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QUANTUM HEISENBERG CHAIN**

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United Nations Educational Scientific and Cultural Organization
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Jean-Pierre Nguenang
*Laboratoire de Mécanique, Faculté des Sciences, Université de Yaoundé I,
BP 812, Yaoundé, Cameroun*
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

and

Timoléon C. Kofane
*Laboratoire de Mécanique, Faculté des Sciences, Université de Yaoundé I,
BP 812, Yaoundé, Cameroun.*

Abstract

We investigate both analytically and numerically non-linear coupled magnetic and elastic excitations of compressible Heisenberg Chains. From a shallow water wave treatment of perturbation terms, one can derive two types of coupled equations which are coupled Boussinesq and non-linear Schrödinger (NLS) equations and coupled Boussinesq and NLS-like equations. We also simulate collisions between magnetic and elastic solitons in the compressible Heisenberg chain when a nonlinearized approach is performed to deal with the magnetic modes in the presence of harmonic as well as anharmonic interactions. Finally from a Fast Fourier Transform (FFT) algorithm, the dynamical structure factor is computed for all the numerical solutions of the elastic mode for the model under consideration.

MIRAMARE TRIESTE

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I INTRODUCTION

The dynamics of solitary structures in quasi-one-dimensional magnets is of great interest in theoretical [1-15], experimental [16-21], and computational [22,23] physics. Solitons for spin chain have been studied by general different approaches. Presently, two important techniques exist for the analysis of Heisenberg chains. First, is the classical method [3-7] in which general solutions are obtained for the continuum version of the classical linear Heisenberg chains. Second, is the case of quantum spin systems from which, a bosonic representation of the spin operators turns out to be a very suitable method because it permit one to include quantum corrections [2] in a systematic way. In XY system of classical spins, soliton has been predicted to give rise to central peak in dynamic structure factor as observed with neutron [16-18]. For antiferromagnets, soliton corresponds to a π rotation of the spin causing a flipping of the spin which is easily detected in experiments [20]. The soliton dynamics in these compounds does not consider the spin-lattice-spin interactions which are inherent in some real magnetic systems.

The system we are considering consists of a compressible Heisenberg one-dimensional magnetic chain. The behaviour of a compressible Heisenberg magnetic system in which higher order interaction between the processing spin through the indirect spin-lattice-spin interactions has been investigated since the work by Pushkarov et al [8], and also by Nayar and Murtaza [24]. They gave an example of interaction between two different species of non-linear dynamics of a coupled magnon system in a one-dimensional Heisenberg ferromagnet where both lattice and the spin system are taken into account with their respective non-linear interactions. Their approach, based upon the application of a semiclassical approximation for the phonon subsystem, analogously to the larger polaron problem, leads to a physical picture which corresponds to the coexistence and simultaneous propagation of coupled magnetic and lattice solitons. However, in the above papers, the different authors have used a severely truncated Holstein-Primakoff approximation. This leads to the presence of inconsistent terms in the equation of motion. So, the purpose of the present paper is to use an approximate study to probe, by a different method, the idea developed by Nayar and Murtaza. Here, we have used the Ursell [11,13] theory of the shallow water waves, in order to avoid this inconsistency.

The paper is organised as follows. In section 2, the model Hamiltonian is introduced and the general non-linear equation for the compressible Heisenberg magnetic chain is derived. In section 3, using the continuum approximation on different variables, in connection with the shallow water wave treatment of the perturbation terms, one can derive two types of coupled Boussinesq and NLS equations when $\eta \cong \varepsilon$, and coupled Boussinesq and NLS-like equations when $\eta \cong \varepsilon^{1/2}$. For the first system, we obtain four types of analytical solutions and their corresponding energies. Section 4 is devoted to the numerical investigations of the different equations obtained in section 3. In the present case we are able to test the validity of various analytical solutions, which can form the basis for an understanding of the rather complex soliton structures and their dynamic. Also investigated are the collisions between magnetic and elastic modes for different values of the magneto-elastic coupling parameter, and also computed is the dynamical structure factor by the means of a (FFT) algorithm. Finally, the conclusion is given in section 5.

II THE MODEL AND EQUATIONS OF MOTION

The ferromagnetic Hamiltonian for the spin degree of freedom including the spin-lattice interaction is taken to be [24]

$$H = \sum_{i\delta}^N \left(\frac{P_{0i}^2}{2m} + \frac{mv_0^2(u_i - u_{i+\delta})^2}{2} + \frac{\kappa\alpha(u_i - u_{i+\delta})^3}{6} \right) + \frac{1}{4} \sum_{i\delta} J(u_i - u_{i+\delta}) (S_i^+ S_{i+\delta}^+ + S_i^- S_{i+\delta}^-) - \frac{1}{2} \sum_{i\delta} \tilde{J}(u_i - u_{i+\delta}) S_i^z S_{i+\delta}^z - A \sum_{i=1}^N (S_i^z)^2 - \mu H \sum_{i=1}^N S_i^z \quad (2.1)$$

where m is the mass of atom that are supposed to lie close to their mean positions due to harmonic as well as anharmonic interactions with their nearest-neighbours. Here, P_{0i} is the linear momentum of the i^{th} atom, $K = mv_0^2$ is the spring constant, α is the strength of non-linear coupling between atomic displacements u_i , J and $\tilde{J}$ are the exchange coupling coefficients in the (x,y) plane and in the z direction, respectively. A , represents a single ion anisotropy parameter, and $\mu = g\mu_B$ is the magnetic constant. Here, the spin-phonon coupling arises as a result of the exchange integral dependence on the instantaneous separation of magnetic ions. So, $J(u_i - u_{i+\delta})$ will be expanded in Taylor series about its equilibrium value J_0 and will be approximated by the first two terms as

$$J(u_i - u_{i+\delta}) \cong J_0 - (u_{i+\delta} - u_i) J_1 \quad (2.2a)$$

$$\tilde{J}(u_i - u_{i+\delta}) \cong \tilde{J}_0 - (u_{i+\delta} - u_i) \tilde{J}_1 \quad (2.2b)$$

where

$$J_1 = \frac{\partial J}{\partial (u_i - u_{i+\delta})}; \quad \tilde{J}_1 = \frac{\partial \tilde{J}}{\partial (u_i - u_{i+\delta})} \quad (2.2c)$$

It follows that the total Hamiltonian (2.1) can be written as:

$$H = H_p + H_m + H_{mp} \quad (2.3)$$

where

$$H_p = \sum_{i\delta} \left\{ \frac{P_{0i}^2}{2m} + \frac{mv_0^2(u_i - u_{i+\delta})^2}{2} + \frac{k\alpha(u_i - u_{i+\delta})^3}{6} \right\} \quad (2.4)$$

is the lattice Hamiltonian and the magnetic Hamiltonian is

$$H_m = -\frac{1}{4} J_0 \sum_{i\delta} (S_i^+ S_{i+\delta}^+ + S_i^- S_{i+\delta}^-) - \frac{1}{2} J_0 \sum_{i\delta} S_i^z S_{i+\delta}^z - A \sum_i (S_i^z)^2 - \mu H \sum_i S_i^z \quad (2.5)$$

The interaction term between lattice and spin is given by the Hamiltonian

$$H_{mp} = \frac{1}{4} J_1 \sum_{i\delta} (u_{i+\delta} - u_i) (S_i^+ S_{i+\delta}^+ + S_i^- S_{i+\delta}^-) + \frac{1}{2} \tilde{J}_1 \sum_{i\delta} (u_{i+\delta} - u_i) S_i^z S_{i+\delta}^z \quad (2.6)$$

Since we are using a semiclassical treatment, we continue this study by expressing the spin and displacement operators in terms of boson creation and annihilation operators through the following Holstein-Primakoff transformation spin operators

$$S_i^+ = (2S)^{1/2} \left(1 - \frac{1}{2S} a_i^+ a_i \right)^{1/2} a_i \quad (2.7a)$$

$$S_i^- = (2S)^{1/2} a_i^+ \left(1 - \frac{1}{2S} a_i^+ a_i \right)^{1/2} \quad (2.7b)$$

$$S_i^z = S - a_i^+ a_i \quad (2.7c)$$

for the spin, where $S_i^\pm = S_i^x \pm jS_i^y$ are the cyclic spin components with $j^2 = -1$. To follow our study, we confine it to a system of sufficiently low temperatures and with only a few spin waves of sufficiently large magnitudes which are excited, so that the first non-linear term in the expansion of the Holstein-Primakoff representation cannot be neglected. From the above point of view, we are going to build our semiclassical treatment by introducing the following classical quantities $S_c = \hbar S$, $J_c = J/\hbar^2$ and the condition

$$S_c = \lim_{S \rightarrow \infty} \frac{\hbar \rightarrow 0}{S} (\hbar S) \quad (2.7d)$$

We will also use a truncated Holstein-Primakoff expansion with a small parameter $\varepsilon = \frac{1}{\sqrt{S}}$ which leads to a properly truncated Hamiltonian, and the amplitude of the Glauber's coherent-state representation expanded with $\eta = a_0/\lambda$ (a_0 is the lattice constant and λ is the characteristic wavelength). As far as our magnetic system is concerned, there are two parameters, namely, ε and η , coming from different species which is the proof that in the system, two perturbations coexist. So, the relative ratio of these two small parameters plays an important role in obtaining the proper non-linear wave equation in such a system. We introduce then the dimensionless variables [11,13]

$$\tilde{H} = H / JS_c^2 \quad (2.8a)$$

$$\tilde{S}_i = S_i / JS_c \quad (2.8b)$$

$$\tilde{x} = x / \lambda \quad (2.8c)$$

With these considerations, equations (2.7) can be rewritten as follows

$$\begin{aligned} \tilde{S}_i^+ &= (2)^{1/2} \left(1 - \varepsilon^2 \frac{a_i^+ a_i}{4} \right)^{1/2} \varepsilon a_i \\ &= (2)^{1/2} \left(1 - \varepsilon^2 \frac{a_i^+ a_i}{4} - \varepsilon^4 \frac{a_i^+ a_i a_i^+ a_i}{32} \right) \varepsilon a_i - O(\varepsilon^7) \end{aligned} \quad (2.9a)$$

$$\begin{aligned} \tilde{S}_i^- &= (2)^{1/2} \varepsilon a_i^+ \left(1 - \varepsilon^2 \frac{a_i^+ a_i}{4} \right)^{1/2} \\ &= (2)^{1/2} \varepsilon a_i^+ \left(1 - \varepsilon^2 \frac{a_i^+ a_i}{4} - \varepsilon^4 \frac{a_i^+ a_i a_i^+ a_i}{32} \right) - O(\varepsilon^7) \end{aligned} \quad (2.9b)$$

$$\tilde{S}_i^z = 1 - \varepsilon^2 a_i^+ a_i \quad (2.9c)$$

Following the same idea, the exchange coupling is also considered to be expanded about it's equilibrium value as

$$J(u_i - u_{i+\delta}) \cong J_0 + \eta a_0 J_1 + O(\eta^2) \quad (2.10a)$$

$$\tilde{J}(u_i - u_{i+\delta}) \cong \tilde{J}_0 + \eta a_0 \tilde{J}_1 + O(\eta^2) \quad (2.10b)$$

For the magnon field, we have

$$u_i = \sum_k \left(\frac{\hbar}{2m\omega_k} \right)^{1/2} (b_k + b_{-k}^*) \exp(ikx_i) \quad (2.11)$$

for the displacement case. Here, ω_k are the k_{th} phonon normal mode frequencies

$$\omega_k^2 = 4 \frac{k}{m} \sin^2 \left(\frac{a_0 k}{2} \right) \quad (2.12)$$

We observed that

$$H_p = H_p^0 + H_p^1, H_m = H_m^0 + H_m^1, H_{mp} = H_{mp}^0 + H_{mp}^1 + H_{mp}^2 \quad (2.13)$$

where

$H_p^0, H_p^1, H_m^0, H_m^1, H_{mp}^0, H_{mp}^1$ and H_{mp}^2 are expressed in Appendix I. The bosonic operators act on the lattice sites and they can be obtained in the Heisenberg picture using equation (2.3) in

$$i\dot{B}_k(t) = [B_k(t), H] \quad (2.14a)$$

$$i\dot{a}_k(t) = [a_k(t), H] \quad (2.14b)$$

where the dot denotes $\frac{\partial}{\partial t}$ and $\hbar=1$, $B_k = b_k + b_{-k}^*$ is related to the phonon field and a_k is related to the magnon field. In order to solve system (2.14), and as far as the solitary excitations induced by the magnon-magnon and magnon-phonon interactions are concerned, a physically acceptable candidate for quantum states formed by many magnons and phonons is a coherent state [25]

$$|\Psi(t)\rangle = |\psi(t)\rangle |\phi(t)\rangle \quad (2.15)$$

where

$$|\Psi_k(t)\rangle = \exp(-|\psi_k|^2) \cdot \sum_{n_k} \frac{(\psi_k)^{n_k}}{\sqrt{n_k!}} |n_k\rangle \quad (2.16a)$$

$$|\phi_k(t)\rangle = \exp\left(\frac{1}{i\hbar} \sum_m (\beta_m(t) \hat{p}_m - \pi_m(t) \hat{u}_m)\right) |m_k\rangle \quad (2.16b)$$

Here, $|\psi_k(t)\rangle$ describes the coherent magnon state while $|\phi_k(t)\rangle$ is a coherent phonon state. The expectation value of the co-ordinate and the conjugate momentum are given by

$$\beta_n(t) = \langle \psi_k(t) | \hat{u}_n | \psi(t) \rangle, \pi_n(t) = \langle \psi_k(t) | \hat{p}_n | \psi(t) \rangle \quad (2.17)$$

Minimising the expectation value of the total Hamiltonian within the coherent state yields a set of coupled ordinary differential equations for the magnon and phonon wave functions, $\psi_n(t)$ and $\beta_n(t)$. The system (2.14) becomes

$$i\hbar \frac{\partial \psi_n}{\partial t} = F_1(\psi_n, \psi_n^*, \psi_{n+\rho}, \psi_{n+\rho}^*, \beta_n, \beta_{n+\rho}) \quad (2.18a)$$

$$m \frac{\partial^2 \beta_n}{\partial t^2} = F_2(\psi_n, \psi_n^*, \psi_{n+\rho}, \psi_{n+\rho}^*, \beta_n, \beta_{n+\rho}) \quad (2.18b)$$

where the asterix represents complex conjugation.

III SOLITARY WAVE SOLUTIONS

If we consider the small parameter $\eta = a/\lambda$ representing the long-wave approximation, and taking into account the fact that the magnon wave functions ψ and ψ^* are smooth functions of the position $\tilde{x}$ and going over the continuum approximation

$$\psi_n(t) \rightarrow \psi(\tilde{x}, t), \psi_n^*(t) \rightarrow \psi^*(\tilde{x}, t) \quad (3.1a)$$

$$\sum_n \rightarrow \frac{1}{d} \int dx \rightarrow \frac{1}{\eta} \int d\tilde{x} \quad (3.1b)$$

one can decompose them in powers of small parameter and obtain in second order

$$\psi_{n+\rho} \rightarrow \psi + \eta \rho \psi_{\tilde{x}} + \frac{1}{2} \eta^2 \psi_{\tilde{x}\tilde{x}} + O(\eta^3) \quad (3.2a)$$

$$\psi_{n+\rho}^* \rightarrow \psi^* + \eta \rho \psi_{\tilde{x}}^* + \frac{1}{2} \eta^2 \psi_{\tilde{x}\tilde{x}}^* + O(\eta^3) \quad (3.2b)$$

The above development is done at $O(\eta^3)$ because we follow by retaining the terms $\eta^m \varepsilon^n$ of order $O(m+n=7)$. Subscripts represent partial derivatives. We assume now that η and ε have the same order ($\eta \equiv \varepsilon$). Substituting equations (3.2a) and (3.2b) into equations (2.18a) and (2.18b), and after retaining terms of equivalent $O(\varepsilon^5)$ of the functions F_1 and F_2 , we obtain a system of coupled Boussinesq and non-linear Schrödinger equations

$$\beta_{tt} = c_0^2 \beta_{\tilde{x}\tilde{x}} + \lambda_0 \beta_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \lambda_1 (\beta^2)_{\tilde{x}\tilde{x}} + \lambda_2 (|\psi|^2)_{\tilde{x}\tilde{x}} \quad (3.3a)$$

$$\left(\frac{j}{S_c} \right) \psi_t = A_0 \psi + B_0 \beta \psi + A_1 \psi_{\tilde{x}\tilde{x}} + A_2 |\psi|^2 \psi \quad (3.3b)$$

These coupled Boussinesq and nonlinear Schrödinger equations have been derived by many authors in various ways. Hence, it plays an important role in many branches of physics.

If $A_2 = \lambda_0 = \lambda_1 = A_0 = 0$, then, the system (3.3) reduces to the Zakharov equations [26] of ion-acoustic waves interacting with electron plasma waves. If $A_2 = 0$, it is reduced to the equations studied by Makhankov [27] for coupled ion sound and Langmuir plasma waves, and by Gaididei, Christiansen and Mingaleev for a soliton charge and energy transport in anharmonic molecular systems [28], and in which the complete integrability using the technique of Painlevé analysis has been investigated by Chanda and Chowdhury [29]. If $A_0 = B_0 = 0$, it also reduces to the equations studied by Nishikawa et al [30], to describe the interactions of a nonlinear ion-acoustic wave with nonlinear electron plasma wave, and Yajima and Satsuma [31], for a diatomic lattice system where there is a trapping of an optical mode by an acoustic mode, using one-soliton solution. It has also been shown by Hase and Satsuma [32] that the NLS equation coupled to the Boussinesq equation has N-soliton solutions for a particular choice of the parameters. Our equations (3.3a) and (3.3b) are similar to those obtained by Kundu and Makhankov for such a magnetic chain [33]. Simo and Kofané have also obtained such a system in the study intramolecular vibrations in molecular chains [34]. The solutions of equations (3.3a) and (3.3b), describing coupled solitary waves, can be obtained by a

$$\beta = \psi = 0 \text{ at } |x| \rightarrow \infty \quad (3.4)$$

and

$$\psi = \phi(s) \exp(i[kx - \omega t]) \quad (3.5)$$

$$\beta = \beta(s), s = x - vt \quad (3.6)$$

The energy which is transferred by a soliton is expressed as

$$E = E_{ph} + E_{int} + E_{mag} \quad (3.7)$$

To find this energy, the Hamiltonian terms E_{ph} , E_{int} , E_{mag} , will be used in the continuum limit, respectively.

Then, we obtain

$$E_{ph} = \int \frac{d\tilde{x}}{\eta} \left\{ f_0 \beta^2 + f_1 \left(\frac{\beta \beta_{\tilde{x}\tilde{x}}}{3} + \frac{(\beta_{\tilde{x}})^2}{4} \right) + f_2 \beta^2 \beta_{\tilde{x}} \right\} \quad (3.8)$$

$$E_{mag} = \int \frac{d\tilde{x}}{\eta} \left\{ \rho_0 |\psi|^2 + D_1 |\psi|^4 + D_2 (\psi^* \psi_{\tilde{x}\tilde{x}} + \psi \psi_{\tilde{x}\tilde{x}}^*) + D_3 |\beta_{\tilde{x}}|^2 \right\} \quad (3.9)$$

$$E_{int} = \int \frac{d\tilde{x}}{\eta} \left\{ g_0 \beta \phi^2 + g_1 \beta_{\tilde{x}} \phi \phi_{\tilde{x}} + g_2 \phi^4 \beta_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} \right\} \quad (3.10)$$

The substitution of equation (3.4) – (3.6) into equations (3.3a) and (3.3b) yields

$$(v^2 - v_0^2) \beta = \lambda_0 \beta_{ss} + \lambda_1 \beta^2 + \lambda_2 \phi^2 \quad (3.11)$$

$$\omega \phi = A_0 \phi + A_1 \phi_{ss} - k^2 A_1 \phi + B_0 \beta \phi + A_2 \phi^3 \quad (3.12)$$

$$2kA_1 - v = 0 \quad (3.13)$$

The solution of the above equations are obtained by supposing that

$$\beta = G \operatorname{sech}^2\left(\frac{s}{\Delta}\right) \quad (3.14)$$

where G is the amplitude of the pulse soliton and Δ it's width. When substituting equation (3.14) into equations (3.11) and (3.12), it comes that there are four types of solutions.

Case I

$$\phi = 0, \beta = G \sec h^2(\eta/\Delta) \quad (3.15a)$$

$$\Delta = \left(\frac{6A_1}{B_0 G} \right)^{1/2} \quad (3.15b)$$

$$k = \frac{v}{2A_1}, \quad \omega = A_0 + \frac{A_1}{\Delta^2} - \frac{v^2}{4A_1^2} \quad (3.15c)$$

This solution is equivalent to the acoustic soliton and is always possible for $G < 0$ and $A_1/B_1 < 0$.

The energy of the system is given by

$$E = \frac{4G^2 \Delta f_0}{3} - \frac{4G^2 f_1}{45\Delta} \quad (3.15d)$$

Case II

$$\phi = \phi_{01} \sec h(\eta/\Delta), \beta = G \sec h^2(\eta/\Delta) \quad (3.16a)$$

where

$$\phi_{01} = \left\{ \frac{1}{\lambda_2} \left[G(v^2 - v_0^2) + \frac{Gv_0^2 a_0^4}{6\Delta^2} - \frac{\alpha v_0^2 a_0^2 G^2}{2} \right] \right\}^{1/2} \quad (3.16b)$$

$$\Delta = \left\{ \frac{12\lambda_2 \Delta_1 - Gv_0^2 a_0^4 A_2}{3[A_2(2G(v^2 - v_0^2)) - \alpha v_0^2 a_0^2 G^2 - 2\lambda_2 B_0 G]} \right\}^{1/2} \quad (3.16c)$$

$$\omega = A_0 + \frac{A_1}{\Delta^2} - \frac{v^2}{4A_1^2}, \quad K = \frac{v}{2A_1} \quad (3.16d)$$

The analysis of equations (3.16b) and (3.16c) shows that the amplitude ϕ_{01} and the soliton width Δ are positive provided that the conditions $G > 0$ and $\alpha G < \left(2^{(s_1^2-1)} \frac{1}{a_0^2} + \frac{a_0^2}{3\Delta^2} \right)$ or $G > 0$ and $\alpha G < \left(2^{(s_1^2-1)} \frac{1}{a_0^2} + \frac{a_0^2}{3\Delta^2} \right)$ with $s_1 = v/v_0$ are satisfied.

The energy is given by

$$E = \frac{4G^2 \Delta}{3} f_0 - \frac{4G^2}{45\Delta} f_1 + 2D_0 \Delta \phi_{01}^2 + \frac{4}{3} D_1 \Delta \phi_{01}^4 + \left(\frac{2}{3\Delta} + \frac{\Delta v^2}{2A_1^2} \right) \cdot (D_3 - D_2) \phi_{01}^2 \\ + \frac{4G\Delta g_0 \phi_{01}^2}{3} + \frac{8G\Delta g_1 \phi_{01}^2}{15\Delta} - \frac{9768G\Delta g_2 \phi_{01}^4}{\Delta^3} \quad (3.16e)$$

Case III

$$\phi = \phi_{01} \sec h(\eta/\Delta) \tanh(\eta/\Delta), \beta = G \sec h^2(\eta/\Delta) \quad (3.17a)$$

where

$$\phi_{01} = \left\{ \frac{1}{A_2} \left(\frac{6A_1}{\Delta^2} - B_0 G \right) \right\}^{\frac{1}{2}} \quad (3.17b)$$

$$\Delta = \left(\frac{6A_1}{B_0 G} \right)^{\frac{1}{2}}, \quad \omega = A_0 + \frac{A_1}{\Delta^2} - \frac{v^2}{4A_1^2} \quad (3.17c)$$

This solution occurs upon the fulfillment of the inequality $G < \frac{6A_1}{B_0 \Delta^2}$.

The total energy is given by:

$$E = \frac{4G^2 \Delta}{3} f_0 - \frac{4G^2}{45\Delta} f_1 + \left(\frac{2}{3} \right) D_0 \Delta \phi_{01}^2 + \left(\frac{4}{35} \right) (D_3 - D_2) \phi_{01}^2 D_1 \Delta \phi_{01}^4 + \left(\frac{14}{15\Delta} + \frac{\Delta v^2}{6A_1^2} \right) (D_3 - D_2) \phi_{01}^2 \\ + \frac{4G\Delta}{15} \phi_{01}^2 g_0 - \frac{8G}{105\Delta} \phi_{01}^2 g_1 - \frac{61208G}{1287\Delta^3} \phi_{01}^4 g_2 \quad (3.17d)$$

Case IV

$$\phi = \phi_{01} \sec h^2 \left(\frac{y}{\Delta} \right), \beta = G \sec h^2 \left(\frac{y}{\Delta} \right) \quad (3.18a)$$

where

$$\phi_{01} = \left\{ \frac{1}{\lambda_2} \left[4G/\Delta^2 + \frac{Gv_0^2 a_0^4}{6\Delta^2} - \frac{\alpha v_0^2 a_0^3 G^2}{2} \right] \right\}^{\frac{1}{2}} \quad (3.18b)$$

$$\Delta = \left\{ \frac{6A_1}{B_0 G} \right\}^{\frac{1}{2}}, \quad \omega = A_0 + \frac{4A_1}{\Delta^2} - \frac{v^2}{4A_1^2} \quad (3.18c)$$

This solution is possible only for the inequalities:

$G > 0$ and $\alpha G > \left(8/v_0^2 a_0^2 \Delta^2 + a_0^2/3\Delta^2 \right)$ and the corresponding energy is given by:

$$E = \frac{4G^2 \Delta}{3} f_0 - \frac{4G^2}{45\Delta} f_1 + \left(\frac{4}{3} \right) D_0 \Delta \phi_{01}^2 + \left(\frac{32}{35} \right) D_1 \Delta \phi_{01}^4 + \left(\frac{16}{15\Delta} + \frac{\Delta v^2}{3A_1^2} \right) (D_3 - D_2) \phi_{01}^2 \\ + \frac{4G\Delta}{3} \phi_{01}^2 g_0 - \frac{64G}{105\Delta} \phi_{01}^2 g_1 - \frac{126976G}{3003\Delta^3} \phi_{01}^4 g_2 \quad (3.18d)$$

Now, we discuss the other long-wave excitation that can be set by assuming that η and $\varepsilon^{\frac{1}{2}}$ have the same order. After retaining terms of equivalent order 0 (ε^7), the system (2.18) becomes

$$\beta_{tt} = c_0^2 \beta_{xx} + \lambda_0 \beta_{xxxx} + \lambda_1 (\beta^2)_{xx} + \lambda_2 (\psi^2)_{xx} \quad (3.19a)$$

$$\left(\frac{j}{S_c} \right) \psi_t = (A_0 + B_0 \beta) \psi + (A_1 + B_1 \beta) \psi_{xx} + (A_2 + B_2 \beta) |\psi|^2 \psi + (A_3 + B_3 \beta) |\psi_x|^2 \psi \\ + (A_4 + B_4 \beta) \psi^2 \psi_{xx}^* + (A_5 + B_5 \beta) |\psi|^2 \psi_{xx} + (A_6 + B_6 \beta) \psi^* \psi_{xx}^2 \quad (3.19b)$$

The coefficients of the above coupled Boussineq and NLS-like equations are expressed in Appendix II. It is difficult to find solutions of equations (3.19a,b). So, numerical investigations of the above system will be the main object of the following section.

IV- NUMERICAL INVESTIGATIONS AND RESULTS

In the absence of analytical methods, numerical simulation has been the traditional route to the understanding of the strongly nonlinear phenomena which are expressed in partial differential equations or a system of partial differential equations in order to approach their solutions. So said, we have in this paper the system (3.19) from which we have not been able to solve analytically. In order to approach the system (3.19), it is useful to recall that we have to have to introduce the relation

$$\psi = P(x, t) + jr(x, t) \quad (4.1)$$

But we will first begin by equations (3.3a,b) in order to test the validity of the analytical solution obtained in the previous section. So, the original system (3.3) will be transformed in a new system of three coupled equations which is presented in Appendix III. After this, since we want to use a finite-difference procedure for solving our system, let us replace the continuous problem domain with a finite-difference mesh containing a finite number of grid points. So, let us consider the interval $x \in [L_1, L_2]$, this is in the assumption that the solution has a compact support and that it is zero outside some interval $[L_1, L_2]$, with

$$x_i = (i-1) * h_1, \text{ and } h_1 = \frac{L_1 + L_2}{N-1} \quad (4.2)$$

where N is the total number of grid points in the interval under consideration. The number of grid points corresponds to the number of atomic sites. We can then treat a very-long-time evolution of solitons in a conservative scheme which reflect the conservation of the total number of particles and also the energy. In the following, we choose $L_1 = 0$ and $L_2 = 16$.

For the system (3.3) we have

$$\begin{aligned} \frac{p_i^{n+1} - p_i^n}{\tau} = & \frac{A_0}{2} [r_i^{n+1} + r_i^n] + \frac{B_0}{4} [\beta_i^{n+1} + \beta_i^n] [r_i^{n+1} + r_i^n] \\ & + \frac{A_1}{4h_1^2} [r_{i+1}^{n+1} - 2r_i^{n+1} + r_{i-1}^{n+1} + r_{i+1}^n - 2r_i^n + r_{i-1}^n] \\ & + \frac{A_2}{4} [r_i^{n+1} + r_i^n] [(p_i^{n+1})^2 + (p_i^n)^2 + (r_i^{n+1})^2 + (r_i^n)^2] \end{aligned} \quad (4.3a)$$

$$\begin{aligned} \frac{r_i^{n+1} - r_i^n}{\tau} = & \frac{A_0}{2} [p_i^{n+1} + p_i^n] + \frac{B_0}{4} [\beta_i^{n+1} + \beta_i^n] [p_i^{n+1} + p_i^n] \\ & + \frac{A_1}{4h_1^2} [p_{i+1}^{n+1} - 2p_i^{n+1} + p_{i-1}^{n+1} + p_{i+1}^n - 2p_i^n + p_{i-1}^n] \\ & + \frac{A_2}{4} [p_i^{n+1} + p_i^n] [(p_i^{n+1})^2 + (p_i^n)^2 + (r_i^{n+1})^2 + (r_i^n)^2] \end{aligned} \quad (4.3b)$$

$$\begin{aligned} \frac{\beta_i^{n+1} - 2\beta_i^n + \beta_i^{n-1}}{2\tau^2} = & \frac{c_0^2}{2h_1^2} [\beta_{i+1}^{n+1} - 2\beta_i^{n+1} + \beta_{i-1}^{n+1} + \beta_{i+1}^n - 2\beta_i^n + \beta_{i-1}^n] \\ & + \frac{\lambda_0}{2h_1^2} [\beta_{i+2}^{n+1} - 4\beta_{i+1}^{n+1} + 6\beta_i^{n+1} - 4\beta_{i-1}^{n+1} + \beta_{i-2}^{n+1} + \beta_{i+1}^n - 4\beta_i^n + 6\beta_{i-1}^n - 4\beta_{i-2}^n + \beta_{i-3}^n] \end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda_1}{2h_1^2} \left[(\beta_{i+1}^{n+1})^2 - 2(\beta_i^{n+1})^2 + (\beta_{i-1}^{n+1})^2 + (\beta_{i+1}^n)^2 - 2(\beta_i^n)^2 + (\beta_{i-1}^n)^2 \right] \\
& + \frac{\lambda_2}{2h_1^2} \left[(p_{i+1}^{n+1})^2 - 2(p_i^{n+1})^2 + (p_{i-1}^{n+1})^2 + (p_{i+1}^n)^2 - 2(p_i^n)^2 + (p_{i-1}^n)^2 \right]
\end{aligned} \tag{4.3c}$$

Notice that, since the conservative scheme is nonlinear, it requires linearization which can be done by a Newton's quasi-linearization (see Appendix III). After some algebra, it is shown that the difference

approximation of the energy is obtained after setting $\beta_i = \frac{v_i - v_{i-1}}{h}$,

$$\begin{aligned}
\epsilon^{n+1} = & \frac{A_0}{4} \sum_{i=2}^N \left[(r_i^{n+1})^2 + (p_i^{n+1})^2 + (r_i^n)^2 + (p_i^n)^2 \right] \\
& - \frac{A_1}{4} \sum_{i=2}^N \left[\left(\frac{r_i^{n+1} - r_{i-1}^{n+1}}{h} \right)^2 + \left(\frac{p_i^{n+1} - p_{i-1}^{n+1}}{h} \right)^2 + \left(\frac{p_i^n - p_{i-1}^n}{h} \right)^2 + \left(\frac{r_i^n - r_{i-1}^n}{h} \right)^2 \right] \\
& + \frac{A_2}{4} \sum_{i=2}^N \left[(r_i^{n+1})^2 + (p_i^{n+1})^2 \right]^2 + \frac{B_0}{4} \sum_{i=2}^N (v_{i+1} - v_{i-1}) \left[(r_i^{n+1})^2 + (p_i^{n+1})^2 \right] \\
& + \frac{1}{2} \sum_{i=2}^N \left(\frac{v_i^{n+1} - v_i^n}{\tau} \right)^2 + \frac{c_0^2}{4} \sum_{i=2}^N \left[\left(\frac{v_i^{n+1} - v_{i-1}^{n+1}}{h_1} \right)^2 + \left(\frac{v_i^n - v_{i-1}^n}{h_1} \right)^2 \right] \\
& + \frac{T_0}{4} \sum_{i=2}^N \left[\left(\frac{v_{i+1}^{n+1} - v_{i-1}^{n+1}}{h_1} \right)^2 + \left(\frac{v_{i+1}^n - v_{i-1}^n}{h_1} \right)^2 \right]
\end{aligned} \tag{4.4a}$$

where

$$T_0 = \lambda_0 + \frac{7}{6} \lambda_1 a_0 \tag{4.4b}$$

which is conserved in the sense that $\epsilon^{n+1} = \epsilon^n$. The next step is to introduce an iterative algorithm and conduct it until convergence in the sense that:

$$\max(N_p, N_r, N_\beta) \ll \varepsilon \tag{4.5}$$

$$N_p = \max_i \left[\frac{p_i^{n+1} - p_{i-1}^{n+1}}{p_i^{n+1}} \right] \tag{4.6}$$

For our treatment, we require $\varepsilon = 10^{-10}$. After the iterations converge, one obtains a time step of the nonlinear scheme which has the desired property of conservativeness.

The dynamical stability of states of the coupled Boussinesq and NLS solitons Eqs(3.3a, b), and the coupled Boussinesq and NLS-Like solitons Eqs (3.19a, b) were studied numerically. For a lattice composed of $N=150$ grid points, we use free boundary conditions, that is, $V(x_0) = V(x_{N+1}) = 0$, and $\psi(x_0) = \psi(x_{N+1}) = 0$. It would be possible to use other boundary conditions if some were available. This hypothesis of compact support is not exact in general. However, many solutions decrease rapidly at infinity and it seems reasonable to use such an approximation. The initial condition is provided by analytical solutions when they are available. On the other hand, the knowledge of good analytical solutions gives us the possibility to do accurate numerical experiments to study the interaction properties between solitons. Solitons

excitations for simulations were created by placing the solutions (Eqs (3.4) and (3.5)) in the chain at $\tau = 0$ and letting the system evolve. For the system (3.19a, b) we have not derived analytic expressions for soliton solutions. However, simulations with arbitrary initial conditions can give us some insight of the type of excitations that one should expect for some complicated nonlinear equations like equations (3.19a, b). Using solutions (3.4,5) as an input, we solved Eqs. (3.19a, b) numerically. Thus, the initial conditions (3.4,5) within a very short propagation time adjust to a wave form with a new shape (see for example Fig 0).

For our numerical simulation we have applied it on a crystal of CsNiF_3 which structure belongs to hexagonal symmetry ($P6_3/mmc$). The lattice constant in Ni^{2+} linear ferromagnetic chain is equal to $a_0 = 5.21 \text{ \AA}$. The exchange interaction along the chain is $J_0/k_B = 23.6K$ [15]. The trigonal distortion of cubic crystalline field at the Ni^{2+} ions causes positive single ion type anisotropy along the c axis with $A/k_B = 9.0K$, $\mu = 0.16K/kG$. Our numerical simulations have been done at low temperatures and at constant magnetic field of $H = 15kG$. The anharmonic spring constant is chosen as $\alpha = 0.1$ and the sound velocity $v_0 = 0.08$. Also note that we have taken $\tilde{J}_0 \equiv 0.1J_0$, and $\tilde{J}_1 \equiv 0.1J_1$, $J_1 \equiv J_0/a_0$.

The result of numerical calculations for the system (4.3) are shown in Figs 1 (a,a')-(d,d'), at different moments of the dimensionless time τ of the magnetic excitation probability distribution, $|\psi_i|^2$, and atomic displacements, β_i . As initial state, we choose the case II (see Eqs. 3.16). As time increases, the solution is remarkably stable. It is also seen that well defined moving pulse solutions exist in the case III. This can be seen in Figs. 2 (a,a')-(d,d'). From Figs. 3 (a,a')-(d,d'), we observed that the lattice soliton still have a stable behavior while in the magnetic mode, as time increases a two-bell soliton is created but with relatively different size. Note that here initial state is chosen as the solutions given in case IV (see Eqs. 3.18).

When initial conditions for system (3.19a,b) are the solutions of system (3.3a, b), it comes that, from Figs. 4 (a,a')-(d,d') which is related to case II, (Eqs.3.16a), they soon adjust themselves to the lattice excitations which are stable, while for the magnetic mode, we observe progressively the formation of a two bell-soliton even if it is asymmetric but we notice that as time further increases it recovers its initial shape.

Let's now chose solution (3.17a) as an initial state. One more remarkable stability is observed but we notice that the magnetic mode appears as a two-bell shape solution with relatively different size but the distance between the bells doesn't change as the wave propagates. The lattice soliton here appears as a one bell with a very large width (see Figs. 5 (a,a')-(d,d')).

Choosing then solution IV (Eqs.3.18a) as an initial state, we note that both magnetic and phonon modes behave as the solutions obtained from system (4.3a,b)(see Fig6). Before looking for another phenomenon we just have to mention that when we chose equations (3.15) as an initial state it comes that for both systems we really obtained no shape of every wave type which belongs to the model under consideration. So, we can only say that this solution is not stable and thus, cannot be seen as a soliton of our compressible chain. This can also be seen as an immobile texture of the compressible chain.

In the following, it is still of interest in more realistic situations to look at another solitonic behavior, in particular during collision or overtaking of waves as the system may not be too "bad", i.e., it may be somewhat nearly integrable. This is why, in the following stage the numerical simulation of collisions of solitary waves which could be solutions of system (3.19) will be done. From Figs.7 (a)-(d), we present the

result of the collision between the two species of waves which are present in such a magnetic chain. For this purpose, the initial state is that of case II (see Eqs. 3.16a,b), where the two waves have the same amplitude and the same velocity. We thus observe that, the system is really inelastic but the magnetic mode appears to be more affected by the collision than by the lattice mode.

Figs.8 (a)-(d) present for the initial states chosen in Eqs.(3.17a,b), a relatively elastic system, because the shape of both the magnetic mode or the lattice mode are not really affected by the collision.

Finally, in Figs.9, 10 and 11, we present the collision of different species of waves when the initial state is that of equations (3.18a,b) that is the case IV, for different values of the magneto-elastic coupling parameters. Then it comes from these figures that the system is progressively losing its elasticity as the magneto-elastic coupling parameter increases.

In order to identify the real nature of the lattice wave modes we are then interested in the dynamical structure factor for the lattice field displacement which gives the probability distribution of energies at which the lattice can absorb energy at a wave vector $\vec{q}$.

For this purpose, we perform a double Fourier transform of every β_i^n lattice field. So said, we first use the dynamical variable:

$$X_q(l') = \frac{1}{N} \sum_{n=0}^{N-1} \beta_n(l'h) \exp(ilqn) \quad (4.7)$$

and the dynamical structure of displacement field is

$$S(q, \omega) = \lim_{M \rightarrow \infty} 2Mh |X_k(r)|^2 \quad (4.8)$$

$$\text{with } X_k(l') = \frac{1}{M} \sum_{l'=0}^{2M} X(l') \exp(-i\pi l l' / M) \quad (4.9)$$

The continuous frequency and wave vector have been discretized according to

$$\omega = \frac{2\pi l}{2MH}; l = 0, 1, \dots, 2M-1 \quad (4.10)$$

and

$$q = 2m\pi k / N; k = 0, 1, \dots, N-1 \quad (4.11)$$

respectively.

Note that the lattice field is taken from numerical computation results of system (3.19). We then observe in figure 12 that the result of this transformation is that, for the case II, we observe peaks which are mainly split into two peaks while in the case III, peaks are always centered at the range of zero frequency, and case IV presents only one major peak for each wave vector which are not concentrated at the range of zero frequency.

V CONCLUSION

In conclusion, we have analyzed the nonlinear dynamics of a compressible Heisenberg ferromagnetic chain by taking a model which includes harmonic as well as anharmonic interactions. We have demonstrated by means of shallow water wave theory that equations obtained in the previous study of this type contain inconsistent terms. It also appears from our results that the dynamics of solitary waves in a compressible chain could be described by two types of coupled Boussinesq and NLS equations or coupled Boussinesq and NLS-like equations. Numerical investigations of soliton propagation in the chain shows that solitons have stable behavior. Numerical simulations of collisions between different species of waves when they are stable, show that a compressible chain appears as an elastic chain when the magneto-elastic coupling parameter is lower. When it increases, the crystal loses its elasticity. We have also computed the dynamical structure factor of the lattice displacement which shows the presence of the central peak in the case III, which is the manifestation of the presence of solitons of breathers type; the reason for this choice is that, in the model under consideration, we have taken an anharmonic term of ϕ^3 type. So, it can only present one minimum. Hence, elastic soliton of kink type cannot exist in such a model.

In the case II the peak appears to be split into double peak structures but not centered at zero frequency for all the wave vectors. This situation can be seen as a manifestation of soliton (of breather type) in addition to the soft mode phonon side band. In case IV, it is worth pointing out that the peaks observed do not provide definitive proof of the existence of a well defined soliton mode, just because the dynamical structure factor appears to be centered out of the zero frequency zone, and the apparent peak appears to be split for some wave vectors. Finally, when higher order interactions are considered, the number of elastic soliton decreases

APPENDIX I

$$H_p^0 = \sum_k \hbar \omega(k) \left[b_k^\dagger b_k + \frac{1}{2} \right]$$

$$H_p^1 = \sum_{k_1, k_2, k_3} \phi(k_1, k_2, k_3) \delta(k_1 + k_2 + k_3) (b_{k_1} + b_{-k_1}^\dagger) (b_{k_2} + b_{-k_2}^\dagger) (b_{k_3} + b_{-k_3}^\dagger) \quad (\text{AI.1})$$

with

$$\phi(k_1, k_2, k_3) = \frac{k\alpha}{6} \left(\frac{\hbar}{2mN} \right)^{3/2} \frac{(2i)^3 \exp[-i(k_1 + k_2 + k_3)a_0]}{[\omega(k_1)\omega(k_2)\omega(k_3)]^{3/2}} \sin\left(\frac{k_1 a_0}{2}\right) \sin\left(\frac{k_2 a_0}{2}\right) * \sin\left(\frac{k_3 a_0}{2}\right) \quad (\text{AI.2})$$

$$H_m^0 = -N\tilde{J}_0 S^2 - g\mu_B S N H - A N S^2 + (g\mu_B H + 2AS + 2\tilde{J}_0 S) \sum_i a_i^\dagger a_i - J_0 S \sum_i (a_{i+1}^\dagger a_i + a_i a_{i+1}^\dagger) \quad (\text{AI.3})$$

$$H_m^1 = \frac{J_0}{4} \sum_i (a_i^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1} + a_i^\dagger a_i^\dagger a_i a_{i+1} + a_i a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} + a_i^\dagger a_i a_i a_{i+1}^\dagger) - \tilde{J}_0 \sum_i a_i^\dagger a_i a_{i+1}^\dagger a_{i+1} - A \sum_i a_i^\dagger a_i a_i a_i^\dagger \quad (\text{AI.4})$$

$$H_{mp}^1 = \frac{S}{N^{1/2}} \sum_i \sum_k \left\{ \left(\frac{\hbar}{2m\omega(k)} \right)^{1/2} (b_k \cdot b_{-k}^\dagger) \exp(ikx_i) J_1(\exp(ika_0) - 1) * (a_{i+1}^\dagger a_i + a_i a_{i+1}^\dagger) + 4\tilde{J}_1 \sin^2\left(\frac{ka_0}{2}\right) a_{i+1}^\dagger a_i \right\} \quad (\text{AI.5})$$

$$H_{mp}^2 = \frac{S}{N^{1/2}} \sum_i \sum_k \left\{ \left(\frac{\hbar}{2m\omega(k)} \right)^{1/2} (b_k \cdot b_{-k}^\dagger) \exp(ikx_i) J_1(\exp(ika_0) - 1) (a_i^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1} + a_i^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1} + a_i a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} + a_{i+1}^\dagger a_i a_i a_{i+1}^\dagger) + \tilde{J}_1 (\exp(ika_0) - 1) a_{i+1}^\dagger a_i a_{i+1}^\dagger a_{i+1} \right\} \quad (\text{AI.6})$$

we begin by defining a variable $y_k(t)$ as

$$y_k(t) = \left(\frac{\hbar}{2m\omega_k} \right)^{1/2} B_k(t) \quad (\text{AI.7})$$

Now we consider $y_k(t)$ to be a classical variable describing the dynamics of a macroscopic state, to obtain, in the long-wavelength limit [20] ($ka_0 \ll 1$) by neglecting the term of order $O(k^6)$,

$$\frac{\partial^2 Y_k^{(t)}}{\partial t^2} + v_0^2 a_0^2 \left(k^2 - \frac{k^4 a_0^2}{12} \right) Y_k^{(t)} + \frac{ik\alpha a_0^3}{\sqrt{N}} \sum_{k'} k' (k - k')^2 \exp(ik'x) Y_{k'}^{(t)} Y_k^{(t)} = \lambda_2 (|a|^2) x \quad (\text{AI.8})$$

where v_0 is the sound velocity along the chain. Further, we make use of

$$V(k, t) = Y_k(t) \text{ and } V(k, t) = \frac{1}{\sqrt{N}} \sum_{k'} V(k, t) \exp(ikx_i) \quad (\text{AI.9})$$

where one obtains after putting $\beta = V_x$ the following equation

$$\frac{\partial^2 \beta(x, t)}{\partial t^2} + v_0^2 a_0^2 \frac{\partial^2 \beta(x, t)}{\partial x^2} - \frac{v_0^2 a_0^2}{12} \frac{\partial^2 \beta(x, t)}{\partial x^4} - \frac{\alpha v_0^2 a_0^2}{2} \frac{\partial^2 [\beta(x, t)]^2}{\partial x^2} = \lambda_2 (|\psi|^2)_{xx} \quad (\text{AI.9})$$

APPENDIX II

Introducing equation (24) it comes that

$$\beta_{tt} = c_0^2 \beta_{xx} + \lambda_0 \beta_{xxxx} + \lambda_1 (\beta^2)_{xx} + \lambda_2 (\psi^2)_{xx} \quad (\text{AII.1a})$$

$$\left(\frac{1}{S_c} \right) p_t = A_0 r + B_0 \beta r + A_1 r_{xx} + A_2 [p^2 + r^2] r \quad (\text{AII.1b})$$

$$\left(\frac{1}{S_c} \right) r_t = A_0 p + B_0 \beta p + A_1 p_{xx} + A_2 [p^2 + r^2] p \quad (\text{AII.1c})$$

the coefficients of equations (3.22a,b) are

$$\lambda_0 = \frac{v_0^2 a_0^4}{12}; \lambda_1 = \frac{\alpha v_0^2 a_0^3}{2}; \lambda_2 = (J_0 - J_1) a_0 S; c_0^2 = a_0 v_0^2 \quad (\text{AII.2a})$$

$$A_0 = g\mu_B H + 2AS + 2(J_0 - \tilde{J}_1)S; A_0 = 2a_0(J_0 - \tilde{J}_1)S \quad (\text{AII.2b})$$

$$A_1 = -J_0 S; A_1 = a_0 J_0 S \quad (\text{AII.2c})$$

$$A_2 = 2(\tilde{J}_0 - J_1 - A); B_2 = 2a_0(-\tilde{J}_1 + J_1) \quad (\text{AII.2d})$$

$$A_3 = J_0/2; B_3 = -a_0 J_1/2 \quad (\text{AII.2e})$$

$$A_4 = J_0/2 - \tilde{J}_0; B_4 = \left(-J_1/2 + \tilde{J}_1 \right) a_0 \quad (\text{AII.2f})$$

$$A_5 = J_0 - \tilde{J}_0; B_5 = (-J_1 - \tilde{J}_1) a_0; \quad (\text{AII.2g})$$

$$A_6 = J_0/2; B_6 = -a_0 J_1/2 \quad (\text{AII.2h})$$

The other coefficients in the energies term are

$$D_0 = (-2J_0 + \tilde{J}_0 + 2A + \mu H) \epsilon^2 \quad (\text{AII.3a})$$

$$D_1 = \left(J_0 - \frac{\tilde{J}_0}{2} - A \right) \epsilon^4 \quad (\text{AII.3b})$$

$$D_2 = \left(\frac{\tilde{J}_0 - J_0}{2} \right) \epsilon^4 \quad (\text{AII.3c})$$

$$D_3 = \tilde{J}_0 \epsilon^4 \quad (\text{AII.3d})$$

APPENDIX III

The linearization of Newton-type is suitable to be used here because it doesn't spoil the conservative properties. Since our conservative scheme is non-linear, this technique requires that we substitute the sought values at the (n+1)-st stage by their approximations

$$(p^{n+1,k})^1 \cong 3(p^{n+1,k-1})^2 p^{n+1,k} - 2(p^{n+1,k+1})^3 \quad (\text{AIII.1a})$$

$$(p^{n+1,k})^2 \cong 2p^{n+1,k-1} p^{n+1,k} - 2(p^{n+1,k+1})^2 \quad (\text{AIII.1b})$$

$$(p^{n+1,k})^2 q^{n+1,k} \cong 2(p^{n+1,k-1})^2 q^{n+1,k} + 2p^{n+1,k} p^{n+1,k-1} q^{n+1,k-1} - 2(p^{n+1,k+1})^2 q^{n+1,k-1} \quad (\text{AIII.1c})$$

$$p^{n+1,k} q^{n+1,k} \cong p^{n+1,k-1} q^{n+1,k} + p^{n+1,k} q^{n+1,k-1} - p^{n+1,k-1} q^{n+1,k-1} \quad (\text{AIII.1d})$$

Then one can begin, after this, the initial conditions $p^{n+1,0} = p^n$. Thus the iteration is conducted until convergence.

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FIGURE CAPTIONS

Fig.0: Schematic plot of a lattice soliton propagation in a compressible Heisenberg chain.

Figs.1 (a,a')---(d,d'): Numerical Calculations of the magnetic excitation probability distribution, $|\psi_i|^2$, and the lattice displacements β_i , for the system 4.3 at different moment of the dimensionless time as far as the case II solitons are taken as the initial state.

Figs.2 (a,a')---(d,d'): Numerical Calculations of the magnetic excitation probability distribution, $|\psi_i|^2$, and the lattice displacements β_i , for the system 4.3 at different moment of the dimensionless time as far as the case III solitons are taken as the initial state.

Figs.3 (a,a')---(d,d'): Numerical Calculations of the magnetic excitation probability distribution, $|\psi_i|^2$, and the lattice displacements β_i , for the system 4.3 at different moment of the dimensionless time as far as the case IV solitons are taken the initial state.

Figs.4 (a,a')---(d,d'): Numerical Calculations of the magnetic excitation probability distribution, $|\psi_i|^2$, and the lattice displacements β_i , for the system 3.19 at different moment of the dimensionless time as far as the case II solitons are taken as the initial state.

Figs.5 (a,a')---(d,d'): Numerical Calculations of the magnetic excitation probability distribution, $|\psi_i|^2$ and the lattice displacements β_i , for the system 3.19 at different moment of the dimensionless time as far as the case III solitons are taken as the initial state.

Figs.6 (a,a')---(d,d'): Numerical Calculations of the magnetic excitation probability distribution, $|\psi_i|^2$, and the lattice displacements β_i , for the system 3.19 at different moment of the dimensionless time as far as the case IV are chosen for the initial state.

Figs.7 (a)-(d): Head on collision for equal amplitude and velocity between magnetic excitations and the lattice displacements β_i , for the system 3.19 at different moment of the dimensionless time as far as the case II solitons are taken the initial state.

Figs.8 (a)-(d): Head on collision for equal amplitude and velocity between magnetic excitations and the lattice displacements β_i , for the system 3.19 at different moment of the dimensionless time as far as the case III solitons are taken as the initial state.

Figs.9 (a)-(d): Head on collision equal amplitude and velocity between magnetic excitations and the lattice displacements β_i , for the system 3.19 for $\lambda = 0.1$ at different moment of the dimensionless time as far as the case IV solutions are taken as the initial state.

Figs.10 (a)-(d): Head on collision for equal amplitude and velocity between magnetic excitations and the lattice displacements β_i , for the system 3.19 for $\lambda = 0.41$ at different moment of the dimensionless time as far as the case IV solitons are taken as the initial state.

Figs.11 (a)-(d): Head on collision for equal amplitude and velocity between magnetic excitations and lattice displacements β_i , for the system 3.19 for $\lambda = 0.8$ at different moment of the dimensionless time as far as the case IV solitons are taken as the initial state.

Figs.12 Dynamical structure factors for the displacement field at five wave vectors

$$q = 0, \pi/4, \pi/2, 3\pi/4, \pi.$$

The solitons for the case II, for $\lambda = 0.1$.

The solitons for the case III, for $\lambda = 0.1$.

The solitons for the case IV, for $\lambda = 0.1$.

Figure. 0

Fig.1

Fig. 1

Fig. 2

Fig.2

Fig. 3

Fig. 3

Fig. 4

Fig. 4

Fig. 5

Fig. 5

Fig.6

Fig. 6

Fig. 7

Fig. 7

Fig. 8

Fig. 8

Fig. 9

Fig. 9

Fig.10

Fig.10

Fig.11

Fig. 11

Figure 12