_1^0 + k_2^0 \right) \delta \left[ (\vec{p} + k_1 + k_2)^2 \right],$$

(17)
with the 4-momentum $\vec{p} = (|\vec{p}|, \vec{p})$. This integral no longer has any explicit dependence on the finite density term $\mu$, and is identical to the integral obtained in the zero density case. We see that $p_0 - \mu$ in the finite density case merely plays the role of the energy $p_0$ in the zero density case. Hence the singularity structure is unchanged at finite density.

Finally, we would like to understand why the Fermi surface degeneracy does not affect the anomaly calculation. To this end, let us examine the on-shell kinematic conditions satisfied by the quasiparticles in (16). Suppose we regard the triangle graph as a spacetime process, with the $k_1$ vertex earliest in time. Let the incoming gauge boson momentum $k_1$ be null. In order that the quasiparticles with momenta $p$ and $p + k_1$ both be on-shell, $\vec{p}$ and $\vec{k}_1$ must be parallel. Thus, a particular point on the Fermi surface is selected by $\vec{k}_1$: the rest of the surface cannot contribute to the singularity (see figure 1). (One of the particles emerging from the $k_1$ vertex is actually a quasihole on the opposite side of the Fermi surface from the quasiparticle. Which is which depends on whether $\vec{k}_1$ and $\vec{p}$ are aligned or anti-aligned.) Finally, we note that if we take the energy of the gauge boson $k_1^0 \rightarrow 0$, we will not be able
to produce quasiparticles whose energy spectrum has a gap: only gapless quasiparticles can reproduce the anomaly in the entire physical region.

\[ \vec{p} \parallel \kappa_1 \]

Figure 1: The particle-hole pair contributing to the anomaly singularity is determined by the external momentum \( k_1 \).

4 Discussion

As shown, 't Hooft’s anomaly matching conditions are as valid at non-zero matter density as at zero density. They provide a non-trivial check on recent results on QCD at asymptotic density. Moreover, they apply even at strong coupling, where dynamical calculations cannot be reliably performed.

Acknowledgements

The authors would like to thank Krishna Rajagopal for useful discussions and comments. S.H. and M.S. thank the Institute for Nuclear Theory at the University of Washington, where this work was begun, for its hospitality. F.S. would also like to thank J. Schechter and Z. Duan for helpful discussions. This work was supported in part under DOE contracts DE-FG02-91ER40676, DE-FG-02-92ER-40704 and DE-FG06-85ER40224.
References


