Perturbation of junction condition and doubly gauge-invariant variables

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The junction condition across a singular surface in general relativity, formulated by Israel, has double covariance. In this paper, a general perturbation scheme of the junction condition around an arbitrary background is given in a doubly covariant way. After that, as an application of the general scheme, we consider perturbation of the junction condition around a background with the symmetry of a \((D-2)\)-dimensional constant curvature space, where \(D\) is the dimensionality of the spacetime. The perturbed junction condition is written in terms of doubly gauge-invariant variables only. Since the symmetric background includes cosmological solutions in the brane-world as a special case, the doubly gauge-invariant junction condition can be used as basic equations for perturbations in the brane-world cosmology.

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I. INTRODUCTION

Brane-world scenario proposed by Randall and Sundrum [1] has been attracting a lot of interests. In this scenario, our 4-dimensional universe is considered as a world volume of a 3-brane, or a timelike hypersurface, in 5-dimensional bulk spacetime. It was shown that, in a 5-dimensional anti de-Sitter background, gravity on the 3-brane can be effectively described by 4-dimensional Newton’s law if the tension of the 3-brane is fine-tuned. Since this scenario may be realistic and may give drastic changes to our understanding of 4-dimensional gravity, many works have been done from various points of view. For example, 4-dimensional effective Einstein equation on the 3-brane was derived [2]; gravitational perturbations in the Randall-Sundrum background were discussed [3–8]; black holes in the brane-world were discussed [9,10]; cosmology based on this scenario was also discussed [11–19]. Recently, formalisms to treat perturbations in the brane-world cosmology were proposed [20–24].

In most of works on the brane-world scenario, the junction condition formulated by Israel [25] is used to treat a singular surface, or the world volume of the 3-brane. The junction condition relates a jump of extrinsic curvature of the singular surface to a surface energy momentum tensor\(^1\). One of its advantages is the manifest double covariance of the formalism. Coordinates outside the singular surface and those intrinsic to the singular surface can be completely independent, and the formalism is covariant under coordinate transformations both outside and on the singular surface. The double covariance of the junction condition is actually convenient for the purpose of the analysis of the brane-world scenario: because of the double covariance, coordinates intrinsic to our universe can be disentangled from those in the higher dimensional spacetime and covariance in our universe will be manifestly realized.

However, as far as the author knows, there is no manifestly doubly covariant scheme of perturbation of the junction condition. Although perturbations of the junction condition around symmetric backgrounds were analyzed by some authors [27–29,21–24], their formalism is not manifestly doubly covariant: coordinates intrinsic to the singular surface are entangled with those outside.

The main purpose of this paper is to formulate a general perturbation scheme of the junction condition around an arbitrary background in a doubly covariant way. After that, as an application of the general scheme, we consider perturbation of the junction condition around a background with the symmetry of a \((D-2)\)-dimensional constant curvature space, where \(D\) is the dimensionality of the spacetime. The perturbed junction condition is written in terms of doubly gauge-invariant variables only. This result is equivalent to that in ref. [21]. Since the symmetric background includes cosmological solutions in the brane-world as a special case, the doubly gauge-invariant junction condition can be used as basic equations for perturbations in the brane-world cosmology.

In Sec. II the junction condition formulated by Israel is reviewed in a doubly covariant form. In Sec. III a general scheme of perturbation of the junction condition is given. In Sec. IV, as an application of the general scheme, we

\(^1\)An equivalent condition can be derived from an action principle with a delta function source. See, for example, [26].
Let us consider a $D$-dimensional spacetime $\mathcal{M}$ and a spacelike or timelike hypersurface $\Sigma$ which separates $\mathcal{M}$ into two regions $\mathcal{M}_+$ and $\mathcal{M}_-$. Suppose that in each region the embedding of the hypersurface $\Sigma$ is specified by the following parametric equations

$$\Sigma : x_\pm^M = Z_\pm^M (y),$$

where $\{x_\pm^M\}$ denote spacetime coordinates in the region $\mathcal{M}_\pm$, respectively, and $y$ denotes the set $\{y^\mu\}$ of $D - 1$ parameters corresponding to intrinsic coordinates of $\Sigma$. In order to make $\Sigma$ well-defined the two sets of functions $\{Z_+^M (y)\}$ and $\{Z_-^M (y)\}$ should transform as

$$Z_\pm^M (y) \rightarrow Z_\pm^M (y) = F_\pm^M (Z_\pm^M (y))$$

under the $D$-dimensional coordinate transformations

$$x_\pm^M \rightarrow x_\pm^M = F_\pm^M (x_\pm^M),$$

respectively. We shall call these coordinate transformations $D$-coordinate transformations. The $D$-coordinate transformations can be taken independently in the regions $\mathcal{M}_+$ and $\mathcal{M}_-$. Hereafter, in most cases, we shall omit the sign $\pm$ which distinguishes quantities in $\mathcal{M}_+$ and those in $\mathcal{M}_-$. The induced metric on $\Sigma$ is

$$q_{\mu\nu} \equiv e^M_{\mu} e^N_{\nu} g_{MN},$$

where $g_{MN}$ is the spacetime metric and $\{e^M_{\mu}\}$ are tangential vectors defined by $e^M_{\mu} \equiv \partial Z^M / \partial y^\mu$. Since $\partial_y Z^M (y) = 0$, we have the equation $[e^M_{\mu} , e^N_{\nu}]^M = 0$. The extrinsic curvature of $\Sigma$ is defined by

$$K_{\mu\nu} = \frac{1}{2} e^M_{\mu} e^N_{\nu} \mathcal{L}_n g_{MN},$$

where $\mathcal{L}$ represents the $D$-dimensional Lie derivative and $n$ is the unit normal of $\Sigma$ determined by

$$n_M e^M_{\mu} = 0, \quad n_M n^M = \epsilon,$$

where $\epsilon = 1$ for timelike $\Sigma$ and $\epsilon = -1$ for spacelike $\Sigma$. Here, we mention that $e^M_{\mu}$ transforms as a $D$-vector under the $D$-coordinate transformation (3) because of the transformation (2). Hence, the induced metric $q_{\mu\nu}$ and the extrinsic curvature $K_{\mu\nu}$ transform as $D$-scalars under the $D$-coordinate transformation (3). Therefore, they are invariant under the $D$-coordinate transformation, provided that they are considered as functions of $y$. On the other hand, they transform as $(D - 1)$-tensors under the reparameterization of $\Sigma$

$$y^\mu \rightarrow y'^\mu = f^\mu (y).$$

We shall call this reparameterization $(D - 1)$-coordinate transformation since $\{y^\mu\}$ can be considered as coordinates in the $(D - 1)$-dimensional manifold $\Sigma$. It seems worth while stressing that the $(D - 1)$-coordinate transformation is NOT a part of the $D$-coordinate transformation. They are completely independent.

Since the intrinsic geometry of $\Sigma$ should be regular, the induced metric calculated from one side and another should be identical:

$$q_{\mu\nu}^+ = q_{\mu\nu}^- \equiv q_{\mu\nu}.$$

Then, the junction condition formulated by Israel [25] relates the jump of the extrinsic curvature to the surface energy momentum tensor $S_{\mu\nu}$ associated with $\Sigma$ as

$$K_{\mu\nu}^+ - K_{\mu\nu}^- = -\kappa^2 \left( S_{\mu\nu} - \frac{1}{D - 2} S q_{\mu\nu} \right),$$

where $S = q^{\mu\nu} S_{\mu\nu}$, $q^{\mu\nu}$ is the inverse of the induced metric $q_{\mu\nu}$, and $\kappa^2$ is the $D$-dimensional gravitational coupling constant. Here, we have assumed that the unit normal $n_M$ is directed from $\mathcal{M}_-$ to $\mathcal{M}_+$. Note that the surface energy momentum tensor $S_{\mu\nu}$ is invariant under the $D$-coordinate transformation, while it transforms as a $(D - 1)$-tensor under the $(D - 1)$-coordinate transformation.
Now let us consider general perturbations of \( g_{MN} \) and \( Z^M(y) \) around an arbitrary background specified by \( g_{MN}^{(0)} \) and \( Z^M(y)^{(0)} \):  

\[
\begin{align*}
  g_{MN} &= g_{MN}^{(0)} + \delta g_{MN}, \\
  Z^M(y) &= Z^{(0)}(y) + \delta Z^M(y). 
\end{align*}
\]  

The unperturbed functions \( Z^M(y)^{(0)} \) determine the unperturbed hypersurface \( \Sigma^{(0)} \), while the perturbed functions \( Z^M(y) \) determine the perturbed hypersurface \( \Sigma \). Therefore, if we define \( e_\mu^{(0)M} \) by \( \partial \Sigma^{(0)} / \partial y^\mu \) then it is well-defined only on \( \Sigma^{(0)} \), while \( e_\mu^M \) is well-defined only on \( \Sigma \). Hence, in order to develop perturbation formalism, it seems convenient to extend those unperturbed tangent vectors \( e_\mu^{(0)M} \) off \( \Sigma^{(0)} \) so that both \( e_\mu^M \) and \( e_\mu^{(0)M} \) become well-defined on \( \Sigma \). The final expressions of the perturbed junction condition will be independent of the way of the extension. For the purpose of extension of \( e_\mu^{(0)M} \), we consider one-parameter family of hypersurfaces \( \Sigma^{(0)}_\varphi \):  

\[
\Sigma^{(0)}_\varphi : \ x^M = Z^{(0)M}(y). 
\]  

We assume that \( Z_0^{(0)M}(y) = Z^{(0)M}(y) \) so that \( \Sigma_0^{(0)} = \Sigma^{(0)} \). By using the one-parameter family of hypersurfaces, we can define a set of tangent vectors by  

\[
e_\mu^{(0)M} = \partial \Sigma^{(0)} / \partial y^\mu. 
\]  

Since \( \partial_\mu \partial_\nu Z^{(0)M}_\varphi(y) = 0 \), we have the equation \( [e_\mu^{(0)}(0), e_\nu^{(0)}(0)]^M = 0 \). With this definition, the perturbed tangent vectors are written as follows up to the linear order.  

\[
e_\mu^M = e_\mu^{(0)M} - L\delta Z e_\mu^{(0)M}. 
\]  

### A. Perturbations of induced metric and extrinsic curvature

Correspondingly, the induced metric of \( \Sigma \) can be written up to the linear order as  

\[
q_{\mu\nu} = \tilde{q}_{\mu\nu}^{(0)} + e_\mu^{(0)M} e_\nu^{(0)N} \delta g_{MN} - L\delta Z (e_\mu^{(0)M} e_\nu^{(0)N}) g_{MN}^{(0)}, 
\]  

where  

\[
\tilde{q}_{\mu\nu}^{(0)} = e_\mu^{(0)M} e_\nu^{(0)N} g_{MN}^{(0)}. 
\]  

However, \( \tilde{q}_{\mu\nu}^{(0)} \) in eq. (14) is not the unperturbed induced metric \( q_{\mu\nu}^{(0)}(y) = q_{\mu\nu}^{(0)}|_{x=Z^{(0)}(y)} \) on \( \Sigma^{(0)} \) since the former should be evaluated on \( \Sigma \) at \( x^M = Z^M(y) \). Rather, they are related as  

\[
\tilde{q}_{\mu\nu}^{(0)}|_{x=Z(y)} = q_{\mu\nu}^{(0)}(y) + \delta Z^M(y) \partial_M \tilde{q}_{\mu\nu}^{(0)}|_{x=Z^{(0)}(y)}, 
\]  

where we have used the fact that \( \tilde{q}_{\mu\nu}^{(0)} \) is a \( D \)-scalar and Taylor-expanded it. Therefore, eq. (14) can be written as \( q_{\mu\nu} = \tilde{q}_{\mu\nu}^{(0)} + \delta q_{\mu\nu} \), where  

\[
\delta q_{\mu\nu} = e_\mu^{(0)M} e_\nu^{(0)N} \delta g_{MN} + L\delta Z \tilde{g}_{MN}^{(0)}. 
\]  

Note that \( \delta q_{\mu\nu} \) may be evaluated on \( \Sigma^{(0)} \) since the difference from the value on \( \Sigma \) is the second order. The unit normal of \( \Sigma \) is written up to the linear order as \( n_M = n_M^{(0)} + \delta n_M \), where \( n_M^{(0)} \) is determined by  

\[
\begin{align*}
  e_\mu^{(0)M} n_M^{(0)} &= 0, \\
  n_M^{(0)} n_M^{(0)} &= 0, \\
  n_M^{(0)} n_M^{(0)} &= \epsilon, 
\end{align*}
\]
and $\delta n_M$ is given by

$e^{(0)M}_\mu \delta n_M = n^{(0)}_M \mathcal{L} Z e^{(0)M}_\mu,$

$n^{(0)M} \delta n_M = \frac{1}{2} n^{(0)M} n^{(0)N} \delta g_{MN},$  \hspace{1cm} (19)

or

$\delta n^M = \frac{\epsilon}{2} n^{(0)M} n^{(0)N} L \delta g_{NL} + e^{(0)M}_\mu q^{(0)\mu\nu} n^{(0)}_N \mathcal{L} Z e^{(0)N}_\nu.$  \hspace{1cm} (20)

Here, $n^{(0)M} = g^{(0)MN} n^N_M$, $\delta n^M = g^{(0)MN} \delta n_N$ and $q^{(0)\mu\nu}$ is the inverse of $g^{(0)\mu\nu}$. The extrinsic curvature of $\Sigma$ is

$K_{\mu\nu} = \tilde{K}_{\mu\nu}^{(0)} + \frac{1}{2} e^{(0)M}_\mu e^{(0)N}_\nu (L \delta n^M g^{(0)}_{MN} - 2 n^{(0)N} \delta \Gamma_{LMN}) - \frac{1}{2} \mathcal{L} Z e^{(0)M} e^{(0)N} \mathcal{L} n^{(0)M} g^{(0)}_{MN}$  \hspace{1cm} (21)

up to the linear order, where

$\tilde{K}_{\mu\nu}^{(0)} = \frac{1}{2} e^{(0)M}_\mu e^{(0)N}_\nu L n^{(0)M} g^{(0)}_{MN},$

$\delta \Gamma_{LMN} = \frac{1}{2} (\delta g_{LMN} + \delta g_{LN,M} - \delta g_{MN,L}),$  \hspace{1cm} (22)

and the semicolon denotes the $D$-dimensional covariant derivative compatible with the unperturbed metric $g^{(0)MN}$. The quantity $\tilde{K}_{\mu\nu}^{(0)}$ in eq. (21) should be evaluated on $\Sigma$ at $x^M = Z^M(y)$ and is related to the unperturbed extrinsic curvature $K_{\mu\nu}^{(0)}(y) = \tilde{K}_{\mu\nu}^{(0)}|_{x = Z(y)}$ of $\Sigma^{(0)}$ as follows.

$\tilde{K}_{\mu\nu}^{(0)}|_{x = Z(y)} = K_{\mu\nu}^{(0)}(y) + \delta Z^M(y) \partial_M \tilde{K}_{\mu\nu}^{(0)}|_{x = Z(y)}.$  \hspace{1cm} (23)

where we have used the fact that $\tilde{K}_{\mu\nu}^{(0)}$ is a $D$-scalar and Taylor-expanded it. Therefore, eq. (21) can be rewritten as

$K_{\mu\nu} = K_{\mu\nu}^{(0)} + \delta K_{\mu\nu},$  \hspace{1cm} (24)

Note that $\delta K_{\mu\nu}$ may be evaluated on $\Sigma^{(0)}$ since the difference from the value on $\Sigma$ is the second order.

From the expression (24) we see that it is not necessary to extend $\delta Z^M$ off $\Sigma^{(0)}$. Nonetheless, for explicit calculation of $\delta K_{\mu\nu}$, it turns out to be convenient to rewrite the expression into another form by extending $\delta Z^M$ off $\Sigma^{(0)}$ as a $D$-vector. Off course, the value of $\delta K_{\mu\nu}$ does not depend on the way of extension off $\Sigma^{(0)}$. The following is an alternative expression of $\delta K_{\mu\nu}$.

$\delta K_{\mu\nu} = \frac{\epsilon}{2} n^{(0)M} n^{(0)N} (\delta g_{MN} + 2 \delta Z_{MN}) K_{\mu\nu}^{(0)}$

$- \frac{1}{2} n^{(0)M} e^{(0)M}_\mu e^{(0)N}_\nu \left[ 2 \delta \Gamma_{LMN} + \delta Z_{LMN} + \delta Z_{LNM} + (R_{LMN}^{(0)} + R_{LNM}^{(0)}) \delta Z_{MN} \right],$  \hspace{1cm} (25)

where $\delta Z_{MN} = g_{MN}^{(0)} \delta Z^{(0)},$ and $R_{LMN}^{(0)}$ is the Riemann tensor of the unperturbed metric $g^{(0)MN}$. To obtain this expression, we have used the following identity for an arbitrary $D$-vector $V^M$.

$e^{(0)M}_\mu e^{(0)N}_\nu \left[ \mathcal{L}_V L_{n^M g^{(0)}_{MN}} - \mathcal{L}_{V[N] g^{(0)}_{MN}} + n^{(0)L} V_{L:MN} + V_{L:NM} + R_{LMN} V_L^{(0)} + R_{LNM} V^L \right] = 0,$  \hspace{1cm} (26)

where $V_M = g^{(0)MN} V^N$ and

$W^{LM} V = e^{(0)M}_\mu n^{(0)N}_\mu n^{(0)L}_N V_{N:L} - e^{(0)M}_\mu q^{(0)\mu\nu} n^{(0)}_N \mathcal{L} V e^{(0)N}_\nu.$  \hspace{1cm} (27)

The reason why we had to extend $\delta Z^M$ off $\Sigma^{(0)}$ is only because we had to make covariant derivatives of $\delta Z^M$ well-defined on $\Sigma^{(0)}$ in this expression. The value of $\delta K_{\mu\nu}$ is, off course, independent of the way of extension off $\Sigma^{(0)}$, since this expression is equivalent to the previous expression (24) and no assumptions with respect to properties of the extension have been needed. Off course, the expression (25) holds for any choices of the one-parameter family of hypersurfaces (11), since no assumptions have been needed with respect to the one-parameter family of hypersurfaces so far.

If we assume some properties with respect to the one-parameter family of hypersurfaces (11), we can obtain other expressions of $\delta K_{\mu\nu}$. One of them is given in appendix A, although we will not use it in the main body of this paper.
Now let us investigate gauge transformation of $\delta q_{\mu \nu}$ and $\delta K_{\mu \nu}$. We have two kinds of gauge transformations. The first one is the infinitesimal version of the $D$-coordinate transformation (3),

$$x^M \rightarrow x'^M = x^M + \xi^M(x).$$

(28)

We call it $D$-gauge transformation. Under the $D$-gauge transformation, $\delta g_{MN}$ and $\delta Z^M$ transform as

$$\delta g_{MN} \rightarrow \delta g_{MN} - \xi_{M;N} - \xi_{N;M},$$

$$\delta Z^M \rightarrow \delta Z^M + \tilde{\xi}^M.$$  

(29)

Hence, by using the expressions (17) and (25), it is easy to show that $\delta q_{\mu \nu}$ and $\delta K_{\mu \nu}$ are invariant under the $D$-gauge transformation. Of course, the $D$-gauge transformation can be taken independently in the regions $\mathcal{M}_+$ and $\mathcal{M}_-$. The second kind of gauge transformation is the infinitesimal version of the $(D-1)$-coordinate transformation (7),

$$y^\mu \rightarrow y'^\mu = y^\mu + \xi^\mu(y).$$

(30)

We call it $(D-1)$-gauge transformation. Under the $(D-1)$-gauge transformation,

$$\delta g_{MN}|_{x=Z^\mu(y)} \rightarrow \delta g_{MN}|_{x=Z'^\mu(y)},$$

$$\delta Z^M \rightarrow \delta Z^M - \tilde{\xi}^M.$$  

(31)

Hence, by using the expressions (17) and (24), it is easy to show that $\delta q_{\mu \nu}$ and $\delta K_{\mu \nu}$ transform as follows under the $(D-1)$-gauge transformation.

$$\delta q_{\mu \nu} \rightarrow \delta q_{\mu \nu} - \bar{L}\kappa q_{\mu \nu}^{(0)},$$

$$\delta K_{\mu \nu} \rightarrow \delta K_{\mu \nu} - \bar{L}\kappa K_{\mu \nu}^{(0)},$$

(32)

where $\bar{L}$ denotes the Lie derivative defined in the $(D-1)$-dimensional manifold $\Sigma^{(0)}$, and we have used the identity $[e^{(0)}_\mu, e^{(0)}_\nu]^M = 0$. Therefore, as expected, $\delta q_{\mu \nu}$ and $\delta K_{\mu \nu}$ actually transform as perturbations of $(D-1)$-tensors under the $(D-1)$-gauge transformation.

C. Junction condition

Finally, by decomposing the surface energy momentum $S_{\mu \nu}$ into the unperturbed part $S_{\mu \nu}^{(0)}$ and the perturbation $\delta S_{\mu \nu}$, we obtain the following junction condition.

$$q_{\mu \nu}^{(0)} = q_{\mu \nu}^{(0)},$$

$$K_{\mu \nu}^{(0)} - K_{\mu \nu}^{(0)} = -\kappa^2 \left( S_{\mu \nu}^{(0)} - \frac{1}{D-2} S^{(0)\mu \nu} \delta q_{\mu \nu}^{(0)} \right).$$

(33)

for the background and

$$\delta q_{\mu \nu}^{(0)} = \delta q_{\mu \nu}^{(0)},$$

$$\delta K_{\mu \nu}^{(0)} - \delta K_{\mu \nu}^{(0)} = -\kappa^2 \left( \delta S_{\mu \nu}^{(0)} - \frac{1}{D-2} \delta S^{(0)\mu \nu} \delta q_{\mu \nu}^{(0)} \right)$$

(34)

for the perturbation, where

$$\delta \tilde{K}_{\mu \nu} = \delta K_{\mu \nu} - \frac{1}{2} (K_{\mu \nu}^{(0)\rho} \delta q_{\rho \nu}^{(0)} + K_{\nu \mu}^{(0)\rho} \delta q_{\rho \mu}^{(0)}),$$

$$\delta \tilde{S}_{\mu \nu} = \delta S_{\mu \nu} - \frac{1}{2} (S_{\mu \nu}^{(0)\rho} \delta q_{\rho \nu}^{(0)} + S_{\nu \mu}^{(0)\rho} \delta q_{\rho \mu}^{(0)}),$$

(35)

and $S^{(0)\mu \nu} = q^{(0)\mu \nu} S_{\mu \nu}^{(0)}$ and $\delta \tilde{S} = q^{(0)\mu \nu} \delta \tilde{S}_{\mu \nu}$. Here, we have assumed that the unperturbed unit normal $n^{(0)}_M$ is directed from $\mathcal{M}_-$ to $\mathcal{M}_+$. This perturbed junction condition can be applied to general perturbations around an arbitrary background. The quantity $\delta q_{\mu \nu}$ is given by (17), while $\delta K_{\mu \nu}$ is given by (24) or (25). These quantities are invariant under the $D$-gauge transformation, and they transform as perturbations of $(D-1)$-tensors under the $(D-1)$-gauge transformation.
In this section, as an application of the general scheme developed in the previous section, we consider perturbation of the junction condition around a background with the symmetry of a \((D-2)\)-dimensional constant curvature space. The perturbed junction condition will be written in terms of doubly gauge-invariant variables only.

Let us consider a background metric with the symmetry of a \((D-2)\)-dimensional constant curvature space.

\[
y^{(0)}_{MN} dx^M dx^N = \gamma^{ab} dx^a dx^b + r^2 \Omega_{ij} dx^i dx^j,
\]

where \(\Omega_{ij}\) is the metric of the \((D-2)\)-dimensional constant curvature space with the curvature constant \(K\), \(\gamma^{ab}\) is a 2-dimensional metric. It is supposed that \(\gamma^{ab}\) and \(r\) depend only on the 2-dimensional coordinates \(\{x^a\}\). In this background we have three kinds of covariant derivatives. The first one is the \((0)\)-dimensional covariant derivative \(D\) compatible with \(\Omega_{ij}\). The second one is the 2-dimensional covariant derivative \(\nabla_a\) compatible with \(\gamma^{ab}\). The final one is the \((D-2)\)-dimensional covariant derivative \(D_i\) compatible with \(\Omega_{ij}\). Relations among these covariant derivatives are easily obtained. (For example, see ref. [20].) It is easy to show by explicit calculation that Riemann tensor of the background metric \(g^{(0)}_{MN}\) has the following components.

\[
R^{(0)ij}_{\ k\ l} = \left( \frac{K}{r^2} - \gamma^{ab} \partial_a \ln r \partial_b \ln r \right) \left( \delta^i_k \delta^j_l - \delta^i_l \delta^j_k \right),
\]

\[
R^{(0)i}_{\ ajb} = -\delta^i_j \left( \nabla_a \nabla_b \ln r + \partial_a \ln r \partial_b \ln r \right),
\]

\[
R^{(0)abcd} = R^{(\gamma)abcd},
\]

and \(R^{(0)}_{abc} = R^{(0)}_{abj} = R^{(0)}_{aijk} = 0\), where \(R^{(\gamma)}_{abcd}\) is the Riemann tensor of \(\gamma^{ab}\).

Further, let us assume that the unperturbed hypersurface \(\Sigma^{(0)}\) also has the same symmetry. In this case, we can choose functions \(Z^{(0)\mu}\) so that \(Z^{(0)a}\) depends only on \(y^0\) and that \(Z^{(0)i} = y^i\). The unperturbed induced metric on \(\Sigma^{(0)}\) is

\[
y^{(0)}_{\mu\nu} dx^\mu dx^\nu = -\epsilon N^2 dy^0 dy^0 + r^2 \Omega_{ij} dy^i dy^j,
\]

where \(N^2\) defined as follows and \(r^2\) are written as functions of \(y^0\) by using \(x^a = Z^{(0)a}(y^0)\), and \(\Omega_{ij}\) is written in terms of \(\{y^a\}\) by using \(x^i = y^i\).

\[
N^2 = -\epsilon \gamma^{ab} e^a e^b,
\]

\[
e^a = \partial Z^{(0)a}/\partial y^0.
\]

As for the unit normal of \(\Sigma^{(0)}, i\)-components are zero and \(a\)-components are determined by

\[
e^a n^{(0)}_a = 0,
\]

\[
n^{(0)a} n^{(0)}_a = \epsilon,
\]

where \(n^{(0)a} = \gamma^{ab} n^{(0)}_b\) and \(\gamma^{ab}\) is the inverse of \(\gamma^{ab}\). Correspondingly, the unperturbed extrinsic curvature is

\[
K^{(0)}_{\mu\nu} dy^\mu dy^\nu = N^2 K dy^0 dy^0 + r^2 \tilde{K} \Omega_{ij} dy^i dy^j,
\]

where

\[
K = N^{-2} e^a e^b \nabla_a n^{(0)}_b,
\]

\[
\tilde{K} = n^{(0)a} \partial_a \ln r.
\]

Because of the symmetry assumed above, we can write the unperturbed surface energy momentum tensor in the following form.

\[
S^{(0)}_{\mu\nu} dy^\mu dy^\nu = N^2 \rho dy^0 dy^0 + r^2 p \Omega_{ij} dy^i dy^j,
\]

where \(\rho\) and \(p\) are functions of \(y^0\) only. Thence, the unperturbed junction condition is

\[
N^2 - N^2 = r^2 - r^2 = 0,
\]

\[
K^+ - K^- = -K^2 \left( \frac{n-1}{n} \rho + ep \right),
\]

\[
\tilde{K}^+ - \tilde{K}^- = -K^2 ep/n.
\]

Here, we have assumed that the unperturbed unit normal \(n^{(0)}_a\) is directed from \(\mathcal{M}^-\) to \(\mathcal{M}^+\).
Perturbations $\delta q_{\mu\nu}$ and $\delta K_{\mu\nu}$ of the induced metric and the extrinsic curvature around the background can be calculated by using the expressions (17) and (25).

\[
\begin{align*}
\delta q_{00} &= e^a e^b (\delta g_{ab} + 2 \nabla_a \delta Z_b), \\
\delta q_{0i} &= e^a (\delta g_{ai} + \partial_i \delta Z_a + \partial_a \delta Z_i - 2 \delta Z_i \partial_a \ln r), \\
\delta q_{ij} &= \delta g_{ij} + D_i \delta Z_j + D_j \delta Z_i + 2 r^2 \Omega_{ij} \delta Z^a \partial_a \ln r, \\
\delta K_{00} &= \frac{c}{2} n^{(0)a} n^{(0)b} (\delta g_{ab} + \nabla_a \delta Z_b) N^2 K \\
&- \frac{1}{2} n^{(0)a} e^c e^d \left[ (2 \nabla_c \delta g_{ab} - \nabla_a \delta g_{bc}) + 2 \left( \nabla_b \nabla_c \delta Z_a + R_{\delta g ac} \delta Z^a \right) \right], \\
\delta K_{0i} &= -\frac{1}{2} n^{(0)a} e^b (\partial_i \delta g_{ab} + \nabla_b \delta g_{ai} - \nabla_a \delta g_{bi} - 2 \delta g_{ai} \partial_b \ln r) \\
&+ 2 \partial_i \nabla_b Z_a - 2 \partial_i \delta Z_a \partial_b \ln r - 2 \delta Z_i \partial_a \ln r + 4 \delta Z_i \partial_a \ln r \partial_b \ln r, \\
\delta K_{ij} &= \frac{c}{2} n^{(0)a} n^{(0)b} (\delta g_{ab} + \nabla_a \delta Z_b) r^2 \delta \Omega_{ij} \\
&- \frac{1}{2} n^{(0)a} \left[ D_i \delta g_{aj} + D_j \delta g_{ai} + 2 \partial_a \Omega_{ij} \delta g_{ab} \partial_b \ln r - \partial_a \delta g_{ij} \right] \\
&+ 2 D_i D_j \delta Z_a - 2 (D_i \delta Z_j + D_j \delta Z_i) \partial_a \ln r + 2 \partial_a \Omega_{ij} \nabla^b \delta Z_a \partial_b \ln r - \Omega_{ij} \delta Z^b \nabla_a \nabla_b \ln r, \tag{45}
\end{align*}
\]

where $\delta g_{ab} = \gamma^{bc} \delta g_{ac}$. Here, we have used relations among three kinds of covariant derivatives derived in ref. [20]. (The relations among covariant derivatives obtained in ref. [20] hold for a general metric of the form (36) since the relations are purely kinematical ones.)

Now, since the background has the symmetry of a $(D-2)$-dimensional constant curvature space, it is convenient to expand perturbations by using harmonics on the constant curvature space as follows.

\[
\begin{align*}
\delta g_{MN} dx^M dx^N &= \sum_k \left[ h_{ab} Y dx^a dx^b + 2 (h_{(T)a} V_{(T)i}) dx^a dx^i \\
&+ (h_{(T)} T_{(T)ij}) dx^i dx^j \right], \\
\delta Z_M dx^M &= \sum_k \left[ z_{a} Y dx^a + (z_{(T)} V_{(T)i}) dx^i \right], \\
\delta S_{\mu\nu} dy^\mu dy^\nu &= \sum_k \left[ t_{00} Y dy^0 dy^0 + 2 (t_{(T)0} V_{(T)i}) dy^0 dy^i \\
&+ (t_{(T)} T_{(T)ij}) dx^i dx^j \right], \tag{47}
\end{align*}
\]

where $Y$, $V_{(T)i}$, and $T_{(T,L,L,L,Y)i}$ are scalar, vector, and tensor harmonics, respectively, and all coefficients are supposed to depend only on the 2-dimensional coordinates $\{x^a\}$. Hereafter, $k$ denotes continuous ($K = 0$, $-1$) or discrete ($K = 1$) eigenvalues, and we omit them in most cases. In this respect, the summation with respect to $k$ should be understood as an integration for $K = 0$, $-1$. For definitions and properties of these harmonics, see Appendix B of ref. [20]. Here, we only mention that $V_{(L)i}$, $T_{(L,L)ij}$, and $T_{(Y)ij}$ are constructed from the scalar harmonics $Y$ and that $T_{(T,L)ij}$ is constructed from the vector harmonics $V_{(T)i}$.

### B. D-gauge-invariant variables

The next task might be to obtain the corresponding harmonic expansions of $\delta q_{\mu\nu}$ and $\delta K_{\mu\nu}$. However, before doing it, we shall introduce those linear combinations of perturbation variables which are invariant under the $D$-gauge transformation (28). Such linear combinations are usually called gauge-invariant variables. However, since we also have another kind of gauge transformation (the $(D-1)$-gauge transformation (30)), we shall call those combinations $D$-gauge-invariant variables. It is expected that each coefficient of the harmonic expansions of $\delta q_{\mu\nu}$ and $\delta K_{\mu\nu}$ can be written in terms of the $D$-gauge-invariant variables only, since $\delta q_{\mu\nu}$ and $\delta K_{\mu\nu}$ should be invariant under the $D$-gauge transformation from the general arguments in the previous section.
We can construct $D$-gauge-invariant variables corresponding to perturbations of physical position of the hypersurface $\Sigma$

\[
\phi_a = z_a + X_a, \\
\phi_{(L)} = z_{(L)} + h_{(LL)}, \\
\phi_{(T)} = z_{(T)} + h_{(LT)}
\] (48)
as well as $D$-gauge-invariant variables introduced in ref. [20],

\[
F_{ab} = h_{ab} - \nabla_a X_b - \nabla_b X_a, \\
F = h_{(Y)} - X^a \partial_a r^2 + \frac{2k^2}{n} h_{(LL)}, \\
F_a = h_{(T)a} - r^2 \partial_a (r^{-2} h_{(LT)}), \\
F_{(T)} = h_{(T)},
\] (49)
where $X_a = h_{(L)a} - r^2 \partial_a (r^{-2} h_{(LL)})$. (Note that the gauge transformation and the construction of the $D$-gauge-invariant variables in ref. [20] can be applied to a general metric of the form (36) since they are purely kinematical. In ref. [20] these $D$-gauge-invariant variables (49) were simply called gauge-invariant variables since only the $D$-gauge-transformation was considered in that paper.)

Now, it can actually be shown by using the harmonic expansion of $\delta g_{MN}$ and $\delta Z_M$ that $\delta q_{\mu\nu}$ and $\delta K_{\mu\nu}$ are written in terms of $D$-gauge-invariant variables only.

\[
\delta q_{\mu\nu} dy^\mu dy^\nu = \sum_k [\sigma_{00} Y dy^0 dy^0 + 2 (\sigma_{(T)} Y_{(T)i}) dy^0 dy^i] + (\sigma_{(T)} T_{(T)ij} + \sigma_{(LL)} T_{(LT)ij} + \sigma_{(Y)} T_{(Y)ij}) dy^i dy^j], \\
\delta K_{\mu\nu} dy^\mu dy^\nu = \sum_k [\kappa_{00} Y_0 dy^0 dy^0 + 2 (\kappa_{(T)} Y_{(T)i}) + \kappa_{(L)} Y_{(L)i}) dy^0 dy^i] + (\kappa_{(T)} T_{(T)ij} + \kappa_{(LT)} T_{(LT)ij} + \kappa_{(LL)} T_{(LL)ij} + \kappa_{(Y)} T_{(Y)ij}) dy^i dy^j],
\] (50)

where

\[
\sigma_{00} = e^a e^b (F_{ab} + 2 \nabla_e \phi_b), \\
\sigma_{(T)0} = e^a [F_a + r^2 \partial_a (r^2 \phi_{(T)})], \\
\sigma_{(L)0} = e^a [\phi_a + r^2 \partial_a (r^2 \phi_{(L)})], \\
\sigma_{(T)} = F_{(T)}, \\
\sigma_{(LT)} = \phi_{(T)}, \\
\sigma_{(LL)} = \phi_{(L)}, \\
\sigma_{(Y)} = F + \phi_a \partial_a r^2 - \frac{2k^2}{n} \phi_{(L)},
\] (51)

and

\[
\kappa_{00} = \frac{\epsilon}{2} \eta_{(0)a} \eta_{(0)b} (F_{ab} + 2 \nabla_e \phi_b) N^2 \kappa - \frac{1}{2} \eta_{(0)a} e^b e^c (2 \nabla_e F_{ab} - \nabla_a F_{bc} + 2 \nabla_b \nabla_c \phi_a + 2 R^{(\gamma)}_{dbac} \phi^d), \\
k_{(T)0} = \frac{1}{2} \eta_{(0)a} e^b [2 \nabla_b (r^{-2} F_a) - \nabla_a F_b - \partial_a r^2 \partial_b (r^{-2} \phi_{(T)})], \\
k_{(L)0} = \frac{1}{2} \eta_{(0)a} e^b [F_{ab} + 2 r \nabla_b (r^{-1} \phi_a) - \partial_a r^2 \partial_b (r^{-2} \phi_{(L)})], \\
k_{(T)} = \frac{1}{2} \eta_{(0)a} (\partial_a F_{(T)}) \\
k_{(LT)} = \frac{1}{2} \eta_{(0)a} (F_a - 2 \phi_{(T)} \partial_a \ln r) \\
k_{(LL)} = \frac{1}{2} \eta_{(0)a} (\phi_a - 2 \phi_{(L)} \partial_a \ln r) \\
k_{(Y)} = \frac{\epsilon}{4} \eta_{(0)a} \eta_{(0)b} \eta_{(0)c} (F_{ab} + 2 \nabla_e \phi_b) \partial_c r^2 \\
- \frac{1}{2} \eta_{(0)a} \left( F_{ab} \partial_b r^2 - \partial_a F + \nabla^b \phi_a \partial_b r^2 - \phi_b \nabla_a \nabla_b r^2 - \frac{2k^2}{n} \phi_a + \frac{4k^2}{n} \phi_{(L)} \partial_a \ln r \right),
\] (52)
where $\phi^a = \gamma^{ab}\phi_b$ and $F^a_a = F_{ac}\gamma^{cb}$. Correspondingly, the quantities $\tilde{K}_{\mu\nu}$ and $\tilde{S}_{\mu\nu}$, which are defined by (35) and which play important roles in the perturbed junction condition (34), are expanded as

$$
\delta \tilde{K}_{\mu\nu} dy^\mu dy^\nu = \sum_k [\tilde{K}_{00} Y dy^0 dy^0 + 2(\tilde{K}_{(T)0} V_{(T)i} + \tilde{K}_{(L)0} V_{(L)i}) dy^0 dy^i] + (\tilde{K}_{(T)ij} + \tilde{K}_{(L)ij} + \tilde{K}_{(Y)ij}) dy^i dy^j, \\
\delta \tilde{S}_{\mu\nu} dy^\mu dy^\nu = \sum_k [\tilde{S}_{00} Y dy^0 dy^0 + 2(\tilde{S}_{(T)0} V_{(T)i} + \tilde{S}_{(L)0} V_{(L)i}) dy^0 dy^i] + (\tilde{S}_{(T)ij} + \tilde{S}_{(L)ij} + \tilde{S}_{(Y)ij}) dy^i dy^j, 
$$

(53)

where $\tilde{K}$'s are defined by

$$
\tilde{K}_{00} = K_{00} + \epsilon K \sigma_{00}, \\
\tilde{K}_{(T)0} = K_{(T)0} - \frac{1}{2}(\hat{K} - \epsilon K) \sigma_{(T)0}, \\
\tilde{K}_{(L)0} = K_{(L)0} - \frac{1}{2}(\hat{K} - \epsilon K) \sigma_{(L)0}, \\
\tilde{K}_{(T)} = K_{(T)} - \hat{K} \sigma_{(T)}, \\
\tilde{K}_{(L)} = K_{(L)} - \hat{K} \sigma_{(L)}, \\
\tilde{K}_{(LL)} = K_{(LL)} - \hat{K} \sigma_{(LL)}, \\
\tilde{K}_{(Y)} = K_{(Y)} - \hat{K} \sigma_{(Y)},
$$

(54)

and $\tilde{S}$'s are defined by the same relations with $\tilde{K}$'s, $\sigma$'s, $K$, $\hat{K}$ replaced by $\tilde{S}$'s, $\sigma$'s, $S$, $\hat{S}$.

C. $(D - 1)$-gauge-invariant variables

As already stressed several times, we have two kinds of independent gauge transformations: $D$-gauge transformation and $(D - 1)$-gauge transformation. As for the $D$-gauge transformation, we have defined $D$-gauge-invariant variables as quantities invariant under that and have shown that $\delta q_{\mu\nu}$ and $\delta K_{\mu\nu}$ (and thus $\delta \tilde{K}_{\mu\nu}$) are written in terms of the $D$-gauge-invariant variables only. Now let us investigate the second: the $(D - 1)$-gauge transformation (30). In what follows, we shall construct those linear combinations of perturbations which are invariant under that, and shall call them $(D - 1)$-gauge-invariant variables. It is evident that a $D$-gauge-invariant variable is not necessary a $(D - 1)$-gauge-invariant variable. This is because the $(D - 1)$-gauge transformation is not a part of the $D$-gauge transformation. In particular, the expansion coefficients $\sigma$'s, $\tilde{K}$'s, and $\tilde{S}$'s are not $(D - 1)$-gauge-invariant variables although they are $D$-gauge-invariant variables.

First, $\sigma$'s transform as follows under the $(D - 1)$-gauge transformation.

$$
\sigma_{00} \rightarrow \sigma_{00} - 2N \frac{\partial}{\partial y^0} (N^{-1} \zeta_0), \\
\sigma_{(T)0} \rightarrow \sigma_{(T)0} - \gamma^2 \frac{\partial}{\partial y^0} (N^{-1} \zeta_{(T)}), \\
\sigma_{(L)0} \rightarrow \sigma_{(L)0} - \zeta_0 - \gamma^2 \frac{\partial}{\partial y^0} (N^{-1} \zeta_{(L)}), \\
\sigma_{(T)} \rightarrow \sigma_{(T)}, \\
\sigma_{(LT)} \rightarrow \sigma_{(LT)} - \zeta_{(T)}, \\
\sigma_{(LL)} \rightarrow \sigma_{(LL)} - \zeta_{(L)}, \\
\sigma_{(Y)} \rightarrow \sigma_{(Y)} + \epsilon N^{-2} \frac{\partial^2}{\partial y^0^2} + \frac{2K^2}{n} \zeta_{(L)},
$$

(55)

where we have expanded $\tilde{\zeta}_\mu = \tilde{q}_{\mu\nu} \zeta^\nu$ as

$$
\tilde{\zeta}_\mu dy^\mu = \sum_k [\zeta_0 Y dy^0 + (\zeta_{(T)} V_{(T)i} + \zeta_{(L)} V_{(L)i}) dy^i].
$$

(56)
Therefore, we can construct $(D-1)$-gauge-invariant variables $f_{00}$, $f$, $f_0$ and $f(T)$ by
\[ f_{00} = \sigma_{00} - 2N \frac{\partial}{\partial y^0} (N^{-1} \chi), \]
\[ f = \sigma_{(Y)} + \epsilon N^{-2} \frac{\partial r^2}{\partial y^0} + \frac{2k^2}{n} \sigma_{(LL)}, \]
\[ f_0 = \sigma_{(T)0} - r^2 \frac{\partial}{\partial y^0} (r^{-2} \sigma_{(LT)}), \]
\[ f(T) = \sigma_{(T)}, \] (57)
where $\chi = \sigma_{(L)0} - r^2 \partial (r^{-2} \sigma_{(LL)})/\partial y^0$. It is evident that these $(D-1)$-gauge-invariant variables can be written in terms of $D$-gauge-invariant variables since $\sigma$'s have already been written by $D$-gauge-invariant variables in (51). In fact, we can show that
\[ f_{00} = e^a e^b F_{ab} - 2N^2 \epsilon_\chi \phi_a, \]
\[ f = F + \epsilon \rho \phi_a \partial_b r^2 \]
\[ f_0 = e^a \phi_a, \]
\[ f(T) = F(T). \] (58)

Next, $\tilde{t}$'s transform as follows under the $(D-1)$-gauge transformation.
\[ \tilde{t}_{00} \rightarrow \tilde{t}_{00} + \epsilon \rho \frac{\partial}{\partial y^0}, \]
\[ \tilde{t}_{(T)0} \rightarrow \tilde{t}_{(T)0} - \frac{1}{2} (p + \epsilon \rho) r^2 \frac{\partial}{\partial y^0} (r^{-2} \zeta_{(T)}), \]
\[ \tilde{t}_{(L)0} \rightarrow \tilde{t}_{(L)0} + \frac{1}{2} (p + \epsilon \rho) \left[ \zeta_0 - r^2 \frac{\partial}{\partial y^0} (r^{-2} \zeta_{(L)}) \right], \]
\[ \tilde{t}_{(T)} \rightarrow \tilde{t}_{(T)}, \]
\[ \tilde{t}_{(LT)} \rightarrow \tilde{t}_{(LT)}, \]
\[ \tilde{t}_{(LL)} \rightarrow \tilde{t}_{(LL)}, \]
\[ \tilde{t}_{(Y)} \rightarrow \tilde{t}_{(Y)} + \epsilon N^{-2} r^2 \zeta_0 \frac{\partial}{\partial y^0}. \] (59)

Correspondingly, we can define the following set of $(D-1)$-gauge-invariant variables.
\[ \tau_{00} = \tilde{t}_{00} + \epsilon \chi \frac{\partial}{\partial y^0}, \]
\[ \tau_{(T)0} = \tilde{t}_{(T)0} - \frac{1}{2} (p + \epsilon \rho) \sigma_{(T)0}, \]
\[ \tau_{(L)0} = \tilde{t}_{(L)0} + \frac{1}{2} (p + \epsilon \rho) \left[ \chi - r^2 \frac{\partial}{\partial y^0} (r^{-2} \sigma_{LL}) \right], \]
\[ \tau_{(T)} = \tilde{t}_{(T)}, \]
\[ \tau_{(LT)} = \tilde{t}_{(LT)}, \]
\[ \tau_{(LL)} = \tilde{t}_{(LL)}, \]
\[ \tau_{(Y)} = \tilde{t}_{(Y)} + \epsilon N^{-2} r^2 \chi \frac{\partial}{\partial y^0}. \] (60)

Thirdly, $(D-1)$-gauge transformations of $\tilde{\kappa}$'s are the same as those of $\tilde{t}$'s with $\tilde{t}$'s, $\rho$, $p$ replaced by $\tilde{\kappa}$'s, $\kappa$, $\bar{\kappa}$. Thus, we can construct $(D-1)$-gauge-invariant variables as follows.
\[ k_{00} = \tilde{\kappa}_{00} + \epsilon \chi \kappa \frac{\partial \kappa}{\partial y^0}, \]
\[ k_{(T)0} = \tilde{\kappa}_{(T)0} - \frac{1}{2} (\bar{\kappa} + \epsilon \kappa) \sigma_{(T)0}. \]
\[
\begin{align*}
\kappa_{(L)0} &= \tilde{\kappa}_{(L)0} + \frac{1}{2}(\bar{\kappa} + \epsilon \kappa) \left[ \chi - r^2 \frac{\partial}{\partial y_0} (r^{-2} \sigma_{LL}) \right], \\
\kappa_{(T)0} &= \tilde{\kappa}_{(T)0} \\
\kappa_{(LT)} &= \tilde{\kappa}_{(LT)} \\
\kappa_{(LL)} &= \tilde{\kappa}_{(LL)} \\
\kappa_{(Y)} &= \tilde{\kappa}_{(Y)} + \epsilon N^{-2} \frac{r^2}{\partial y_0} 
\end{align*}
\]

(61)

It is evident that these \((D - 1)\)-gauge-invariant variables can be written in terms of \(D\)-gauge-invariant variables since \(\tilde{\kappa}\)’s and \(\sigma\)'s have already been written by \(D\)-gauge-invariant variables in (51), (52) and (54). Explicit expressions are as follows.

\[
\begin{align*}
k_{00} &= \frac{1}{2} n^{(0) a} e^b (2 \nabla_c F_{ab} - \nabla_d F_{bc}) + \frac{\epsilon}{2} (n^{(0) a} n^{(0) b} + 2 N^{-2} e^a e^b) F_{ab} N^2 \kappa \\
&\quad - N e^b \partial_b \left[ N^{-1} e^c \partial_c (n^{(0) a} \phi_a) \right] + \frac{1}{2} N^2 (\epsilon R^{(\gamma)} + 2 \bar{\kappa}^2) n^{(0) a} \phi_a, \\
k_{(T)0} &= \frac{1}{2} r^2 n^{(0) a} e^b \left[ \nabla_b (r^{-2} F_a) - \nabla_a (r^{-2} F_b) \right], \\
k_{(L)0} &= \frac{1}{2} r^2 n^{(0) a} e^b F_{ab} - r e^a \nabla_a (r^{-1} n^{(0) b} \phi_b), \\
k_{(T)} &= \frac{1}{2} r^2 n^{(0) a} \partial_a (r^{-2} F(T)), \\
k_{(LT)} &= -\frac{1}{2} n^{(0) a} F_a, \\
k_{(LL)} &= -\frac{1}{2} n^{(0) a} \phi_a \\
k_{(Y)} &= \frac{\epsilon}{4} F_{ab} n^{(0) a}(2 N^{-2} e^b e^c - n^{(0) b} n^{(0) c}) \partial_c r^2 + \frac{1}{2} r^2 n^{(0) a} \partial_a (r^{-2} F) \\
&\quad + \frac{\epsilon}{2} N^{-2} e^b \partial_b (n^{(0) a} \phi_a) e^c \partial_c r^2 + \left( \epsilon r^2 n^{(0) b} n^{(0) c} \nabla_b \nabla_c \ln r + \frac{k^2}{n} \right) n^{(0) a} \phi_a,
\end{align*}
\]

(62)

where \(R^{(\gamma)}\) is the Ricci scalar of \(\gamma_{ab}\).

**D. Doubly gauge-invariant junction condition**

Now we can write down junction conditions for doubly gauge-invariant variables \(f\)'s, \(\tau\)'s and \(k\)'s, where \(f\)'s are given by (58), \(\tau\)'s are given by (60), and \(k\)'s are given by (62). By **doubly gauge-invariant variables**, we mean variables invariant under both \(D\)-gauge transformation and \((D - 1)\)-gauge transformation.

In ref. [20], a classificatory criterion was introduced for gravitational perturbations around a maximally symmetric spacetime. Namely, expansion coefficients of harmonics \(Y\), \(V_{(L) i}\), \(T_{(LL) i j}\) and \(T_{(Y) i j}\) are called **scalar perturbations**; expansion coefficients of harmonics \(V_{(T) i}\) and \(T_{(LT) i j}\) are called **vector perturbations**; expansion coefficients of harmonics \(T_{(Y) i j}\) are called **tensor perturbations**. This classification can be applied to the situation in this paper, too. From the orthogonality between different kinds of harmonics (see Appendix B of ref. [20]), it is evident that perturbations belonging to different categories should be decoupled from each other. We can easily confirm this general conclusion explicitly from eqs. (58), (60), and (62). Therefore, in what follows, we can write down the doubly gauge-invariant junction conditions for each category independently.

First, the junction condition for scalar perturbations is

\[
\begin{align*}
f_{00+} - f_{00-} &= f_+ - f_- = 0, \\
k_{00+} - k_{00-} &= -\kappa^2 \left( \frac{n - 1}{n} \tau_{00} + \epsilon N^2 r^{-2} \tau_{(Y)} \right), \\
k_{(Y)+} - k_{(Y)-} &= -\kappa^2 \frac{\epsilon r^2}{n N^2} \tau_{00}, \\
k_{(L)0+} - k_{(L)0-} &= -\kappa^2 \tau_{(L)0}, \\
k_{(LL)+} - k_{(LL)-} &= -\kappa^2 \tau_{(LL)}. 
\end{align*}
\]

(63)
Levi-Civita tensor is reduced to variables cosmology. It was recently shown first in ref. [20] that for a maximally symmetric background the
This off course includes the anti de-Sitter spacetime which can be used as a bulk geometry of the brane-world. (Curvature constants, or cosmological constants, in two regions $2$-dimensional spacetime. First, we formulated a general scheme of perturbation of the junction condition around an arbitrary background in a doubly covariant way. The junction condition is given by (34), where the quantity $\delta q_{\mu \nu}$ is given by (17) and $\delta K_{\mu \nu}$ is given by (24) or (25). It was shown that $\delta q_{\mu \nu}$ and $\delta K_{\mu \nu}$ are invariant under $D$-gauge transformation and that they transform as perturbations of $(D - 1)$-tensors under $(D - 1)$-gauge transformation. The $D$-gauge transformation is an infinitesimal coordinate transformation of the spacetime, and the $(D - 1)$-gauge transformation is an infinitesimal reparameterization of $\Sigma$. It is important that the latter is not a part of the former but they are completely independent.

Next, as an application of the general formalism, we analyzed perturbation of the junction condition around a symmetric background which has the symmetry of a $(D - 2)$-dimensional constant curvature space. By applying the general formalism to the symmetric background, we have obtained the doubly gauge invariant junction condition for perturbations. Junction conditions for scalar, vector, and tensor perturbations are given by (63), (64) and (65), respectively, where doubly gauge-invariant variables $f$'s, $\tau$'s and $k$'s are given by (58), (60), and (62), respectively.

Now let us discuss about a special case in which the background metric $g^{(0)}_{M N}$ is maximally symmetric or, equivalently, of constant curvature. (Curvature constants, or cosmological constants, in two regions $M_+$ and $M_-$ may be different.) This off course includes the anti de-Sitter spacetime which can be used as a bulk geometry of the brane-world cosmology. It was recently shown first in ref. [20] that for a maximally symmetric background the $D$-gauge-invariant variables $F_{ab}$ and $F$ are described by a common scalar-type master variable $\Phi(S)$ and that the $D$-gauge-invariant variable $F_a$ is also described by another scalar-type master variable $\Phi(V)$.

\[
\begin{align*}
D^{-4} F_{ab} &= \nabla_a \nabla_b \Phi(S) - \frac{D - 3}{D - 2} \nabla^2 \Phi(S) \gamma_{ab} - \frac{2(D - 4)\Lambda}{(D - 1)(D - 2)} \Phi(S) \gamma_{ab}, \\
D^{-6} F &= \frac{1}{D - 2} \left[ \nabla^2 \Phi(S) + \frac{4\Lambda}{(D - 1)(D - 2)} \Phi(S) \right], \\
D^{-4} F_a &= \epsilon_a^b \partial_b \Phi(V),
\end{align*}
\]

where $\epsilon_a = \gamma_{ac}e^c$, and $\epsilon_{ab}$ is the Levi-Civita tensor defined by $\epsilon_{01} = -\epsilon_{10} = \sqrt{|\det \gamma_{ab}|}$ and $\epsilon_{00} = \epsilon_{11} = 0$. (The Levi-Civita tensor is reduced to $\epsilon_{ab} = \epsilon N^{-1}(n^{(0)a}a^{b} - e^a n^{(0)b})$ on the hypersurface $\Sigma_0$.) By substituting the relations (66) into eqs. (62), we can obtain expressions for $k$'s in terms of the master variables. The corresponding junction conditions, combined with the wave equations for the master variables given in ref. [20], can be considered as basic equations for perturbations of the brane-world cosmology. Actually, in the brane world scenario, if we assume the $Z_2$ symmetry then the curvature constants in the two regions $M_+$ and $M_-$ should be the same and the junction conditions (63), (64) and (65) are reduced to

\[
\begin{align*}
2k_{00+} &= -\kappa^2 \left( \frac{n - 1}{n} \tau_{00} + \epsilon N^2 r^{-2} \tau_Y \right), \\
2k_{(Y)+} &= -\kappa^2 \frac{\epsilon r^2}{nN^2} \tau_{00}, \\
2k_{(L)0+} &= -\kappa^2 \tau_{(L)0}, \\
2k_{(LL)+} &= -\kappa^2 \tau_{(LL)},
\end{align*}
\]
\[2k(T)_{0+} = -\kappa^2 \tau(T)_0,\]
\[2k(LT)_{+} = -\kappa^2 \tau(LT),\]
\[2k(T)_+ = -\kappa^2 \tau(T).\]
\[(67)\]

These junction condition are equivalent to those obtained in ref. [21]. (In appendix B, some relations between variables defined in the present paper and those in other papers are given.) In future publications we would like to analyze perturbations of the brane-world cosmology in detail by using the basic equations.

If the brane-world scenario is realistic then it might also be interesting to analyze black hole perturbations in the brane-world. We may be able to expect that the brane-world scenario might give non-standard predictions for gravitational waves emitted from, say, a binary black hole system. For this purpose, of course, we have to analyze background black hole solutions in the brane-world scenario thoroughly. After that, the general doubly covariant formalism developed in Sec. III may be useful for the analysis of perturbations.

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APPENDIX A: ANOTHER EXPRESSION OF $\delta K_{\mu\nu}$

In the main body of this paper, no assumptions have been needed with respect to the one-parameter family of hypersurfaces (11). In particular, the expression (25) holds for any choices of the one-parameter family of hypersurfaces. On the other hand, in order to obtain another expression of $\delta K_{\mu\nu}$ which is similar to the expression used in ref. [21], we have to assume that $\partial_{[M^{(0)}_\mu]} = 0$. Once this is assumed, we can obtain the following expression for $\delta K_{\mu\nu}$.

\[
\delta K_{\mu\nu} = \frac{\epsilon}{2} \delta g_{\perp\perp} K_{\mu\nu}^{(0)} - \delta \Gamma_{\mu\perp\nu} - \nabla_\mu \nabla_\nu \delta Z_{\perp} + \epsilon \delta Z_{\perp} (K_{\rho\mu}^{(0)} K_{\nu\rho}^{(0)} - R_{\perp\mu\nu\perp}^{(0)}) + \mathcal{L}_\delta Z_{\perp} K_{\mu\nu}^{(0)},
\]
\[(A1)\]

where

\[
\delta g_{\perp\perp} = n^{(0)M}_\mu n^{(0)N}_\delta g_{MN},
\]
\[
\delta \Gamma_{\mu\nu} = n^{(0)\rho}_{\mu} \epsilon^{(0)}_{\nu} \delta \Gamma_{\rho MN},
\]
\[
R_{\perp\mu\nu\perp}^{(0)} = n^{(0)\rho}_{\mu} \epsilon^{(0)}_{\nu} R_{\rho \perp MN},
\]
\[
K_{\mu\nu}^{(0)} = K_{\mu\nu}^{(0)} q^{(0)\sigma\rho},
\]
\[
\delta Z_{\perp} = n^{(0)M}_\perp \delta Z^M,
\]
\[
\delta Z^\mu = q^{(0)\mu\nu} \epsilon^{(0)}_{\nu} g^{(0)M}_{MN} \delta Z^N,
\]
\[(A2)\]

and $\nabla$ denotes the $(D - 1)$-dimensional covariant derivative compatible with the unperturbed induced metric $q^{(0)}_{\mu\nu}$ on $\Sigma^{(0)}$, $\mathcal{L}$ denotes the Lie derivative defined in the $(D - 1)$-dimensional manifold $\Sigma^{(0)}$, and $R_{\rho \perp MN}^{(0)}$ is the Riemann tensor of the unperturbed metric $g^{(0)}_{MN}$. Note that all defined in eqs. (A2) are $D$-scalars while they transform as $(D - 1)$-scalars, $(D - 1)$-vectors, or $(D - 1)$-tensors under the $(D - 1)$-coordinate transformation. In particular, $\delta Z^\mu$ is a $(D - 1)$-vector-valued $D$-scalar.

In the expression (25) used in the main body of this paper, all derivatives are $D$-dimensional covariant derivatives compatible with $q^{(0)}_{\mu\nu}$. Hence, it is easier to calculate $\delta K_{\mu\nu}$ by using the expression (25) than the expression (A1).

APPENDIX B: RELATIONS TO VARIABLES DEFINED IN OTHER REFERENCES

For the sake of readers who like to compare the results of this paper with those in other references, in this appendix, we shall give some relations between variables defined in this paper and those in other papers.

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The \((D - 1)\)-gauge-invariant variables defined in this paper can be related to gauge-invariant variables used in cosmology. Here, we give relations between the \((D - 1)\)-gauge-invariant variables and those gauge-invariant variables defined by Kodama and Sasaki [30]. (In ref. [30], \(\epsilon = + \) and \(N = r\).) In the following, the subscript \((KS)\) represents a quantity defined in ref. [30], and we use the relations \((Y, V_{(T)i}, V_{(L)i}, T_{(LT)ij}, T_{(LL)ij}, T_{(Y)ij}) = (Y, Y_i^{(1)}, -\sqrt{k^2}Y_i, Y_{ij}^{(2)}, -2\sqrt{k^2}Y_{ij}^{(1)}, 2k^2Y_{ij}, Y\Omega_{ij})_{(KS)}\). For scalar perturbations,

\begin{align}
N^{-2}f_{00} &= -2\Psi_{(KS)}, \\
f &= 2r^2\Phi_{(KS)}, \\
N^{-2}\tau_{00} &= \rho\Delta_{(KS)} - N^{-1}\frac{\partial \rho}{\partial y^0}rV_{(KS)} - \sqrt{k^2}, \\
\tau_{(Y)} &= r^2(\rho\Gamma_{(KS)} + c_s^2N^{-2}\tau_{00}), \\
N^{-1}\tau_{(L)0} &= -(p + \rho)rV_{(KS)}, \\
\tau_{(LL)} &= r^2\rho\Pi_{(KS)} - \sqrt{k^2},
\end{align}

(B1)

where \(c_s\) is the sound velocity defined by \(c_s^2 = (\partial p/\partial y^0)/(\partial \rho/\partial y^0)\). Relations for vector perturbations are

\begin{align}
N^{-1}f_{0} &= r\sigma_{(KIS)}^{(1)}, \\
N^{-1}\tau_{(T)0} &= (p + \rho)rV_{(KIS)}^{(1)}, \\
\tau_{(LT)} &= r^2p\Pi_{(KS)}^{(1)} - 2\sqrt{k^2}.
\end{align}

(B2)

For tensor perturbations,

\begin{align}
f_{(T)} &= 2r^2H_{T(KIS)}^{(2)}, \\
\tau_{(T)} &= r^2p\Pi_{(KIS)}^{(2)}.
\end{align}

(B3)

Next, let us give relations between the \(D\)-gauge-invariant variables defined in this paper and those defined in ref. [21]. (In ref. [21], \(\epsilon = + \) and \(N = 1\).) In the following, the subscript \((KIS)\) represents a quantity defined in ref. [21].

\begin{align}
F_{ab} &= F_{ab(KIS)}, \\
F &= 2r^2F_{(KIS)}, \\
F_a &= rF_a(KIS), \\
F_{(T)} &= 2r^2H_{T(KIS)}.
\end{align}

(B4)

Finally, we point out some relations between definitions in ref. [30] and ref. [21].

\begin{align}
p\Gamma_{(KIS)} &= \Gamma_{(KIS)}, \\
p\Pi_{(KIS)} &= \pi_{T(KIS)}, \\
p\Pi_{(KIS)}^{(1)} &= \pi_{T(KIS)}^{(1)}, \\
p\Pi_{(KIS)}^{(2)} &= \pi_{T(KIS)}^{(2)}.
\end{align}

(B5)
