A higher harmonic cavity to increase the bunch length in LEP-70

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1. Introduction

A higher harmonic cavity is considered here for LEP, to increase the bunch length. With such a cavity the slope of the total RF voltage seen by the beam can be made small or even zero, the bunch length thereby being increased and the higher mode losses reduced. The cavity also provides Landau damping against coupled bunch mode instabilities¹,²,³,⁴,⁵).

2. Theory

We assume a main RF system with amplitude \( V_o \), frequency \( \omega_{RF} \) and synchronous phase angle \( \phi_s \) and in addition a higher frequency system with amplitude \( V_1 = k V_o \), frequency \( n \omega_{RF} \) and synchronous phase angle \( \phi_n \) (measured with respect to \( \phi_{RF} \)). The meaning of these parameters is illustrated in Fig. 1. Using \( \phi \) for the phase angle measured from the operating point

\[
\phi = \phi_{RF} - \phi_s = \omega_{RF} t - \phi_s
\]

we get for the voltage \( V(\phi) \) seen by the particles in the beam

\[
V(\phi) = V_o \left[ \sin(\phi + \phi_s) + k \sin(n(\phi + \phi_n)) \right]
\]  

(1)

The voltage \( U \) applied to the bunch has to replace the synchrotron radiation loss \( eU_r \) and the higher mode losses \( eU_{h\ell} \)

\[
U = U_r + U_{h\ell} = V_o \left[ \sin(\phi_s) + k \sin(n \phi) \right]
\]

(2)
To obtain a large increase of the bunch length we make the slope of the RF wave form zero at the bunch

$$\frac{dV}{d\Phi}(0) = V_0 \left[ \cos \Phi_s + nk \cos(n \Phi_n) \right] = 0$$

(3)

Furthermore we would like to avoid the wave form having a maximum or minimum at the bunch. This could form a small bucket inside the normal bucket. Although this would probably do no harm regarding the operation of the cavity, it makes the analysis more complicated. To avoid this we demand

$$\frac{d^2V}{d\Phi^2}(0) = -V_0 \left[ \sin \Phi_s + n^2 k \sin(n \Phi_n) \right] = 0$$

From this and (3) we get the two conditions

$$nk \cos(n \Phi_n) = -\cos \Phi_s$$

$$n^2 k \sin(n \Phi_n) = -\sin \Phi_s$$

(4)

which lead to

$$k = \frac{1}{n} \sqrt{\cos \Phi_s + \frac{1}{n^2} \sin^2 \Phi_s} > 0$$

$$\Phi_n = \frac{1}{n} \tan^{-1} \left( \frac{1}{n} \tan \Phi_s \right)$$

With the condition $k > 0$ we find

$$\pi < n \Phi_n < 2\pi$$

With the conditions (4) we obtain for the RF wave form

$$V(\Phi) = V_0 \left[ \sin \Phi_s \left( \sin \Phi - \frac{1}{n} \sin(n \Phi) \right) + \sin \Phi_s \left( \cos \Phi_s - \frac{1}{n^2} \cos(n \Phi_n) \right) \right]$$

(5a)

and

$$U = V_0 \left( 1 - \frac{1}{n^2} \right) \sin \Phi_s$$

(5b)
The equation of motion is

$$\frac{d^2 \Phi}{dt^2} = -\frac{\omega_c^2 \eta e}{2\pi \beta^2 E} (V(\Phi) - U)$$  \hspace{1cm} (6)$$

with $\omega_c$ = revolution frequency,

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_T^2},$$

$E$ = synchronous energy.

Equation (6) can be integrated once with the result

$$\frac{\dot{\Phi}^2}{\Omega_o^2} + 2 Y^2(\Phi, \Phi_s) = \frac{\dot{\Phi}_0^2}{\Omega_o^2}$$  \hspace{1cm} (7)$$

with

$$Y^2(\Phi, \Phi_s) = \frac{1}{V_o} \int_0^\Phi (V(\Phi) - U) d\Phi$$

$$= \left(1 - \frac{1}{\eta^2}\right) \cos \Phi_s - \Phi \sin \Phi_s - \cos \Phi \left(\cos \Phi - \frac{1}{\eta^2} \cos(n\Phi)\right)$$

$$+ \frac{1}{\eta^2} \sin \Phi_s \left(\sin \Phi - \frac{1}{\eta^2} \sin(n\Phi)\right)$$  \hspace{1cm} (8)$$

and

$$\Omega_o^2 = \frac{\omega_c^2 \eta e V_o}{2\pi \beta^2 E}.$$

The quantity $\Omega_o$ is the phase oscillation frequency in the absence of a higher harmonic system and for $\sin \Phi_s = 0$. We can replace the variable $\dot{\Phi}$ by the momentum deviation $\Delta p/p$

$$\frac{\Delta p}{p} = \frac{1}{\eta \omega_c} \dot{\Phi} = \sqrt{\frac{e V_o}{2\pi \beta^2 E \eta \Omega_o^2}}$$

and get for equation (7)

$$\left(\frac{\Delta p}{p}\right)^2 + \frac{e V_o Y(\Phi, \Phi_s)}{\eta \pi \beta^2 E} = \left(\frac{\Delta p}{p}\right)_o^2$$

The acceptance in $\dot{\Phi}$ or $\Delta p$ of the bucket is given by the maximum $Y^2(\Phi, \phi)$ of the function $Y(\Phi, \phi)$. 

\[ Y_m^2(\hat{\phi}) = Y_m^2(\hat{\phi}_0, \hat{\phi}_s) \]

where \( \hat{\phi}_0 \) is the unstable fixed point which can be calculated

\[ V(\hat{\phi}_0) = 0 \quad (\hat{\phi}_0 \neq 0) \]

\[ \left( \frac{\Delta p}{P_m} \right) = \sqrt{\frac{e V_c}{h \eta \pi \beta^2 E}} Y_m^2(\hat{\phi}) \]

(9)

The Hamiltonian of the phase motion expressed in \( \phi \), \( \Delta p/p \) is

\[ H = h |\eta| \pi \beta^2 E \left[ \frac{e V_c}{h \eta \pi \beta^2 E} Y^2(\hat{\phi}, \hat{\phi}_s) + \left( \frac{\Delta p}{P} \right)^2 \right] \]

3. Approximation

For small amplitudes of oscillation we can approximate

\[ Y^2(\hat{\phi}, \hat{\phi}_s) \approx \frac{(n^2-1) \cos \hat{\phi}_s}{2 \gamma} \]

and get

\[ H \approx h |\eta| \pi \beta^2 E \left[ \frac{e V_c (n^2-1) \cos \hat{\phi}_s}{2 \gamma h \eta \pi \beta^2 E} \hat{\phi}^y + \left( \frac{\Delta p}{P} \right)^2 \right] \]

The particle trajectory is approximately

\[ \frac{\dot{\hat{\phi}}^2}{\Omega_s^2} + \frac{(n^2-1) \cos \hat{\phi}_s}{12} \hat{\phi}^y = \frac{\dot{\hat{\phi}}^2}{\Omega_s^2} = \frac{(n^2-1) \cos \hat{\phi}_s}{12} \hat{\phi}_0^y \]

or

\[ \left( \frac{\Delta p}{P} \right)^2 + \frac{e V_c (n^2-1) \cos \hat{\phi}_s}{2 \gamma h \eta \pi \beta^2 E} \hat{\phi}^y = \left( \frac{\Delta p}{P} \right)_0^2 \]

The phase oscillation frequency \( \Omega_s \) is in this approximation \(^3,4\)

\[ \Omega_s = \frac{\pi \mid \Omega_s \mid}{2 \sqrt{(n^2-1) \cos \hat{\phi}_s}} \hat{\phi}_0 \]

\[ = \frac{\pi \mid \Omega_s \mid}{2 \sqrt{(n^2-1) \cos \hat{\phi}_s}} \hat{\phi}^y \]

\[ = \frac{\pi \mid \Omega_s \mid}{2 \sqrt{(n^2-1) \cos \hat{\phi}_s}} \frac{\hat{\phi}^y}{\Omega_s^2} \]
where \( K(1/\sqrt{2}) = 1.8541 \) is the complete elliptic integral of modulus \( 1/\sqrt{2} \).

The phase motion (within this approximation) can be expressed by the Jakobian elliptic function \( cn \), of modulus \( 1/\sqrt{2} \) 6,7).

\[
\bar{\Phi} = \bar{\Phi}_0 \ cn(\Omega_K \ell)
\]

with

\[
\Omega_K = \frac{2 \ K(1/\sqrt{2}) \Omega_S}{\pi} = \sqrt{\frac{(n^2-1) \cos \bar{\Phi}_0 \Omega_0}{6} \ \bar{\Phi}_0}
\]

4. Distribution of the particles and bunch length

The distribution of the particles in phase space has the form 8,5)

\[
g(\bar{\Phi}, \Delta p/p) = \text{const.} \ \exp \left( -\frac{e V_0 \gamma^2(\bar{\Phi}_0, \bar{\Phi}_0)}{\Delta p/p} \right) \ \exp \left( -\frac{1}{2}(\Delta p/p)^2 \right)
\]

Here \( \sigma_p \) is the standard deviation of the momentum distribution which is still the same as for a normal RF system 8,9). With the approximation used in the last chapter we obtain

\[
g(\bar{\Phi}, \Delta p/p) = \frac{2 N_b \lambda}{\sqrt{2 \pi} \ Gamma(\gamma) (\sigma_p/p)^{3\gamma}} \ \exp \left( -\frac{\lambda^{\gamma} \bar{\Phi}^{\gamma}}{2(\sigma_p/p)^2} - \frac{1}{2}(\Delta p/p)^2 \right)
\]

with \( \Gamma(\frac{1}{2}) = 3.6256 \),

\[
\lambda^\gamma = \frac{e V_0 (n^2-1) \cos \bar{\Phi}_S}{2 \gamma h \gamma \pi \beta^2 E}
\]

\( N_b \) = number of particles per bunch.

Integrating over \( \Delta p/p \) gives the "line density" with respect to the phase angle \( \bar{\Phi} \)

\[
G(\bar{\Phi}) = \int_{-\infty}^{\infty} g(\bar{\Phi}, \Delta p/p) \ d(\Delta p) = \frac{2 N_b \lambda}{\sqrt{2 \pi} \ Gamma(\gamma) (\sigma_p/p)^{\gamma}} \ \exp \left( -\frac{\lambda^{\gamma} \bar{\Phi}^{\gamma}}{2(\sigma_p/p)^2} \right)
\]

From this we get the instantaneous beam current \( I(\bar{\Phi}) \)

\[
I(\bar{\Phi}) = I_0 \ \frac{2 \pi \ h \ \lambda}{M \ N_b} \ \frac{4 \pi^2 \ h \lambda}{\sqrt{2 \ Gamma(\gamma) M \ \sigma_p/p}} \ \exp \left( -\frac{\lambda^{\gamma} \bar{\Phi}^{\gamma}}{2(\sigma_p/p)^2} \right)
\]
where \( I_0 \) is the average current of all \( M \) bunches.

Since the bunch form (10) is not Gaussian we will often use the \( 1/e \)-half length \( \bar{\Phi} \) to characterize the bunch length

\[
\bar{\Phi} = \frac{\sqrt{2\pi} \sqrt{\Delta \rho / \rho}}{\lambda} = 2 \sqrt{\frac{\epsilon \hbar \pi \sigma^2 E}{eV_0 (n^2 - 1) \cos \delta}} \sqrt{\Delta \rho / \rho} \tag{11}
\]

or else the rms value

\[
\text{rms} (\bar{\Phi}) = \frac{\sqrt{2\pi} \sqrt{\Delta \rho / \rho}}{\Gamma (\frac{1}{4}) \lambda} = \frac{2\sqrt{\frac{2\pi}{\Gamma (\frac{1}{4})}}}{\frac{3}{2}} \sqrt{\frac{\hbar \pi \sigma^2 E}{eV_0 (n^2 - 1) \cos \delta}} \sqrt{\Delta \rho / \rho} \tag{12}
\]

The bunch forms for \( n = 3 \) and \( n = 3.5 \) as well as the natural bunch form for the normal RF system are shown in Fig. 3.

5. Choice of the higher harmonic frequency

The parameters of this double RF system were calculated for the 70 GeV LEP parameters (version 5) \(^{10} \). The higher mode losses are 221 MeV/turn for a bunch with \( C^2 = 2.5 \) cm (\( C^2 = 3.53 \) cm). To calculate these losses for the lengthened bunch it was assumed that they scale inversely proportional to the \( 1/e \)-bunch length. From the necessary voltage per turn \( U \) and the momentum acceptance \( (\Delta \rho / \rho)_m \), the voltages \( V_0 \) and \( V_n \), the phase angles \( \Phi_s \) and \( \Phi_n \), and the bunch length \( C^2 \) were calculated. In Fig. 2 \( V_0 \), \( V_n \) and \( C^2 \) are plotted for different frequency ratios \( n \). Low higher harmonic frequencies give a long bunch but demand high voltages. For higher frequencies the voltages become smaller but the bunch shorter and it becomes difficult to construct cavities with sufficient beam aperture. If \( n \) is not an integer or a half integer the higher harmonic cavity can excite coupled bunch mode instabilities. To avoid this and to have reasonable RF parameters and sufficient bunch length we concentrated on the cases \( n = 3 \) and \( n = 3.5 \). The parameters for these cases are shown in Table 1 and compared with the normal RF system.
Table 1

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<th>V_n</th>
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<th>\phi_n</th>
<th>\rho</th>
<th>U_h\ell</th>
<th>I_{peak}</th>
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<td>\degree</td>
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<td>cm</td>
<td>MeV/turn</td>
<td>A</td>
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<td>119.3</td>
<td>-</td>
<td>3.53*</td>
<td>221</td>
<td>920*</td>
</tr>
</tbody>
</table>

* Here an increase of the natural bunch length is already assumed. The natural bunch length is \( \rho = 1.77 \) cm and would give a larger higher mode loss \( U_{h\ell} \).

The two RF voltages and the total RF wave form, as well as the bucket and phase plane trajectories, are shown in Fig. 1.

6. Conclusions

The beam dynamics in the presence of a higher harmonic voltage has been treated here assuming "stiff" RF systems not influenced by the beam. Within this frame we found that a considerable increase of the bunch length can be obtained and the higher mode losses will be reduced. At the same time a large spread in phase oscillation frequencies is obtained, which provides Landau damping. The smaller value of \( Q_s \) can help to reduce the effect of synchro-betatron resonances; on the other hand the larger spread in \( Q_s \) could make it difficult to stay between such resonances.

The higher harmonic system, operated in a different way, can be used to go to higher energies \(^{11}\); a shorter bunch has to be accepted in this case. This possibility is under investigation \(^{11}\). The effect of beam loading and stability of the whole system \(^{11}\) is also being studied. In the higher harmonic cavity the fields induced decay before the arrival of the next bunch. This
single traversal beam loading and its effect on quantum lifetime is also under study.  

References


6) B. Zotter, private communication to be published soon.


8) H.G. Hereward, PEP Note 53 (1973 PEP Summer Study).


10) E. Keil, LEP-70/19.

11) J. Le Duff (to be published soon).

12) P.B. Wilson (to be published soon).
Fig. 1

$E = 70 \text{ GeV, } n = 3.5$

$\phi_n = 94^\circ$

$\phi_s = 115.5^\circ$
RF- and higher harmonic voltages and bunch length vs. h.h. frequency

$E=70 \text{ GeV}, \ f_{rf}=357 \text{ Mhz}$
Fig. 3

Bunch form

$I$ vs. $z$

- $\eta = 3$
- Normal RF

$I$ vs. $z$

- $\eta = 3.5$