Renormalization Group Approach
to Cosmological Back Reaction Problems

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Abstract

We investigated the back reaction of cosmological perturbations on the evo-
lation of the universe using the second order perturbation of the Einstein’s
equation. To incorporate the back reaction effect due to the inhomogeneity
into the framework of the cosmological perturbation, we used the renormal-
ization group method. The second order zero mode solution which appears
by the non-linearities of the Einstein’s equation is regarded as a secular term
of the perturbative expansion, we renormalized a constant of integration con-
tained in the background solution and absorbed the secular term to this con-
stant. For a dust dominated universe, using the second order gauge invariant
quantity, we derived the renormalization group equation which determines
the effective dynamics of the Friedman-Robertson-Walker universe with the
back reaction effect in a gauge invariant manner. We obtained the solution
of the renormalization group equation and found that perturbations of the
scalar mode and the long wavelength tensor mode works as positive spatial
curvature, and the short wavelength tensor mode as radiation fluid.

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I. INTRODUCTION

Our universe seems to be very close to a Friedman-Robertson-Walker (FRW) spacetime at length scale of the order of the Hubble radius, but the metric and matter content appears to be highly inhomogeneous at smaller scale. The conventional cosmological perturbation approach [1,2] treat such a situation as the homogeneous isotropic background plus small perturbation, and investigate the evolution of linear fluctuations. We must go beyond linear approximation to treat non-linear structure and construct suitable model of inhomogeneous universe which is close to a FRW universe on large scales.

Based on the perturbative approach, the metric of inhomogeneous universe is approximated by perturbative series and expanded as the background FRW metric, the first order linear perturbation and the second order perturbation which carries the non-linearities of the Einstein’s equation. The non-linear effect affects not only the evolution of the fluctuation, but also change the evolution of the background FRW universe. This is the cosmological back reaction effect. Non-linear structure of the universe can affect the dynamics of the background FRW universe and it is important task to understand the effect of the cosmological back reaction theoretically to determine the correct cosmological parameter of our universe.

There are many works treating the back reaction problem so far. Issacson [3] considered the gravitational wave propagating on a slowly varying background spacetime. Assuming high frequency of the gravitational wave, he expanded the vacuum Einstein’s equation perturbatively. The leading order of the expansion gives the propagation equation of the gravitational wave and the next order gives the equation which determines the slowly varying background spacetime on which the gravitational wave propagates. The leading and the next order equations must be solved self-consistent way. Futamase [4,5] discussed the effect of the inhomogeneity on the global expansion rate of the universe using the cosmological post-Newtonian approximation with specific gauge conditions. Russ et al. [6] treated the problem using the second order perturbation with synchronous comoving gauge and derived the evolution equation for the comoving volume. Boersma [7] used the linear perturbation with a uniform Hubble slice and derived the spatially averaged Einstein’s equation. Abramo [8–11] derived the gauge invariant effective energy momentum tensor for the scalar field and the gravitational wave, and discussed the effect of inhomogeneity on the background FRW universe in a gauge invariant manner.

These works are based on perturbation expansion of the Einstein’s equation with the averaging of the metric. The averaging operation of the Einstein’s equation is introduced to obtain the evolution equation of the effective scale factor. Papers [5–7] use the different gauge conditions and the different perturbation scheme, and the relation of their result is not obvious. Abramo [8–11] constructed the gauge invariant effective energy momentum tensor and discussed the effect of inhomogeneity on the background FRW universe, but he did not derived explicit form of the evolution equation of the effective scale factor.

In this paper, we investigate the cosmological back reaction problem using the second order cosmological perturbation and the renormalization group method [12–19]. We treat a dust dominated universe and examine how the inhomogeneity affect the evolution of the background FRW universe. We does not use the averaging procedure for the Einstein’s equation to obtain the evolution equation of the effective scale factor. We only use the solu-
tion of the second order perturbation. Within the framework of the standard cosmological perturbation, we do not know how to include the second order non-linear effect in the background FRW metric because the perturbation of each order separate. We use the method of the renormalization to incorporate the back reaction effect in the perturbation expansion. Using this method, we can interpret the back reaction as renormalization of constants of integration contained in the background FRW solution. Furthermore, it is easy to obtain the approximate analytic solution of the effective scale factor which contains the back reaction effect of inhomogeneity. To circumvent the problem of gauge ambiguity, we construct the second order gauge invariant quantities and evaluate the effect of the back reaction in a gauge invariant manner. We find that the scalar mode and the long wavelength tensor mode perturbation work as positive spatial curvature and the short wavelength tensor mode perturbation works as radiation fluid.

The plan of this paper is as follows. In Sec. II, we introduce the renormalization group method by using FRW model. In Sec. III, we consider the second order cosmological perturbation of the Einstein’s equation and obtain the zero mode solution of the second order perturbation. In Sec. IV, the second order gauge invariant quantity is introduced. The renormalization group method is applied to the solution of the second order zero mode perturbation and the effect of the back reaction is discussed. Sec. V is devoted to summary and discussion. We use the units in which $c = \hbar = 8\pi G = 1$ throughout the paper.

II. RENORMALIZATION GROUP METHOD

In this section, we introduce the renormalization group method [12–16] using a FRW cosmological model. To obtain temporal evolution of the solution of the non-linear differential equation, we usually apply a perturbative expansion. But naive perturbation often yields secular terms due to resonance phenomena. The secular term prevents us from getting approximate but global solutions. The renormalization group method is one of techniques to circumvent the problem. Starting from a naive perturbative expansion, the secular divergence is absorbed into constants of integration contained in the zeroth order solution by the renormalization procedure. The renormalized constants obey the renormalization group equation.

We consider a dust dominated FRW universe with perfect fluid. The Einstein’s equations are

$$\dot{\alpha}^2 = \frac{1}{3} \rho_0 + \frac{1}{3} \rho_1,$$

$$\ddot{\alpha} + \frac{3}{2} \dot{\alpha}^2 = -\frac{1}{2} p_1,$$

where $\alpha$ is logarithm of a scale factor of the universe, $\rho_0$ is the energy density of the dust field, $\rho_1$ and $p_1$ are the energy density and the pressure of perfect fluid with the equation of the state $p_1 = (\Gamma - 1)\rho_1$ where $\Gamma$ is a constant. Conservation equation for fluids are

$$\dot{\rho}_0 + 3 \dot{\alpha} \rho_0 = 0,$$

$$\dot{\rho}_1 + 3 \Gamma \dot{\alpha} \rho_1 = 0,$$
and their solutions are
\[ \rho_0 = \frac{3c_0}{e^{3\alpha}}, \quad \rho_1 = \frac{3c_1}{e^{3\Gamma \alpha}}, \] (3)
where \( c_0, c_1 \) are constants of integration. Substitute this solution to Eq. (1a), we obtain
\[ \dot{\alpha}^2 = \frac{c_0}{e^{3\alpha}} + \frac{c_1}{e^{3\Gamma \alpha}} \] (4)
We solve this equation perturbatively by assuming the second term of the right hand side is small:
\[ \alpha = \alpha_0 + \alpha_1 + \cdots. \] (5)
The solution of the scale factor up to the first order of perturbation is given by
\[ a(t) = a_0 t^{2/3} \left[ 1 + \frac{3c_1}{4(3 - 2\Gamma)} a_0^{-3\Gamma} (t^{2-2\Gamma} - t_0^{2-2\Gamma}) \right], \] (6)
where \( a_0 \) and \( t_0 \) are constants of integration of the zeroth order and the first order, respectively. For \( 0 < \Gamma < 1 \), the first order term grows in time and the perturbation breaks down for large \( t \). We improve this behavior by applying the renormalization group method [12,15,16].
We redefine the zeroth order integration constant \( a_0 \) as
\[ a_0 = a_0^R(\mu) + \delta a_0(t_0; \mu), \] (7)
where \( \mu \) is a renormalization point and \( \delta a_0 \) is a counter term which absorbs the secular divergence of the solution. The naive solution (6) can be written
\[ a(t) = t^{2/3} \left[ a_0^R(\mu) + \delta a_0^R(t_0; \mu) + \frac{3c_1}{4(3 - 2\Gamma)} (a_0^R)^{-3\Gamma} (t^{2-2\Gamma} - a_0^R - t_0^{2-2\Gamma}) \right] \]
\[ = t^{2/3} \left[ a_0^R(\mu) + \frac{3c_1}{4(3 - 2\Gamma)} (a_0^R)^{-3\Gamma} (t^{2-2\Gamma} - \mu^{2-2\Gamma} - t_0^{2-2\Gamma}) \right], \] (8)
where we have chosen the counter term \( \delta a_0 \) so as to absorb the \( (\mu^{2-2\Gamma} - t_0^{2-2\Gamma}) \) dependent term:
\[ \delta a_0(t_0; \mu) = a_0^R(t_0) - a_0^R(\mu) = -\frac{3c_1}{4(3 - 2\Gamma)} (a_0^R)^{-3\Gamma} (\mu^{2-2\Gamma} - t_0^{2-2\Gamma}). \] (9)
This defines the renormalization transformation
\[ R_{\mu,t_0} : a_0^R(t_0) \mapsto a_0^R(\mu) = a_0^R(t_0) - \frac{3c_1}{4(3 - 2\Gamma)} (a_0^R)^{-3\Gamma} (\mu^{2-2\Gamma} - t_0^{2-2\Gamma}), \] (10)
and this transformation forms the Lie group up to the first order of the perturbation. So we can have \( a_0(\mu) \) for arbitrary large value of \( (\mu^{2-2\Gamma} - t_0^{2-2\Gamma}) \) by assuming the property of the Lie group, and this makes it possible to produce a globally uniform approximated solution...
of the original equation. The renormalization group equation is obtained by differentiating Eq.(9) with respect to $\mu$, and setting $t_0 = \mu$:

$$\frac{\partial}{\partial \mu} a_0^R(\mu) = \frac{3 c_1 (1 - \Gamma)}{2 (3 - 2\Gamma)} a_0^{1-3\Gamma} \mu^{1-2\Gamma}.$$  \hfill (11)

The renormalized solution is obtained by equating $\mu = t$ in (8):

$$a^R(t) = t^{2/3} a_0^R(t) = t^{2/3} \left[ c + \frac{9 c_1 \Gamma}{4 (3 - 2\Gamma)} t^{2-2\Gamma} \right]^{1/(3\Gamma)}.$$ \hfill (12)

Now we compare the renormalized solution with the naive solution for $\Gamma = \frac{2}{3}$ and $\Gamma = \frac{4}{3}$ cases. For $\Gamma = \frac{2}{3}$, the energy density of the fluid is $\rho_1 \propto a^{-2}$ and the effect of the fluid is equivalent to spatial curvature. The naive and the renormalized solutions are

$$a(t) = a_0 t^{2/3} \left[ 1 + \frac{9 c_1}{20 a_0^2} (t^{2/3} - t_0^{2/3}) \right], \hfill (13a)$$

$$a^R(t) = t^{2/3} \left( c + \frac{9 c_1}{10} t^{2/3} \right)^{1/2}.$$ \hfill (13b)

To see how the renormalized solution improves the naive solution, we choose $a_0 = 1$, $c_1 = -1$, $c = 1$, $t_0 = 0$. $c_1 < 0$ corresponds to the positive spatial curvature. In this case, the exact solution of Eq.(4) is given by

$$a = \frac{9}{2} (1 - \cos \eta),$$

$$t = \frac{9}{2} (\eta - \sin \eta) \quad (0 \leq \eta \leq 2\pi).$$ \hfill (14)

The scale factor of the exact solution has a maximum value $4/9 \approx 0.4444$ at $t = \frac{2\pi}{9} \approx 0.6981$. The naive solution has a maximum value $5/9 \approx 0.5556$ at $t = (10/9)^{3/2} \approx 1.1712$. The renormalized solution has a maximum value $20/(27\sqrt{3}) \approx 0.4277$ at $t = (20/27)^{3/2} \approx 0.6375$. The renormalized solution gives accurate approximated solution and reproduces the contracting behavior of the universe qualitatively and quantitatively well.

For $\Gamma = \frac{4}{3}$, the energy density of the fluid is $\rho_1 \propto a^{-4}$ and the effect of the fluid is the same as radiation. The naive and the renormalized solutions are

$$a(t) = a_0 t^{2/3} \left[ 1 + \frac{9 c_1}{4 a_0^2} (t^{-2/3} - t_0^{-2/3}) \right], \hfill (15a)$$

$$a^R(t) = t^{2/3} \left( c + 9 c_1 t^{-2/3} \right)^{1/4}.$$ \hfill (15b)

The renormalized solution behaves as $a^R \propto t^{1/2}$ for $t \sim 0$ and $a^R \propto t^{2/3}$ for $t \sim \infty$. Therefore the renormalized solution reproduces the behavior of the scale factor which represents the transition from the radiation dominant era to the matter dominant era. We cannot get such a behavior from the naive solution (15a).
We aim to treat the cosmological back reaction problem using perturbation approach. Let us assume that the metric is expanded as follows:

\[ g_{ab} = (0)g_{ab} + (1)g_{ab} + (2)g_{ab} + \cdots. \tag{16} \]

\( (0)g_{ab} \) is the background FRW metric and represent the homogeneous and isotropic space. \( (1)g_{ab} \) is the metric of the first order (linear) perturbation. This metric represents the small linear fluctuation from the background space time. We can assume that the spatial average of the first order perturbation vanishes:

\[ \langle (1)g_{ab} \rangle = 0, \tag{17} \]

where \( \langle \cdots \rangle \) means the spatial average with respect to the background FRW metric. \( (2)g_{ab} \) is the second order metric and contains non-linear effect caused by the first order linear perturbation. This non-linearity produces homogeneous and isotropic zero mode part of the second order metric. That is

\[ \langle (2)g_{ab} \rangle \neq 0. \tag{18} \]

As we want to interpret the zero mode part of the metric as the background FRW metric, we must redefine the background metric as follows:

\[ (0)g_{ab} \rightarrow (0)g_{ab} + \langle (2)g_{ab} \rangle. \tag{19} \]

This is the back reaction caused by the non-linearities of the fluctuation and it changes the background metric. But in cosmological situations, the second order perturbation term grows in time and dominates the background metric, this simple prescription does not work well in the context of the perturbation expansion. Furthermore, the meaning of the gauge invariance is not obvious in the second order quantity. We cannot adopt Eq.(19) as the definition of the background metric because the gauge transformation changes the definition of the background metric. We will resolve these problems in the next section by constructing the second order gauge invariant quantity and applying the renormalization group method.

In this section, we obtain the solution of the second order perturbation. To extract the effect of the back reaction, it is not necessary to know the full form of the solution. We only need the homogeneous and isotropic zero mode of the second order perturbation. We consider the perturbation of the Einstein’s equation with dust field in comoving synchronous gauge. The background is assumed to be a spatially flat FRW universe. The metric is

\[ ds^2 = -dt^2 + g_{ij}(t, \mathbf{x})dx^i dx^j. \tag{20} \]

The Einstein’s equations are

\[ ^{(3)}R + K^2 - K_{ij}K_{ij} = 2\rho, \tag{21a} \]

\[ K_{ij} \big|_j - K_{ij} \big|_i = 0, \tag{21b} \]

\[ \dot{K}_{ij} + KK_{ij} + ^{(3)}R_{ij} = \frac{\rho}{2} \delta_{ij}. \tag{21c} \]
where \((3)R_{ij}\) is the Ricci tensor of the spatial metric, \(\rho\) is the energy density of the dust and the extrinsic curvature is defined by
\[
K_{ji} = \frac{1}{2} g^{jk} \dot{g}_{ki}.
\]  
(22)

By eliminating \(\rho\), the evolution equation can be written
\[
\dot{K}_{ji} + K^j_\ell K^\ell_{i} - \frac{1}{4} \delta^j_\ell (K^2 - K_m^l K^m_l) + \left( (3)R^j_\ell - \frac{1}{4} \delta^j_\ell (3)R \right) = 0.
\]  
(23)

Using the conformally transformed metric \(\gamma_{ij} = a^{-2}(t) g_{ij}\),
\[
K^j_\ell = H \delta^j_\ell + \frac{1}{2} \gamma^{j\ell} \dot{\gamma}_{\ell i},
\]  
(24)

where \(H = \dot{a}/a\) and \(a\) is the scale factor of the background FRW universe. The evolution equation for the three metric becomes
\[
\dot{k}^j_\ell + 3H k^j_\ell + \frac{2}{a^2} \left( R^j_\ell - \frac{1}{4} \delta^j_\ell R \right) = -\delta^j_\ell (2\dot{H} + 3H^2) - \frac{1}{2} k k^j_\ell + \frac{1}{8} \delta^j_\ell (k^2 - k^l_m k^m_l),
\]  
(25)

where \(R_{ij}\) is the Ricci tensor of the conformally transformed metric \(\gamma_{ij}\) and \(K_{ij} = \dot{\gamma}_{ij}\). We solve this equation perturbatively up to the second order:
\[
\gamma_{ij} = \delta_{ij} + h_{ij} + l_{ij},
\]  
(26)

where \(h_{ij}\) is the metric of the first order and \(l_{ij}\) is the metric of the second order, respectively.

**A. Background solution**

The background evolution equation is
\[
2\dot{H} + 3H^2 = 0,
\]  
(27)

and the solution is
\[
a(t) = a_0 t^{2/3},
\]  
(28)

where \(a_0\) is a constant of integration.

**B. First order solution**

The first order evolution equation is
\[
\dot{h}^j_\ell + 3H h^j_\ell + \frac{2}{a^2} \left( R^j_\ell (h) - \frac{1}{4} \delta^j_\ell R(h) \right) = 0,
\]  
(29)
where \( R_{ij}(h) \) is the linear part of the Ricci tensor:

\[
(1) \quad R_{ij}(h) = \frac{1}{2} (-h_{,ij} - \nabla^2 h_{ij} + h_{ki,j}^k + h_{kj,i}^k).
\]

The momentum constraint is

\[
(31) \quad h_{i,j}^j - h_{,i} = 0.
\]

1. scalar mode

For the scalar mode perturbation, the metric can be written using scalar functions as follows:

\[
(32) \quad h_{ij}^{(s)} = A(t, x) \delta_{ij} + B_{,ij}(t, x).
\]

By using the momentum constraint (31), we have

\[
(33) \quad \dot{A}(t, x) = 0.
\]

Therefore

\[
(34) \quad A(t, x) = \frac{20}{9} \Psi(x),
\]

where \( \Psi \) is an arbitrary function of the spatial coordinate. The evolution equation becomes

\[
(35) \quad (\nabla^2 B)^\cdot + 3H(\nabla^2 B)^\cdot - \frac{20}{27a^2} \nabla^2 \Psi = 0,
\]

and the growing mode solution is

\[
(36) \quad B = \frac{2t^{2/3}}{a_0^2} \Psi.
\]

The scalar mode solution is given by

\[
(37) \quad h_{ij}^{(s)} = \frac{20}{9} \Psi(x) \delta_{ij} + \frac{2t^{2/3}}{a_0^2} \Psi_{,ij}(x).
\]

The density contrast is given by

\[
(38) \quad \frac{(1)}{(0)} \frac{\rho}{\rho_0} = -\frac{4}{9a^2H^2} \nabla^2 \Psi = -\frac{t^{2/3}}{a_0^2} \nabla^2 \Psi \propto a(t).
\]
The evolution equation for the tensor mode is
\[ \ddot{h}^{(GW)}_{ij} + 3H \dot{h}^{(GW)}_{ij} - \frac{1}{a^2} \nabla^2 h^{(GW)}_{ij} = 0, \tag{39} \]
where \( h^{(GW)}_{ij} \) satisfies the transverse traceless condition:
\[ h^j_{\ i,j} = h^i_{\ i} = 0. \tag{40} \]
The solution is given by
\[ h^{(GW)}_{ij} = \sum_k e^{i k \cdot x} A_{ij}(t, k), \tag{41} \]
\[ A_{ij}(t, k) = A_{ij}(k) t^{-1/2} Z_{\pm 3/2}(\frac{3 k}{a_0 t^{1/3}}), \tag{42} \]
where \( Z \) is a Bessel function and \( A_{ij}(k) \) is a constant tensor satisfying the transverse traceless condition.

C. Second order solution

The evolution equation for the second order is
\[ \dddot{l}_{ij} + 3H \ddot{l}_{ij} + \frac{2}{a^2} \left( \dot{R}^{(2) L}_{ij} - \frac{1}{4} \delta_{ij} \dot{R}^{(2) L}_{L} \right) = \]
\[ - \frac{2}{a^2} h^{jk} \left( \delta_{ij} \frac{R}{k^{1/3}} - \frac{1}{4} \delta_{ik} \dot{R}^{(1) L} \right) + \dot{h}^{jk} \dot{h}_{kj} - \frac{1}{2} \dot{h} h^{jk}_{\ ij} + \frac{1}{8} \delta_{ij} \left( \dot{h}^2 - h^{k}_{\ kl} \dot{h}^l_{\ l} \right) \]
\[ - \frac{2}{a^2} \left[ \dot{R}^{(2) NL}_{ij} - h^{jk} \frac{R}{k^{1/3}} - \frac{1}{4} \delta_{ij} \left( \dot{R}^{(2) NL} - h^{kl} \frac{R}{k^{1/3}} \right) \right], \tag{42} \]
where \( \dot{R}^{(2) L}_{ij} \) is the linear part of the Ricci tensor with respect to the second order metric \( l_{ij} \) and \( \dot{R}^{(2) NL}_{ij} \) is the quadratic part of the Ricci tensor with respect to the first order metric \( h_{ij} \).

Due to the homogeneity and the isotropy of the background metric, the zero mode part of the second order metric satisfies
\[ \langle l_{ij} \rangle = \frac{1}{3} \delta_{ij} \langle l \rangle, \tag{43} \]
where the spatial average is defined as
\[ \langle l_{ij} \rangle = \frac{1}{V} \int_V d^3 x \ l_{ij}. \tag{44} \]
\( V \) is the volume of a sufficiently large compact domain and we assume periodic boundary conditions for perturbations. Therefore it is sufficient to consider only the trace part of the
second order metric. The trace part of the evolution equation (42) with the spatial average becomes
\[ \langle l \rangle'' + 3H \langle l \rangle' = \left\langle \frac{5}{8} \dot{h}^{kl} \dot{h}_{kl} - \frac{1}{8} \dot{h}^2 - \frac{2}{a^2} \left( \frac{1}{4} R^{(2)NL} - \frac{1}{4} h R + \frac{3}{4} h^{kl} R_{kl} \right) \right\rangle, \]  
(45)

where
\[ R^{(2)NL} = \frac{1}{2} \left[ \frac{1}{2} h_{kl,i} h^{kl,i} + h^{kl} \left( \nabla^2 h_{kl} + h_{kl} - 2 h_{ki,l} \right) + h^{ik,l} (h_{ik,l} - h_{il,k}) - \left( h^{kl} - \frac{1}{2} h^k \right) (2 h_{ki,l} - h_{k,l}) \right]. \]  
(46)

1. second order zero mode solution : contribution of scalar mode

For the first order scalar mode solution (37), Eq.(45) becomes
\[ \langle l \rangle'' + 3H \langle l \rangle' = \frac{100 t^{-4/3}}{81 a_0^4} \sum_k k^2 \Psi_k^* \Psi_k + \frac{28 t^{-2/3}}{9 a_0^4} \sum_k k^4 \Psi_k^* \Psi_k, \]  
(47)

where \( \Psi_k \) is the Fourier component of \( \Psi(x) \). The solution of this equation is
\[ \langle l \rangle = \frac{10 t^{2/3}}{9 a_0^2} \sum_k k^2 \Psi_k^* \Psi_k + t^{4/3} \sum_k k^4 \Psi_k^* \Psi_k + \text{const.} \]  
(48)

This solution is consistent with the result of Tomita and Kasai [20,21] who obtained the complete form of the second order solution.

2. second order zero mode solution : contribution of tensor mode

For the first order tensor mode solution (41), Eq.(45) becomes
\[ \langle l \rangle'' + 3H \langle l \rangle' = \sum_k \left( \frac{5}{8} \ddot{A}^{kl}(t,k) \dot{A}_kl^*(t,k) - \frac{7 k^2}{8 a^2} A^{kl}(t,k) A_{kl}^*(t,k) \right). \]  
(49)

For the mode whose wavelength is smaller than the horizon scale \( k \gg aH \), the first order solution is approximated to be WKB form:
\[ A_{ij}(t,k) \approx \frac{1}{a} A_{ij}(k) e^{ik \cdot \frac{\dot{a}}{a}}, \quad \dot{A}_{i}^i = k^i A_{ij} = 0. \]  
(50)

For this solution Eq.(49) becomes
\[ \langle l \rangle'' + 3H \langle l \rangle' \approx -\frac{1}{4 a^4} \sum_k k^2 A^{ij}(k) A_{ij}^*(k), \]  
(51)
and the solution is given by
\[ \langle l \rangle = \frac{9}{8} t^{-2/3} \sum_k k^2 A^{ij}(k) A^*_{ij}(k). \] (52)

For the long wavelength mode whose wavelength is longer than the horizon scale \( k \ll aH \), the first order tensor mode solution becomes
\[ A_{ij}(t, k) \approx A_{ij}(k) \left[ 1 - \frac{2}{5} \left( \frac{k}{aH} \right)^2 \right], \] (53)
and the Eq.(49) becomes
\[ \langle l \rangle'' + 3H \langle l \rangle \approx -\frac{7}{8} a^2 \sum_k k^2 A^{ij}(k) A^*_{ij}(k). \] (54)
The solution is
\[ \langle l \rangle = -\frac{63}{80} \frac{t^{2/3}}{a^2} \sum_k k^2 A^{ij}(k) A^*_{ij}(k). \] (55)

**IV. GAUGE INVARIANCE AND RENORMALIZATION OF THE SECOND ORDER ZERO MODE SOLUTION**

The second order perturbation has the zero mode part which comes from the the non-linearity of the Einstein’s equation, and this changes the evolution of the background FRW universe. We want to treat this back reaction effect by using the renormalization method. The background FRW solution has a constant of integration which corresponds to the freedom of constant rescaling of the scale factor. By absorbing the second order zero mode part to this constant using the renormalization method, we can obtain the effective scale factor of the FRW universe which contains the effect of back reaction due to the inhomogeneity. But the existence of the gauge freedom prevents us from direct application of the renormalization method. If we change the gauge condition, the component of the second order zero mode solution may change. We do not know what component or combination of the zero mode solution should be renormalized.

In this section, we first review the method of construction of the second order gauge invariant quantities using Abramo’s argument [8–10]. Then we apply the renormalization group method to the second order gauge invariant quantity for the zero mode. Let us consider the infinitesimal coordinate transformation
\[ x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu, \] (56)
where \( \xi^\mu \) is assumed to be the first order small quantity. An arbitrary tensor \( q \) receives the gauge transformation
\[ q \rightarrow q' = e^{-\mathcal{L}_\xi} q = q - \mathcal{L}_\xi q + \frac{1}{2} \mathcal{L}_\xi^2 q + \cdots, \] (57)
where $\mathcal{L}_\xi$ is the Lie derivative with respect to the vector $\xi$. Now we prepare the vector $X$ which transforms under the coordinate transformation as follows:

$$X^\mu \longrightarrow X'^\mu = X^\mu + \xi^\mu + \frac{1}{2} [X, \xi]^\mu + \cdots.$$  \hspace{1cm} (58)

Then the quantity $Q \equiv e^{\mathcal{L}_X} q$ transforms as

$$Q \longrightarrow Q' = e^{\mathcal{L}_X'} q' = e^{\mathcal{L}_X} e^{-\mathcal{L}_\xi} q = e^{\mathcal{L}_X} q = Q.$$  \hspace{1cm} (59)

Therefore the quantity $Q$ is gauge invariant. If expand the tensor $q$ perturbatively $q = (0) + (1) + (2)$, the gauge invariant quantities in each order are

$$\langle 0 \rangle = (0) q,$$  \hspace{1cm} (60a)

$$\langle 1 \rangle = (1) q + \mathcal{L}_X (0) q,$$  \hspace{1cm} (60b)

$$\langle 2 \rangle = \langle 2 \rangle + \langle \mathcal{L}_X (1) q \rangle + \frac{1}{2} \langle \mathcal{L}_X^2 (0) q \rangle + \langle \mathcal{L}_Y (0) q \rangle.$$  \hspace{1cm} (60c)

$\langle 0 \rangle$ is the background variable and does not receive gauge transformation. $Y$ represents the vector which is needed to construct the gauge invariant combination under the second order coordinate transformation. We do not need explicit form of this vector. For the second order zero mode metric, the effect of $Y$ is to produce the second order time coordinate transformation. We can set $Y = 0$ if we do not mind the freedom of the second order time coordinate transformation. The quantity $\langle 2 \rangle$ is invariant under first order gauge transformation for $Y = 0$.

Now we look for the explicit form of the gauge invariant quantity in a specific gauge. We consider the following form of the first order metric:

$$g^{(1)}_{\mu\nu} = \begin{pmatrix} -2 \phi & a B_i \\ a B_i & a^2 (-2 \psi \delta_{ij} + 2 E_{ij}) \end{pmatrix}.$$  \hspace{1cm} (61)

Under the coordinate transformation, the perturbation variables transform as follows

$$\phi' = \phi - \dot{\xi}^0,$$  \hspace{1cm} (62a)

$$B' = B - a \left( \frac{1}{a^2} \xi \right) + \frac{1}{a} \dot{\xi}^0,$$  \hspace{1cm} (62b)

$$\psi' = \psi + H \xi^0,$$  \hspace{1cm} (62c)

$$E' = E - \frac{1}{a^2} \xi,$$  \hspace{1cm} (62d)

$$\delta \chi' = \delta \chi - \xi^0 \dot{\chi}_0,$$  \hspace{1cm} (62e)

where the scalar function $\chi$ represents the velocity potential for the dust field. From this, we can choose the following vector $X$ which obeys the desired transformation (58):

$$X = \left[ -\frac{\delta \chi}{\dot{\chi}_0}, -\delta_{ij} E_{,j} \right].$$  \hspace{1cm} (63)
Of course, the vector $X$ is not unique. There are infinite possibility of the form of $X$ which corresponds to the infinite possible candidate of the gauge invariant combination of the perturbation variables. We choose this form because we want to impose the comoving gauge condition $\delta \chi = 0$ for the perturbation variables. The first order gauge invariant quantities constructed from $X$ are

\begin{align}
(1) \ Q_{00} &= -2 \phi + 2 \left( \frac{\delta \chi}{\chi_0} \right)^{,} , \\
(1) \ Q_{0i} &= \left( aB - a^2 \dot{E} + \frac{\delta \chi}{\chi_0} \right)^{,i} ,
\end{align}

\begin{align}
(1) \ Q_{ij} &= -2 a^2 \delta_{ij} \left( \psi + \frac{H}{\chi_0} \delta \chi \right) , \\
(1) \ Q_{\chi} &= \delta \chi + X^0 \dot{\chi}_0 = 0.
\end{align}

For comoving synchronous gauge $\phi = B = \delta \chi = 0$, there is no freedom of the coordinate transformation and the each component of the metric perturbation can be written using the combination of these gauge invariant variables.

**A. gauge invariant zero mode quantity: contribution of scalar mode**

Using the first order scalar mode solution (37), the zero mode gauge invariant quantities in comoving synchronous gauge are

\begin{align}
\langle (2) Q_{00} \rangle &= \langle (2) Q_{0i} \rangle = \langle (2) Q_{\chi} \rangle = 0, \\
\langle (0) Q_{ij} + (2) Q_{ij} \rangle &= a^2 \delta_{ij} + \langle (2) g_{ij} \rangle + \langle \mathcal{L}_X (1) g_{ij} \rangle + \frac{1}{2} \langle \mathcal{L}_X^2 (0) g_{ij} \rangle \\
&= a^2 \delta_{ij} \left[ 1 + \frac{2}{3} \langle \psi_k E_k \rangle - \frac{1}{3} \langle E_{ki} E_{ki} \rangle \right] + \langle (2) g_{ij} \rangle \\
&= a^2 \left[ 1 - \frac{10 t^{2/3}}{27 a_0^2} \sum_k k^2 \Psi_k \Psi_k \right] \delta_{ij}.
\end{align}

These quantities are invariant under the first order gauge transformation. To interpret the geometrical meaning of the gauge invariant quantity $\langle (0) Q_{ij} + (2) Q_{ij} \rangle$, we consider the determinant of the three metric:

\begin{align}
\langle \sqrt{g} \rangle &= a^3 \left[ 1 + \langle \psi_i E_i \rangle - \frac{1}{2} \langle E_{ij} E_{ij} \rangle + \frac{3}{2} \langle \psi^2 \rangle + \frac{1}{2} \langle (2) g \rangle \right] \\
&= \left[ \frac{1}{3} \delta^{ij} \langle (0) Q_{ij} + (2) Q_{ij} \rangle + a^2 (\langle \psi^2 \rangle + \text{const.}) \right]^{3/2} \\
&= \left[ \frac{1}{3} \delta^{ij} \langle (0) Q_{ij} + (2) Q_{ij} \rangle \right]^{3/2}.
\end{align}

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Therefore, the trace of the gauge invariant variable \( Q_{ij} + \langle Q_{ij} \rangle \) corresponds to the physical volume element of the universe and we can interpret \( Q + \langle Q \rangle \) as square of the scale factor of the FRW universe:

\[
\begin{align*}
    ds^2 &= -dt^2 + \tilde{a}^2(t) \, dx^2, \\
    \tilde{a}(t) &= \left( (0)Q + \langle Q \rangle \right)^{1/2} = t^{2/3} \left( a_0 - \frac{5 \, t^{2/3}}{27 \, a_0} \sum_k k^2 \Psi_k \Psi_k^* \right).
\end{align*}
\] (67)

At this stage, we renormalize the expression for the effective scale factor \( \tilde{a}(t) \) to improve its behavior for large \( t \). The background FRW solution has the constant of integration \( a_0 \). We regard the second order term in \( \tilde{a}(t) \) as a secular term and apply the renormalization group method. The renormalization group equation for \( a_0 \) becomes

\[
\frac{\partial a_0}{\partial t^{2/3}} = - \frac{5}{27 \, a_0} \sum_k k^2 \Psi_k \Psi_k^*,
\] (68)

and the solution is

\[
a_0(t) = \left( c - \frac{10 \, t^{2/3}}{27} \sum_k k^2 \Psi_k \Psi_k^* \right)^{1/2},
\] (69)

where \( c \) is a constant of integration. The renormalized expression of the line element (67) is

\[
\begin{align*}
    ds^2 &= -dt^2 + (a^R(t))^2 \, dx^2, \\
    a^R(t) &= t^{2/3} \left( c - \frac{10 \, t^{2/3}}{27} \sum_k k^2 \Psi_k \Psi_k^* \right)^{1/2}.
\end{align*}
\] (70)

Comparing with the analysis of the section II, this solution is the same as the FRW equation with positive spatial curvature. We conclude that the effect of the inhomogeneity due to the scalar mode fluctuation is equivalent to positive spatial curvature. This result is the same as Russ et al. [6] who derived the spatially averaged Einstein’s equation for a comoving volume and evaluate its behavior using the solution of the second order perturbation of the Einstein’s equation.

**B. gauge invariant zero mode quantity: contribution of tensor mode**

For the first order tensor mode perturbation, the second order solution \( \langle l_{ij} \rangle \) is invariant under the first order gauge transformation because the first order tensor mode is already gauge invariant. The gauge invariant zero mode quantity is given by

\[
(0)Q_{ij} + \langle (2)Q_{ij} \rangle = a^2 (\delta_{ij} + \langle l_{ij} \rangle).
\] (71)
Using the second order zero mode solution, we have

\[
(0) Q_{ij} + \langle (2) Q_{ij} \rangle = \begin{cases} 
  a^2 \left( 1 + \frac{3}{8} t^{-2/3} a_0^4 \sum_k k^2 A^{ij}(k) A^*_k(k) \right) \delta_{ij} & \text{for } k \gg aH \\
  a^2 \left( 1 - \frac{21}{80} t^{2/3} a_0^2 \sum_k k^2 A^{ij}(k) A^*_k(k) \right) \delta_{ij} & \text{for } k \ll aH
\end{cases}
\]  

(72)

The renormalized scale factor becomes

\[
a^R(t) = \begin{cases} 
  t^{2/3} \left( c + \frac{3}{4} t^{-2/3} \sum_k k^2 A^{ij}(k) A^*_k(k) \right)^{1/4} & \text{for } k \gg aH \\
  t^{2/3} \left( c - \frac{21}{80} t^{2/3} \sum_k k^2 A^{ij}(k) A^*_k(k) \right)^{1/2} & \text{for } k \ll aH
\end{cases}
\]  

(73)

Comparing with the result of section II, the effect of the back reaction of the short wavelength tensor mode is same as radiation fluid, and the long wavelength tensor mode is same as positive spatial curvature. This result is consistent with the analysis of Abramo [11] using the effective energy momentum tensor.

C. gauge invariant zero mode solution in the longitudinal gauge

To compare the result in the comoving gauge with other gauge condition, we use the longitudinal gauge \( E = B = 0 \) here. We present only the result of the calculation. The first order solution for the scalar mode is

\[
(1) g_{ab} = \begin{pmatrix} -\frac{20}{9} \Psi(x) & 0 \\ 0 & -a^2 \frac{20}{9} \Psi(x) \delta_{ij} \end{pmatrix},
\]

(74)

and the second order zero mode solution is

\[
\langle (2) g_{ab} \rangle = \begin{pmatrix} -\frac{50}{9} t^{2/3} \sum_k k^2 \Psi_k \Psi^*_k & 0 \\ 0 & -a^2 \frac{50}{9} t^{2/3} \sum_k k^2 \Psi_k \Psi^*_k \end{pmatrix}.
\]

(75)

In this gauge, each components of the first order metric are gauge invariant quantities and we can set \( X = 0 \) to construct the second order gauge invariant quantities. Therefore the gauge invariant zero mode solution is given by very simple form:

\[
(0) Q_{ij} + \langle (2) Q_{ij} \rangle = (0) g_{ij} + \langle (2) g_{ij} \rangle.
\]

(76)

This means the metric components of the line element

\[
ds_{\text{long}}^2 = - \left( 1 + \frac{50}{7 \cdot 9^2} t^{2/3} \sum_k k^2 \Psi_k \Psi^*_k \right) dt^2 + a^2 \left( 1 - \frac{50}{7 \cdot 9^2} t^{2/3} \sum_k k^2 \Psi_k \Psi^*_k \right) dx^2
\]

(77)
are gauge invariant under the first order gauge transformation. As we have the freedom of the second order coordinate transformation, which is the second order coordinate transformation of time, we can transform this metric to the FRW form. We change the time coordinate as

\[ t = T - \int dt \frac{25}{7 \cdot 9^2} t^{2/3} \sum_k k^2 \Psi_k \Psi^*_k. \]  

(78)

Then the metric becomes

\[ ds^2_{\text{long}} = -dT^2 + T^{4/3} \left( a_0^2 - \frac{10}{81} T^{2/3} \sum_k k^2 \Psi_k \Psi^*_k \right) d\mathbf{x}^2. \]  

(79)

At this stage, we use the renormalization method and the second order perturbation term is absorbed to the constant \( a_0 \) of the zeroth order. The renormalization group equation is

\[ \frac{\partial a_0^2}{\partial T^{2/3}} = -\frac{10}{81} \sum_k k^2 \Psi_k \Psi^*_k, \]  

(80)

and the solution is

\[ a_0(T) = \left( c - \frac{10}{81} T^{2/3} \sum_k k^2 \Psi_k \Psi^*_k \right)^{1/2}. \]  

(81)

The renormalized metric is given by

\[ ds^2_{\text{long}} = -dT^2 + (a^R(T))^2 d\mathbf{x}^2, \]

\[ a^R(T) = T^{2/3} a_0(T). \]  

(82)

This metric give the same result in the comoving gauge. Therefore we have confirmed the gauge invariance of the back reaction effect.

V. SUMMARY AND DISCUSSION

We have investigated the cosmological back reaction problem using the second order perturbation of the Einstein’s equation. To describe the back reaction effect due to the inhomogeneity in the framework of the cosmological perturbation approach, we used the renormalization group method. The second order zero mode solution which appeared by the non-linear effect is regarded as a secular term of the perturbative expansion, we renormalized a constant of integration contained in the background solution and absorbed the secular term to this constant. The renormalized constant obeys the renormalization group equation which determines the dynamics of the effective background universe.

Owing to the gauge freedom of the Einstein’s equation, it is not obvious that what combination of the second order variables should be renormalized. Constants contained in the background solution change if we use a \( N(t) \neq 1 \) lapse function for the background metric. The gauge independent background constants can be obtained by using the Hamilton-Jacobi method [22] and we can recognize the constant \( a_0 \), which is the freedom of rescaling of the
scale factor, has the gauge invariant meaning. For perturbation, using the second order gauge invariant variable, we renormalized the secular term to the background constant in a gauge invariant way. In comoving synchronous gauge, the gauge invariant combination corresponds to a comoving volume of the universe. This means that the requirement of the gauge invariance for inhomogeneous spacetime uniquely picks out the effective background FRW universe. In previous works, the effective scale factor was introduced through the uniform Hubble slicing condition [5,7] and the comoving volume was introduced from the first [6]. Anyway gauge invariance of the back reaction was not manifest.

For the dust dominated universe, we derived the renormalization group equation and its solution is obtained. The scalar mode and the long wavelength tensor mode inhomogeneity work as positive spatial curvature, and the short wavelength tensor mode works as radiation fluid. These results are consistent with previous works [6,7,11]. We also confirmed that these results are gauge independent by comparing the calculation using the comoving and the longitudinal gauge. Besides determining the dynamics of the effective background FRW universe, the renormalization group method gives the evolution of the fluctuation which includes the back reaction effect. As the background evolution is affected by the inhomogeneity, the evolution of the fluctuation is also modified. For the first order perturbation solution of the density contrast (38), we can obtain the renormalized evolution by replacing $a_0$ contained in the solution with the renormalized value $a_0(t)$. For the scalar mode perturbation, we have

$$\frac{(1)}{(0)} \frac{\rho}{\rho} = -\frac{t^{2/3}}{a_0^2(t)} \nabla^2 \Psi \propto \frac{t^{2/3}}{c - \frac{10}{27} t^{2/3} \sum_k k^2 \Psi_k \Psi_k^*}. \quad (83)$$

This indicates that the back reaction effect of the inhomogeneity enhances the growth rate of the density contrast.

Including the scalar field is an interesting task to investigate the back reaction problem in the inflationary cosmology. This was partly done by Abramo et al. [8–10] using the effective energy momentum tensor, but they do not derived explicit form of the evolution equation of the effective scale factor and the scalar field. We will treat this subject in a separate publication.

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