Non-linear gravitational wave interactions with plasmas

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(June 10, 2000)

We consider the interactions of a strong gravitational wave with electromagnetic fields using the 1+3 orthonormal tetrad formalism. A general system of equations is derived, describing the influence of a plane fronted parallel (pp) gravitational wave on a cold relativistic multi-component plasma. We focus our attention on phenomena that are induced by terms that are higher order in the gravitational wave amplitude. In particular, it is shown that parametric excitations of plasma oscillations takes place, due to higher order gravitational non-linearities. The implications of the results are discussed.

I. INTRODUCTION

There have been numerous investigations on the scattering of electromagnetic waves off gravitational fields (see Refs. [1,2]). Previous research has mostly directed its interest towards the effects on vacuum electromagnetic fields (although there are exceptions, see e.g. Refs. [3,4], where the effects of plasmas have been taken into account). Similarly, much work concerning gravitational waves have considered the linearized theory, which is obviously the relevant regime for gravitational wave detectors, or, in general, for distances far away from the gravitational wave source. Alternatively, there has been an interest in exact solutions, and thus a number of exact gravitational wave solutions (see e.g. Ref. [5] and references therein) have been found. In the present paper we will choose an intermediate approach, starting with an exact gravitational wave solution, but focusing on a weak amplitude (but still non-linear) approximation, and studying the effects induced in a plasma.

The question under study in this paper is whether non-linear gravitational wave effects – that may be of significance close to the gravitational wave source – can give rise to qualitatively new phenomena in plasmas that are absent in linearized theory. Close to the source, additional effects apart from non-linearities – due to for example the three dimensional geometry and/or the non-radiative part of the gravitational field – are likely to be important for astrophysical applications. However, in order to focus on the processes directly induced by non-linearities, a somewhat simpler model problem with a unidirectional gravitational wave will be studied: To facilitate the analysis of the non-linear interaction between a plasma and a gravitational wave, we make use of the pp-wave solution of Einstein’s field equations. Furthermore, we introduce a Lorentz tetrad in order to define physical variables in a straightforward manner. With this setup, the governing plasma equations can be written in a simple 3-dimensional form. In Maxwell’s equations, the gravitational effects are given by effective charge- and current densities. Moreover, the fluid equations are given for a cold plasma.

Previously, parametric excitation of a Langmuir wave and an electromagnetic wave by a linearized gravitational wave has been considered [6]. Here we address the question whether higher order terms in the gravitational wave amplitude can result in new effects, using the above mentioned equations for a cold plasma. In order to demonstrate the usefulness of our set of equations, we study the stability properties of a plasma in the presence of a pp-wave. We show that including 2nd order gravitational wave effects may give rise to new phenomena. In particular it is found that electrostatic waves can be excited at a resonant surface where the gravitational wave frequency is equal to the local plasma frequency. Our results are summarized and discussed in the last section of the paper.

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A. Equations for a general space-time

We follow the approach presented in [3] for handling gravitational effects in Maxwell’s equation. Suppose an observer moves with 4-velocity \( u^a \) (\( a = 0, ..., 3 \)). This observer will measure the electric and magnetic fields [7]

\[
E_a = F_{ab} u^b, \quad B_a = \frac{1}{2} \epsilon_{abc} F^{bc},
\]
respectively, where \( F_{ab} \) is the EM field tensor. Here \( \epsilon_{abc} \) is the volume element on hyper-surfaces orthogonal to \( u^a \).

We denote the fluid velocity \( V^a \equiv (\gamma, \gamma u) \), where \( \gamma \equiv (1 - u^2)^{-1/2} \). Let \( q \) be the particle charge and \( n \) the proper number density. Using the split (1) together with \( j^a = q n V^a \), Maxwell’s equations \( \nabla_b F^{ab} = j^a \), \( \nabla_a F_{bc} = 0 \) read

\[
\begin{align*}
\nabla \cdot E &= \rho_\text{e} + \rho, \\
\nabla \cdot B &= \rho_\text{m}, \\
\rho_\text{e} &= -\Gamma^\alpha_{\beta\gamma} e_b^\beta - \epsilon^{\alpha\beta\gamma} \Gamma_\gamma^\alpha B_\beta, \\
\rho_\text{m} &= -\Gamma^\alpha_{\beta\gamma} e_b^\beta + \epsilon^{\alpha\beta\gamma} \Gamma_\beta^\gamma E_\gamma - e^\alpha_\beta \Gamma_\beta^\gamma (\Gamma^0_{\beta\gamma} B_\gamma + \Gamma^\gamma_{\beta\gamma} B_\beta), \\
j^\alpha_\beta &= -\Gamma^\alpha_{\beta\gamma} B_\beta + \Gamma^\beta_{\beta\gamma} B^\alpha + \epsilon^{\alpha\beta\gamma} (\Gamma^0_{\beta\gamma} E_\gamma + \Gamma^\gamma_{\beta\gamma} E_\beta),
\end{align*}
\]

where \( \rho \equiv \sum_{p,s} q^\gamma n \) and \( j \equiv \sum_{p,s} q^\gamma n u \) are the matter charge and current densities, respectively (the sums are over all particle species). Here \( \Gamma^\alpha_{\beta\gamma} \) are the Ricci rotation coefficients with respect to an orthonormal frame (ONF) \( e_a \), and we have introduced the “Cartesian” 3-vector notation \( E \equiv (E^a) = (E^1, E^2, E^3) \) etc., and \( \nabla \equiv (e_1, e_2, e_3) \). The dot- and cross-products are defined in the usual Euclidean way.

The energy-momentum tensor for each particle species is assumed to be in the form of pressure free matter (dust), \( T^{ab} = mn V^a V^b \), where \( m \) is the rest mass of the particles. Then the conservation equations \( \nabla_b T^{ab} = q n F^{ab} V_b \) takes the form

\[
\begin{align*}
\epsilon_0(\gamma n) + \nabla \cdot (\gamma n u) &= -\gamma n (\Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{00} u_\alpha + \Gamma^\alpha_{0\beta} u^\beta) , \\
(e_0 + v \cdot \nabla) \gamma u &= \frac{q}{m} (E + v \times B) - \gamma \left[ \Gamma^\alpha_{00} + (\Gamma^\alpha_{0\beta} + \Gamma^\alpha_{\gamma0}) v^\beta + \Gamma^\gamma_{\beta\gamma} v^\beta v^\gamma \right] e_\alpha .
\end{align*}
\]

B. Basic relations in the field of a pp-wave

Previous examinations of interactions between gravitational radiation and EM waves have focused on linearized gravitation. On the other hand, one may suspect that there will be interesting effects in the non-linear regime, not present to linear order. Below we will show that this is indeed the case.

In order to address the issue of how strong gravitational radiation may be involved in generation of EM waves, we look at the plane fronted parallel (pp) waves (for a discussion, see Refs. [8]), in the special case of a linearly polarized plane wave

\[
ds^2 = -dt^2 + a(u)^2 dx^2 + b(u)^2 dy^2 + dz^2 ,
\]
where \( u = z - t \), and \( a \) and \( b \) satisfies \( ab_{ab} + a_{uu} b = 0 \), the subscript \( u \) denoting derivative with respect to retarded time. Note that we have chosen a vacuum geometry, i.e. we have omitted the influence of the plasma on the metric.

In order to make interpretations simple, we introduce the canonical Lorentz frame

\[
e_0 = \partial_t , \quad e_1 = a^{-1} \partial_x , \quad e_2 = b^{-1} \partial_y , \quad e_3 = \partial_z .
\]

With this frame, the effective charge and current densities (3) read
\[ \rho_E = -(\ln ab)uE^3, \]  
\[ \rho_0 = -(\ln ab)uB^3, \]  
\[ J_E = -(\ln b)_u(E^1 - B^2)e_1 - (\ln a)_u(E^2 + B^1)e_2 - (\ln ab)_uE^3e_3, \]  
\[ J_0 = -(\ln b)_u(E^2 + B^1)e_1 + (\ln a)_u(E^1 - B^2)e_2 - (\ln ab)_uB^3e_3. \]  

Apart from Maxwell’s equations (2a)-(2d) [together with the effective charge and current densities (7a)-(7d)] we also need the fluid equations. From the conservation equations (4) we obtain, the fluid equations using the frame (6):

\[ \frac{\partial \gamma n}{\partial t} + \nabla \cdot (\gamma n \mathbf{v}) = \gamma n(\ln ab)u(1 - v_\parallel), \]  
\[ \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \gamma \mathbf{v} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \gamma \left[ (\ln a)_u v_1 e_1 + (\ln b)_u v_2 e_2 \right] (1 - v_\parallel) + \gamma \left[ (\ln a)_u v_1^2 + (\ln b)_u v_2^2 \right] e_3, \]

where \( v_\parallel \equiv v_3 \) is the velocity parallel to the gravitational wave propagation direction. These equations should be satisfied for each particle species. In the limit of small gravitational wave amplitude and non-relativistic velocities, our set of equations (7)-(8), together with Maxwell’s equations, agree with previous authors [3]. All terms with factors \((\ln ab)_u\) are however new, and – as we will demonstrate in the remainder of this article – they may induce new phenomena as compared to the linear regime.

### III. AN EXAMPLE: PARAMETRIC EXCITATION OF PLASMA OSCILLATIONS

The longitudinal “currents” and “charges” are second order in the gravitational wave amplitude (see Appendix for further details). This can give rise to qualitatively new phenomena compared to the linear regime, as will be demonstrated by a simple, but illustrative, example. In what follows, we will investigate longitudinal perturbations, i.e., \( E = (0, 0, E), \mathbf{v} = (0, 0, v) \) etc., around a cold one-component equilibrium plasma. Compared to the case of weak gravitational waves [3], we now have \( \rho_{E,0} \) different from zero, and we also have a longitudinal part of the effective currents. This means that longitudinal EM- and plasma waves waves can be excited.

In the unperturbed plasma, \( \partial n_0/\partial t = 0, E_0 = 0, \) and \( B_0 = 0 \) [9]. We denote the number density perturbation by \( \bar{n} \), i.e., \( n(z,t) = n_0(z) + \bar{n}(z,t). \) We assume that all perturbed quantities only depend on \( t \) and \( z \). To first order, Maxwell’s equation (2c) is written

\[ \frac{\partial E}{\partial t} = (\ln ab)_u E - \mu_0 q n_0 v, \]

where we have used \( j_m = q n_0 v. \) Furthermore, the momentum equation (8b) becomes

\[ \frac{\partial \mathbf{v}}{\partial t} = \frac{q}{m} \mathbf{E}. \]

Taking the time derivative of Eq. (9) and using Eq. (10), we obtain

\[ \frac{\partial^2 E}{\partial t^2} + \omega_p^2(z)E = \frac{\partial}{\partial t} \left[ (\ln ab)_u E \right], \]

where \( \omega_p(z) = [n_0(z)q^2 \mu_0 / m]^{1/2} \) is the local plasma frequency. Thus the left hand side is the usual equation for plasma oscillations in a cold inhomogeneous plasma, and the right hand side is the modification induced by the the pp-wave. We next focus ourselves on weak periodic deviations from flat space-time (see the Appendix). At the resonant surface where \( \omega_p(z_{\text{res}}) = \omega \), we can then have parametric excitation of plasma oscillations. We let \( E(z_{\text{res}}, t) = \tilde{E}(t) \exp(-i \omega t) + c.c., \) where c.c. denotes the complex conjugate, and we assume that \( |\partial \tilde{E}(t)/\partial t| \ll \omega |\tilde{E}(t)| \).

At the resonant surface, Eq. (11) then reduces to

\[ \frac{d \tilde{E}}{dt} = -\frac{i}{2} \exp(2i \omega z_{\text{res}}) \omega \tilde{h}^2 \tilde{E}^*, \]

where the star denotes complex conjugate. Taking the time-derivative of Eq. (12) and using the complex conjugate of the same equation, we find \( \dot{E} \propto \exp(i \Gamma t) \) where the growth rate is

\[ \Gamma = \frac{1}{2} \omega |\tilde{h}^2|. \]
Note that the threshold value for excitation is zero, since we have not included any dissipation mechanism of the plasma oscillations. Adding electron-ion collision in Eq. (8b), the threshold value $h_{\text{thr}}$ of this instability is of the order $(\nu_{c-1}/\omega_p)^{1/2}$, where $\nu_{c-1}$ is the electron-ion collision frequency.

Clearly, our instability does not occur unless higher order gravitational perturbations are included, in contrast to the results in Ref. [6]. Thus the corresponding growth rate is smaller in our case for a given source of gravitational radiation. There are still two interesting properties of the above instability as compared to the process in Ref. [6], where parametric excitation of a Langmuir wave and an electromagnetic wave was considered:

(i) The frequency matching condition in our case is $\omega = \omega_p$, which requires a rather high gravitational frequency [10], but is less severe than the condition in Ref. [6], where $\omega \geq 2\omega_p$.

(ii) In contrast to most parametric instabilities in plasmas we have no wave vector matching condition, but instead the process takes place at a localized resonance surface $z = z_{\text{res}}$ where $\omega = \omega_p(z_{\text{res}})$. This means that there is no threshold value for the instability introduced by plasma inhomogeneities. Normally the threshold value is inversely proportional to the inhomogeneity scale length [11], and close to a binary system, where the effects of gravitational radiation are likely to be most important, such a condition for parametric excitation may thus be rather severe. Unfortunately, the result of the “no inhomogeneity threshold” depends on the cold plasma approximation, and a finite temperature is likely to change the picture.

### IV. SUMMARY AND DISCUSSION

In the present paper we have investigated a higher order effect of gravitational waves on a plasma. For this purpose we have developed a Lorentz tetrad formalism (similar to the membrane paradigm equations [2], although that approach concerns stationary spacetimes) for a cold plasma in the presence of a strong gravitational wave. The obvious advantage of using a Lorentz tetrad is its direct connection to measurements. It is possible to formulate Maxwell’s equations such that the gravitational contributions takes the form of “charge”- and “current” densities. Similarly, the fluid equations are modified by effective particle sources and gravitational forces. Of course, this is not the physical picture behind the equations, but it still provides a useful tool for predicting the consequences of the gravitational influence.

The main purposes of this study has been to (i) provide a framework for investigating strong gravitational pulse effects in cold multi-component plasmas, and (ii) shown that higher order gravitational wave effects may be of importance, since it introduces effective charges and longitudinal currents, as well as effective “particle sources” and gravitational forces. As demonstrated, this in turn makes new processes – such as parametric generation of electrostatic waves – possible. Since the effect under discussion is of order $\hat{h}^2$, we do not believe that it will be of significance concerning direct earth based observations of gravitational waves. It is possible however that there exists favorable circumstances, e.g. close to a binary merger, for which the higher order gravitational effective charge- and current densities can play an important role. Close to such sources, the gravitational wave amplitudes can reach considerable strength, implying observational possibilities for the induced phenomena.

### ACKNOWLEDGMENTS

M. M. was supported by the Royal Swedish Academy of Sciences. P. K. S. D. was supported by the NRC (South Africa).

### APPENDIX: PERTURBATIVE EXPANSION OF THE PP - WAVE

In many situations of interest the gravitational wave amplitude is small, i.e. $|a - 1| \ll 1$, $|b - 1| \ll 1$ and it is appropriate to make approximations for the factors $(\ln a)_u$, $(\ln b)_u$ and $(\ln ab)_u$ that appears in the gravitational source terms in Eqs. (7) and (8). We will concentrate on approximately periodic gravitational waveforms, such as those generated by binary systems, in order to get definite results. Let $a(u) = \sum_{n=-\infty}^{\infty} \hat{a}_n \exp(i\omega u)$, $b(u) = \sum_{n=-\infty}^{\infty} \hat{b}_n \exp(i\omega u)$, where $\hat{a}_n, \hat{b}_n \ll 1$, $|\hat{a}_n| \sim |\hat{b}_n| \sim |\hat{a}_1|^n$, $\forall n$. Furthermore $\hat{a}_n^* = \hat{a}_{-n}$ and similarly for $b$. Then $a_{uu} b + ab_{uu} = 0$ becomes

\[
\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (n^2 + m^2) \hat{a}_n \hat{b}_m \exp[i\omega(n + m)] = 0 .
\] (A1)
To 0th order, we assume that $\hat{a}_0 = 1 = \hat{b}_0$. To first order, the solution to Eq. (A1) is $\hat{a}_1 = -\hat{b}_1 = \delta$. Clearly, quadratic non-linear terms will generate second harmonic terms proportional to $\exp(2i\omega u)$. Separating the frequencies in Eq. (A1), and concentrating on the second harmonic part we obtain

$$2\hat{b}_2 + 2\hat{a}_2 - \delta^2 = 0.$$  \hfill (A2)

The canonical choice is $\hat{a}_2 = \hat{b}_2 = (1/4)\delta^2$. Physically this means that we minimize the (pseudo) energy density at the second harmonic frequency. Thus for this choice all the oscillations at $2\omega$ are strictly due to the non-linearity of Einstein's equations, and no harmonics are assumed to be initially present, i.e. generated by a varying octopole moment of the binary source. For astrophysical applications this is not necessarily the most accurate choice (since binary systems may indeed have finite octopole moments), but it has the advantage of clearly isolating the effects due to non-linearities. Furthermore, it turns out that including the effect of higher moments of the gravitational source (i.e. octopole moments and higher) do not influence our calculations in Sec. III. The reason is that an alternative solution to Eq. (A2) $\hat{a}_2 = (1/4)\delta^2 + \delta \hat{a}_2$, $\hat{b}_2 = (1/4)\delta^2 - \delta \hat{a}_2$, instead of $\hat{b}_2 = \hat{a}_2 = (1/4)\delta^2$ - does not significantly affect the factor $\ln(ab)_u$, since it is independent of $\delta \hat{a}_2$ to second order in the gravitational amplitude, provided $\delta \hat{a}_2 \sim \delta^2$.

Continuing to 3rd order, a similar calculation shows that we can make the natural choice $\hat{a}_3 = \hat{b}_3 = 0$. For the 4th order terms, Eq. (A1) gives

$$32(\hat{a}_4 + \hat{b}_4) - \delta^4 = 0.$$  \hfill (A3)

where the canonical choice $\hat{a}_4 = \hat{b}_4 = (1/64)\delta^4$ is made. Continuing this procedure, it turns out that all terms odd in $n \neq 1$ disappears, while the terms even in $n$ satisfies $\hat{a}_n = \hat{b}_n \forall n$.

Using the above results, the logarithmic factors in Eqs. (7) and (8) becomes

\begin{align}
(\ln a)_u &= i\omega \hat{a} \exp(i\omega u) - \frac{i}{2} \omega \delta^2 \exp(2i\omega u) + \frac{i}{4} \omega \delta^3 \exp(3i\omega u) - \frac{1}{16} \omega \delta^4 \exp(4i\omega u) + c.c, \\
(\ln b)_u &= -i\omega \hat{b} \exp(i\omega u) - \frac{i}{2} \omega \delta^2 \exp(2i\omega u) - \frac{i}{4} \omega \delta^3 \exp(3i\omega u) - \frac{1}{16} \omega \delta^4 \exp(4i\omega u) + c.c, \\
(\ln ab)_u &= -i\omega \delta^2 \exp(2i\omega u) - \frac{i}{2} \omega \delta^4 \exp(4i\omega u) + c.c
\end{align}

(A4a) \hfill (A4b) \hfill (A4c)

to 4th order in the gravitational amplitude. This procedure may of course be continued to arbitrary order, noting that this in general will result in an asymptotic series, i.e., it does not necessarily converge towards a solution of $a_{uu}b + ab_{uu} = 0$.

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[9] Note that as long as the magnetic field is parallel to the direction of propagation of the gravitational wave it will decouple from the rest of the equations, and our following results are therefore unaffected by the addition of such a field.

[10] For interstellar matter we have $\omega_0 \sim 10^4 \text{ rad/s}$.