Can We See a Rotating Gravitational Lens?

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It is known that the rotation of a gravitational lens affects properties of images. We consider an inverse problem: If the lens is dark, can we infer its rotation from the observed images? We find that, up to the first order in the gravitational constant $G$, a rotating lens is not distinguishable from a non-rotating one. The quadrupole contribution to the deflection angle is also calculated for a general, extended lens object.

\section*{§1. Introduction}

The gravitational lens is an important phenomenon in astrophysics for probing mass distributions.\textsuperscript{1) Most studies of gravitational lensing have assumed that the lens object is non-rotating. However, celestial objects such as stars and galaxies are usually rotating. Hence, it is important to consider the prospect of probing the rotation through the gravitational lensing. In particular, recent rapid advances in observations enable us to see the light passing in the vicinity of neutron stars, black holes, and so on. For this reason, we may obtain a great deal of useful information regarding lens objects.

As for the sun, Epstein and Shapiro estimated the deflection angle due to its spin in the context of the parameterized post-Newtonian (PPN) approximation of gravitational theories.\textsuperscript{2) For more general lens models, observable effects due to the rotation have been discussed by several authors.\textsuperscript{3-5) On the other hand, in the study of the lensing of gravitational waves, Baraldo et al. have found that an interference pattern is shifted translationally in proportional to the Kerr parameter without any change in the shape of the pattern.\textsuperscript{6) We consider an inverse problem: If the lens is dark, can we infer its rotation from the observed images?

The conventional derivation of the lens equation is based on the non-rotating assumption. Hence, in order to clarify the effect due to rotation, we first summarize the general formalism of the gravitational lens in Section 2. We also obtain the lens equation for a rotating extended object that produces a weak gravitational field along the line of sight. In Section 3, we use the multipole expansion in order to take account of the effect due to the extended nature of lens. In particular, the quadrupole contribution, which does not appear in the linearized Kerr metric, is also studied. Section 4 is devoted to a summary and conclusion.

\section*{§2. General formulation of the gravitational lens}

First, we summarize the notation and give the equations for gravitational lensing. We basically employ the notation of Ref.\textsuperscript{1), but the signature is $(-, +, +, +)$.}
2.1. The 3 + 1 splitting of stationary spacetime

The metric of stationary spacetime can be written in the form

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -h \left( cdt - w_i dx^i \right)^2 + h^{-1} \gamma_{ij} dx^i dx^j, \]  

where

\[ h \equiv -g_{00}, \quad w_i \equiv -\frac{g_{0i}}{g_{00}}, \]  

and

\[ \gamma_{ij} dx^i dx^j = -g_{00} \left( g_{ij} - \frac{g_{0i} g_{0j}}{g_{00}} \right) dx^i dx^j \equiv d\ell^2. \]  

This is essentially the same as the Landau-Lifshitz 3 + 1 decomposition of stationary spacetime.\(^7\) The only difference here is in the definition of the spatial metric. In the Landau-Lifshitz case, this is

\[ \tilde{\gamma}_{ij} \equiv \left( g_{ij} - \frac{g_{0i} g_{0j}}{g_{00}} \right) = h^{-1} \gamma_{ij}. \]  

We will hereafter use a conformally rescaled \( \gamma_{ij} \) for the following reason: The spatial distance \( d\ell \) defined by Eq. (2.3) behaves as the affine parameter of the null geodesics in this spacetime, as shown in the Appendix. Therefore, it is convenient to use \( \gamma_{ij} \) when investigating light ray propagation. The conformal factor \( h \) corresponds to the gravitational redshift factor.

2.2. Fermat’s principle in stationary spacetime

For a future-directed light ray, the null condition \( ds^2 = 0 \) gives

\[ c dt = \frac{1}{h} \sqrt{\gamma_{ij} dx^i dx^j + w_i dx^i}. \]  

Since the spacetime is stationary, \( h, \gamma_{ij} \) and \( w_i \) are functions of the spatial coordinates \( x_i \) only. Then, the arrival time of a light ray is given by the integral of Eq. (2.5) from the source to the observer (denoted by the subscripts \( S \) and \( O \), respectively):

\[ t \equiv \int_{t_S}^{t_O} dt = \frac{1}{c} \int_S^O \left( \frac{1}{h} \sqrt{\gamma_{ij} e^i e^j + w_i e^i} \right) d\ell. \]  

Here \( e^i = dx^i/d\ell \) is the unit tangent vector, which represents the direction of the light ray. Hereafter, lowering and raising the indices of the spatial vectors are accomplished by \( \gamma_{ij} \) and its inverse \( \gamma^{ij} \). We note here that the quantity

\[ n \equiv \frac{1}{h} \sqrt{\gamma_{ij} e^i e^j + w_i e^i} \]  

acts as an effective refraction index.

Now Fermat’s principle\(^8\) states that \( \delta t = 0 \). From the Euler-Lagrange equation, we obtain the equation for the light rays as follows:

\[ \frac{de^i}{d\ell} = -\left( \gamma^{ij} - e^i e^j \right) \partial_j \ln h - \gamma^{il} \left( \gamma_{lj,k} - \frac{1}{2} \gamma_{jk,l} \right) e^j e^k + h \gamma^{ij} (w_{k,j} - w_{j,k}) e^k. \]  

This “equation of motion” for light rays is valid in any stationary spacetime.
2.3. Post-Minkowskian metric

To this point, the treatment is fully exact. Hereafter, we use an approximate metric to represent the mildly inhomogeneous spacetime in which the gravitational lensing events take place. We assume that the perturbed spacetime is approximately described by the following post-Minkowskian metric:

\[
\begin{align*}
\text{ds}^2 &= - \left( 1 + \frac{2\phi}{c^2} \right) c^2 dt^2 + 2 c dt \frac{\psi_i dx^i}{c^3} + \left( 1 - \frac{2\phi}{c^2} \right) \delta_{ij} dx^i dx^j. \\
\end{align*}
\] (2.9)

The energy-momentum tensor for a slowly moving, perfect fluid source is

\[
\begin{align*}
T_{00} &\simeq \rho c^2, \\
T_{0i} &\simeq \rho c v_i, \\
T_{ij} &\simeq \rho v_i v_j + p \delta_{ij}.
\end{align*}
\] (2.10)

In the near zone of a slowly moving, extended body, the metric is obtained up to the first-order in the gravitational constant \( G \) from the integrals

\[
\phi = -G \int \frac{\rho(r')}{|r - r'|} d^3 x', \quad (2.11)
\]

\[
\psi_i = -4G \int \frac{\rho v_i(r')}{|r - r'|} d^3 x'. \quad (2.12)
\]

Then, in the usual vector notation, the propagation equation (2.8) becomes

\[
\frac{de}{d\ell} = -\frac{2}{c^2} \left( \nabla - e (e \cdot \nabla) \right) \phi + \frac{1}{c^3} e \times (\nabla \times \psi). \quad (2.13)
\]

2.4. The deflection angle

The deflection angle \( \alpha \) is defined as the difference in the ray directions at the source and the observer, \( \alpha \equiv e_S - e_O \). Using Eq. (2.13), we obtain

\[
\alpha = -\int_S^O de = \int_S^O \left( \frac{2}{c^2} \left( \nabla - e (e \cdot \nabla) \right) \phi - \frac{1}{c^3} e \times (\nabla \times \psi) \right) d\ell. \quad (2.14)
\]

2.5. The lens equation

The lens equation relates the image position \( \xi \) to the source position \( \eta \) as

\[
\eta = \frac{D_{OS}}{D_{OL}} \xi - D_{LS} \alpha(\xi), \quad (2.15)
\]

where \( D_{OS} \) is the distance from the observer to the source, \( D_{OL} \) is that from the observer to the lens, and \( D_{LS} \) is that from the lens to the source. The vectors \( \xi, \eta \) and \( \alpha \) are 2-dimensional vector in the sense that they are orthogonal to the ray direction \( e \) within our approximation. Alternatively, the lens equation is often written in terms of the angular position vectors as

\[
\beta = \theta - \frac{D_{LS}}{D_{OS}} \alpha(D_{OL} \theta), \quad (2.16)
\]

where \( \beta = \eta/D_{OS} \) is the unlensed position angle of the source and \( \theta = \xi/D_{OL} \) is the angular position of the image. In a cosmological situation, the unlensed position \( \beta \) (hence \( \eta \)) is not an observable, because we cannot remove the lens from the observed position.
§3. Multipole expansion analysis

The zeroth-order solution of the propagation equation (2.13) is \( e \equiv \bar{e} = \text{const.} \). Therefore, within our approximation, the integration in Eq. (2.14) is taken over the unperturbed ray, \( x(\ell) = \xi + \ell \bar{e} \). Then Eq. (2.14) can also be written as

\[
\alpha = \frac{2}{c^2} \int_S \nabla \left( \phi - \frac{1}{2c} \bar{e} \cdot \psi \right) d\ell - \int_S (\bar{e} \cdot \nabla) \left( \frac{2}{c^2} \bar{e} \phi - \frac{1}{c^3} \psi \right) d\ell. \tag{3.17}
\]

The second part of the right-hand side of Eq. (3.17) is simply written in terms of the boundary values. Its contribution is negligible in asymptotically flat regions. Consequently, the frame-dragging effect of a rotating lens can be expressed as the change of the Newton potential term in the following way:

\[
\phi \rightarrow \tilde{\phi} = \phi - \frac{1}{2c} \bar{e} \cdot \psi. \tag{3.18}
\]

From Eqs. (2.11) and (2.12), we also have

\[
\tilde{\phi} = -G \int \frac{\tilde{\rho}(r')}{|r - r'|} d^3 x', \tag{3.19}
\]

where

\[
\tilde{\rho} \equiv \rho - \frac{2\rho v \cdot \bar{e}}{c} \tag{3.20}
\]

is the “effective” density for a rotating lens. Therefore, a rotating gravitational lens with density \( \rho \) is equivalent to a non-rotating one with effective density \( \tilde{\rho} \).

Under the assumption \(|r| > |r'|\), the integral representation of the potential Eq. (2.11) is expanded as follows:

\[
\phi = -\frac{GM}{r} - \frac{GM \cdot R}{r^3} - \frac{G}{2} \left( \frac{x_i x_j}{r^5} - \frac{\delta_{ij}}{r^3} \right) I_{ij} + \cdots, \tag{3.21}
\]

where

\[
M = \int \rho d^3 x, \quad R = \frac{1}{M} \int \rho r d^3 x, \quad I_{ij} = \int \rho x_i x_j d^3 x. \tag{3.22}
\]

The stationarity of the spacetime now requires that \( M, R, I_{ij}, \cdots \) are all independent of \( t \). With the help of the continuity equation, \( \partial \rho / \partial t + (\rho v^i)_i = 0 \), we obtain

\[
\int \rho v_i d^3 x = 0, \quad \int \rho v_{(i} x_{j)} d^3 x = 0, \quad \cdots. \tag{3.23}
\]

Therefore, Eq. (2.12) is expanded as

\[
\psi_i = \frac{4Gx^j}{r^3} \int \rho x_{[i} v_{j]} d^3 x - 2G \left( \frac{x^j x^k}{r^5} - \frac{\delta_{jk}}{r^3} \right) \int \rho v_{k}, x_i d^3 x + \cdots. \tag{3.24}
\]

In the above equations, the round brackets denote symmetrization and the square brackets denote skew-symmetrization. Finally, \( \tilde{\phi} \) is expanded as

\[
\tilde{\phi} = \tilde{\phi}_0 + \tilde{\phi}_1 + \tilde{\phi}_2 + \cdots, \tag{3.25}
\]

where \( \tilde{\phi}_0, \tilde{\phi}_1 \) and \( \tilde{\phi}_2 \) represent the monopole, dipole and quadrupole components, respectively.
3.1. **Monopole and dipole contributions**

In the case of a compact rotating lens, calculation up to order $1/r^2$ is sufficient. Then, we find

$$\tilde{\phi}_0 + \tilde{\phi}_1 = -\frac{GM}{r} - \frac{GM}{r^3} \cdot \left( R - \bar{e} \times \frac{L}{Mc} \right),$$  \hspace{1cm} (3.26)

where $L = \int \rho \bar{e} \times \mathbf{v} \, d^3x$ is the angular momentum of the lens object. The dipole term can always be removed with a constant translation of the coordinate system:

$$r \rightarrow r \left( R - \bar{e} \times \frac{L}{Mc} \right) = r - \tilde{R}. \hspace{1cm} (3.27)$$

Therefore, we have an important conclusion: A rotating lens with center of mass position $R$ is equivalent to a non-rotating lens with center of mass position $\tilde{R}$. The frame-dragging term due to the rotation of the lens object produces no additional observable effects, such as image deformation, increase of the separation angle, or magnification of the images. It simply results in a constant shift of the coordinate system. It should be noted that this result is independent of the assumption of a perfect fluid, since the expansion of potentials can always be written as Eq. (3.26) with a suitable definition of the mass and the angular momentum.

As an alternative statement of our conclusion, we find that the frame-dragging effect is separable only when the “true” position of the center of mass $R$ is independently known. In the following, we choose the origin of the coordinates such that the dipole contribution vanishes: $\tilde{\phi}_1 = 0$.

3.2. **Quadrupole contributions**

For extended lenses, quadrupole contributions should also be considered. Equations (3.21) and (3.24) give

$$\tilde{\phi}_2 = -\frac{G}{2} \left( \frac{x^i x^j}{r^5} - \frac{\delta^{ij}}{r^3} \right) \left( I_{ij} - \frac{2}{c} \int \rho \bar{e} \cdot \mathbf{v} x_i x_j \, d^3x \right)$$

$$\equiv -\frac{G}{2} \left[ \frac{x^i x^j}{r^5} - \frac{\delta^{ij}}{r^3} \right] \tilde{I}_{ij}. \hspace{1cm} (3.28)$$

Therefore, even if the “true” quadrupole moment of the mass distribution vanishes, $I_{ij} = 0$, it is possible to have quadrupole contributions due to the effect of the dragging of inertia. It should be noted that to the first order in $G$, the quadrupole moments of the Kerr lens vanish.

3.3. **Multipole expansion of the deflection angle**

The deflection angle is given by

$$\alpha = \frac{2}{c^2} \int_{-\infty}^{+\infty} \nabla \left( \tilde{\phi}_0 + \tilde{\phi}_2 + \cdots \right) \, d\ell, \hspace{1cm} (3.29)$$

where the integration is taken along the path $\mathbf{x} = \xi + \ell \hat{e}$. For convenience, we take the direction of $\hat{e}$ as the $z$-axis, $\hat{e} = (0,0,1)$ and $\xi = (\xi^1, \xi^2, 0)$. Then, straightforward
calculations show that $\alpha = \{\alpha^i\} = (\alpha^1, \alpha^2, 0)$ is

$$
\alpha^i = \frac{4GM}{c^2} \frac{\xi^i}{|\xi|^2} - \frac{4G}{c^2} \sum_{j,k=1}^2 \tilde{I}_{jk} \left( \delta^{jk} - \frac{4 \xi^j \xi^k}{|\xi|^2} \right) \frac{\xi^i}{|\xi|^4} - \frac{8G}{c^2} \sum_{j=1}^2 \tilde{I}_{ij} \frac{\xi^j}{|\xi|^4},
$$

(3.30)

Note that the $z$ components of the quadrupole moment $\tilde{I}_{3i}$ do not appear in the above equations.

§4. Summary and discussion

We have studied the following inverse problem: If a lens is dark, can we infer its rotation from the observed images? We have found that a rotating lens cannot be distinguished from a non-rotating one unless the true mass density is known independently.

In order to consider explicitly the effect due to the extended nature of the lens, we have used a multipole expansion analysis. At the dipole order, we have found that the frame-dragging term due to the rotation of the lens object produces no additional observable effects. It simply results in a constant shift of the coordinate system. Alternatively, the frame-dragging effect is separable only when the “true” position of the center of mass is independently known. The quadrupole order, which does not appear in the linearized Kerr metric, has also been studied. We have shown that the rotation of the lens induces a quadrupole effect in the gravitational lensing which results in an increase in the number of images. Even if the “true” quadrupole moment of the mass distribution vanishes, quadrupole contributions appear owing to the effect of dragging of inertia. However, we cannot recognize these additional phenomena as products of the rotation unless we know the true mass distribution through observational methods other than gravitational lensing.

The sun is an exceptional case for which we do know the true mass distribution. In this case, the light deflection caused by the rotation, quadrupole moment and the second order of the mass are, respectively, about 0.7, 0.2 and 11 $\mu$ arcsec.$^2$.

In this paper, we have not considered the polarization of the light. In Kerr spacetime, the direction of polarization rotates due to the frame-dragging effect.$^{11,12}$ In order to make the effect of the rotation separable, we need additional information such as the position of the lens or the direction of polarization. Our conclusion is valid only to linear order in the gravitational constant. Therefore, it would be interesting to study effects at higher orders, though they are negligible in the astronomical situation.

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Appendix A

Here we prove that the spatial length $\ell$ defined by Eq. (2.3) is an affine parameter of the null geodesics in the stationary spacetime given by Eq. (2.1).

Let us introduce the null geodesic parameterized by the affine parameter $\lambda$. The tangent vector of the null ray is

$$k^\mu = \frac{dx^\mu}{d\lambda}.$$  \hfill (A.1) 

The geodesic equation for $k_\mu$ is

$$\frac{dk_\mu}{d\lambda} = \frac{1}{2} g_{\alpha\beta,\mu} k^\alpha k^\beta.$$ \hfill (A.2) 

Since the spacetime is stationary, the metric does not depend on the time coordinate $x^0$. Therefore, $k_0$ is constant along the geodesic: $dk_0/d\lambda = 0$. The null condition $k^\mu k_\mu = 0$ reads

$$(k_0)^2 = \gamma_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = \left( \frac{d\ell}{d\lambda} \right)^2.$$ \hfill (A.3) 

Hence,

$$\frac{d}{d\lambda} \left( \frac{d\ell}{d\lambda} \right) = 0,$$ \hfill (A.4) 

which implies that $\ell$ is another affine parameter.

References

8) V. Perlick, Class. Quant. Grav. 7 (1990), 1319.