The O-Shell Nucleon Nucleon Amplitude: Why it is Unmeasurable in Nucleon-Nucleon Bremsstrahlung

Abstract. Nucleon-nucleon bremsstrahlung has long been considered a way of probing nucleon-nucleon interactions at O-shell distances. It has been suggested that this process can be used to probe the O-shell part of the nucleon-nucleon amplitude. However, we show that this is not the case. The O-shell amplitude is not measurable as a matter of principle. This follows formally from the invariance of the S-matrix under transformations of the fields. This result is discussed here and illustrated via two simple models, one applying to spin zero, and one to spin one half, processes. The latter model is very closely related to phenomenological models which have been used to study O-shell effects at electromagnetic vertices.

The central point of this talk is to show that this historical motivation for nucleon-nucleon bremsstrahlung is incorrect. In actual fact the O-shell amplitude is not a physically well defined quantity and as a matter of principle cannot be measured in nucleon-nucleon bremsstrahlung, or any other process.
choice. Thus in this normal, but incorrect, scenario, measurement of $p + p \rightarrow p + p + \gamma$ in kinematic regions far enough off-shell will determine the half-off-shell T-matrix and distinguish among potentials.

To understand the problem with this approach we first review the standard potential model description of bremsstrahlung. First observe that in an abstract mathematical sense the off-shell amplitude is well defined as the solution of the Lippman-Schwinger equation for off-shell momenta. It is only part of the full physical bremsstrahlung process however. To get the full process standard potential models include first the so called external radiation graphs in which a photon is emitted from an external leg. These involve off-shell effects both at the strong vertex and at the electromagnetic vertex. Usually the double scattering term in which the photon is emitted from a line in between two full scattering T-matrices is also included. Finally a few ‘contact’ graphs are included in an ad hoc fashion. These may include diagrams with $\Delta$’s or the one with a $\pi \rho \gamma$ or $\pi \omega \gamma$ vertex. Modern nucleon-nucleon potentials are built from diagrams with multi meson exchanges, and in principle diagrams with photons attached to any interior charged line should also be included. There are many such ‘contact’ diagrams which are irreducible and thus can not be included in the double scattering or external radiation graphs. In potential models, these are just dropped.

At the very simplest, qualitative level one can begin to understand the ambiguities inherent in the off-shell amplitude as follows. The amplitude for a typical bremsstrahlung external radiation graph can be written as

$$\{T_{on} + (p^2 - m^2)T_{off}\} \frac{i}{p^2 - m^2}, em$$

corresponding to an on-shell part and an off-shell part of the nucleon-nucleon interaction, together with propagator and electromagnetic interaction. However, since the factor $p^2 - m^2$ appearing in the off-shell part exactly cancels the same factor appearing in the propagator, the amplitude can also be written as

$$T_{on} \frac{i}{p^2 - m^2}, em + T_{off} i, em$$

which has the form of an external radiation graph with on-shell amplitude plus a contact term. Thus the off-shell terms can be written alternatively as contact terms, and there will always be an ambiguity in how one separates from the full measurable process the part associated with an off-shell amplitude.

At a more fundamental level this ambiguity can be related to a theorem \[2\] which tells us that a transformation can be made on the fields appearing in a Lagrangian without changing the S-matrix elements, which correspond to the physical, measurable quantities. Such transformations generally change the ‘equation of motion’ terms in the Lagrangian, which are those leading to the $p^2 - m^2$ factor which gives the off-shell part of the vertex function. \[3\] This means that such transformations can be used to change the off-shell T-matrix for the elastic process in an arbitrary fashion, without changing the full
bremsstrahlung amplitude. Thus one can never measure an off-shell amplitude since the measurable quantity, the bremsstrahlung cross section, corresponds to an infinite number of different off-shell amplitudes.

We can summarize what we have learned so far as follows. There will always be some ambiguity as to what is called off-shell and what is called contact. At a more formal level field transformations can change the coefficient of the off-shell amplitude without changing the measurable bremsstrahlung. Thus the off-shell amplitude is not measurable as there are an infinite number of different such amplitudes which give the same bremsstrahlung amplitude.

Why wasn’t this noticed in the fifty years of work on bremsstrahlung? Perhaps it was because potential model calculations tend to drop the contact terms from the beginning and because field transformation concepts, though well known in some areas of physics, are quite different from the techniques used in most non relativistic potential model approaches.

We turn now to a couple of simple field theory models which will illustrate the concepts so far described only in a qualitative way. Consider first a model of the spin zero bremsstrahlung process, e.g. \( \pi^+ + \pi^- \to \pi^+ + \pi^- + \gamma \), described in detail in ref. [4]. We take as the Lagrangian for this process, \( \mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4^L + \Delta \mathcal{L}_4 \) where \( \mathcal{L}_2 + \mathcal{L}_4^L \) is the usual chiral perturbation theory Lagrangian to \( O(p^4) \) written in terms of \( \chi, U \), which contain the fields, and of the covariant derivative \( D_\mu \). The last term involves the equation of motion and is given by the expression

\[
\Delta \mathcal{L}_4 = \beta_1 T r (\mathcal{O} \mathcal{O}^\dagger) + \beta_2 T r \left[ (\chi U^\dagger - U \chi^\dagger) \mathcal{O} \right].
\]

Here \( \mathcal{O} = 0 \) is the equation of motion, with \( \mathcal{O} \) a somewhat complicated function also of \( \chi, U \), and \( D_\mu \).

This Lagrangian generates a simple effective field theory which allows exact calculations to a given order. Though it was originally motivated by chiral perturbation theory, the result to be obtained from it has absolutely nothing to do with chiral perturbation theory.

Using this Lagrangian we can calculate the elastic amplitude for the process \( \pi^+(p_1) + \pi^-(p_2) \to \pi^+(p_3) + \pi^- (p_4) \), obtaining a result given (schematically) as

\[
\Delta r = \frac{4}{F^2} \left( \ldots (\Lambda_1 + \Lambda_3) + (\ldots) \beta_1 (\Lambda_1 + \Lambda_3) + (\ldots) \beta_1 \Lambda_1 + (\ldots) \beta_1 \Lambda_3 \right)
\]

where \( (\ldots) \) are known factors and \( \Lambda_i = p_i^2 - m_i^2 \). This is the exact analog of the off-shell elastic amplitude one would calculate from a potential by solving the Lippman-Schwinger equation. The off-shell part is proportional to \( \Lambda_1 \) or \( \Lambda_3 \) and for the most part to the parameters \( \beta_1 \) and \( \beta_2 \). The on-shell amplitude is obtained by taking the limit \( \Lambda_1, \Lambda_3 \to 0 \).

Now what happens when we apply a field transformation to the fields in this Lagrangian? To that end consider the transformation \( U' \to \exp(iS)U \), for arbitrary \( a_1, a_2 \), where \( S \) is given by

\[
S = \frac{4i}{F^2} \left[ a_1 \mathcal{O} - a_2 \left( \chi U^\dagger - U \chi^\dagger - \frac{1}{2} T r (\chi U^\dagger - U \chi^\dagger) \right) \right].
\]
Under this transformation $L_2(U') \rightarrow L_2(U) + \delta L_2(U)$ with

$$\delta L_2 = a_1 \text{Tr}(\mathcal{O} \mathcal{O}^\dagger) + a_2 \text{Tr}((\mathcal{U} U^\dagger - U \mathcal{U}^\dagger) \mathcal{O})$$

Clearly $\delta L_2(U) \sim \Delta L_4$. Thus what the transformation does, since $a_1, a_2$ are arbitrary, is to arbitrarily change the values of the coefficients $\beta_1, \beta_2$ in the Lagrangian, i.e., to change the coefficients which give the off-shell elastic amplitude. By the general result such transformation does not change the full measurable amplitude, though it clearly changes the off-shell part of the elastic amplitude.

Now let us apply this model to bremsstrahlung. The external radiation diagrams can be obtained from the off-shell elastic amplitude, a propagator, and the electromagnetic $\pi \pi$ vertex, all consistently calculated using the model Lagrangian. The result is in abbreviated form,

$$M_3 + M_1 = T_{\pi n}(\frac{e \cdot p_3}{k \cdot p_3} - \frac{e \cdot p_1}{k \cdot p_1}) + \text{constant}$$

$$+ \beta_1(...A...) + \beta_2(...B...)$$

where the A and B pieces are known kinematic dependent factors.

$M_3 + M_1$ is the analogue of what would be calculated in a potential model from the half off-shell T-matrix, a non-relativistic propagator, and an electromagnetic vertex. In a potential model it would be compared with data to extract the off-shell information "contained" in $\beta_1$ and $\beta_2$. But, this approach has to be wrong because field transformations change the value of $\beta_1$ and $\beta_2$ arbitrarily without changing the value of the bremsstrahlung amplitude. In fact the bremsstrahlung amplitude must be independent of $\beta_1$ and $\beta_2$ since $0$ is one possible value.

To understand what is happening, consider the contact term, which can be calculated explicitly in this model, unlike in potential model calculations.

$$\delta_{\gamma \pi \gamma} = \text{const} - \beta_1(...A...) - \beta_2(...B...)$$

Comparison of the contact term with $M_3 + M_1$ shows that all terms involving the parameters $\beta_1, \beta_2$, which govern the strength of the off-shell amplitude, cancel in the full result for bremsstrahlung which is thus independent of $\beta_1$ and $\beta_2$ as it must be. A similar result is obtained when this Lagrangian is used for Compton scattering.

The fragment of the amplitude, $M_1 + M_3$, which is analogous to the usual potential model bremsstrahlung result, does however depend on $\beta_1$, $\beta_2$ and if one considers it alone, one is led to the (spurious) conclusion that off-shell amplitudes can be measured by measuring bremsstrahlung.

Let us now consider a different model, one applicable to spin one-half particles. This model corresponds very closely to the types of phenomenological models used in calculations investigating off-shell effects at electromagnetic vertices, e.g., the prototype reaction will be $p + n \rightarrow p + n + \gamma$, i.e., proton-neutron bremsstrahlung. We consider the p-p system, rather than the p-n system just to reduce the number of diagrams to be considered and to avoid the extra algebra required for identical particles.
We thus start with the Lagrangian,

$$\mathcal{L}_0 = \overline{\psi}(iD - m)\psi - \frac{\kappa}{4m} \overline{\psi} \sigma_{\mu\nu} F^{\mu\nu} \psi + g \overline{\psi} \psi \Phi. \quad (1)$$

Here $D$ is the covariant derivative, $e$ and $\kappa$ are the proton charge and anomalous magnetic moment, $A_\mu$, $F_{\mu\nu}$ are photon field and field strength tensor, and $\Phi \sim$ proton, $\Phi \sim$ neutron. This Lagrangian leads to the standard electromagnetic coupling to a spin one-half particle, the standard free equation for the proton and to a standard, but simplified, scalar boson exchange form for the strong interaction. However it generates no off-shell effects to lowest order.

Now consider a field transformation on this Lagrangian of the form

$$\Psi' \rightarrow \Psi + \hat{\alpha} g \overline{\psi} \Phi \Psi + \hat{\beta} \epsilon_{\mu\nu} F^{\mu\nu} \Psi$$

where $\hat{\alpha}$ and $\hat{\beta}$ are arbitrary constants. This transformation generates a new Lagrangian

$$\mathcal{L}_2(\Psi') = \mathcal{L}_0(\Psi) + \Delta_1 \mathcal{L}(\Psi) + \Delta_2 \mathcal{L}(\Psi)$$

The new piece of the Lagrangian $\Delta_1 \mathcal{L}(\Psi)$ generates an off-shell contribution to the strong amplitude proportional to $\hat{\alpha}$ and an off-shell part of the electromagnetic vertex proportional to $\hat{\beta}$ as well as a contact term necessary for gauge invariance. $\Delta_2 \mathcal{L}(\Psi)$ generates only contact terms.

If we now take $\mathcal{L}_0(\Psi) + \Delta_1 \mathcal{L}(\Psi)$ as our Lagrangian we have something corresponding very closely to the Lagrangian used in phenomenological calculations such as those of [8, 10]. It produces off-shell contributions at both strong and electromagnetic vertices. One could compare the predictions of this Lagrangian for bremsstrahlung with data and in principle extract values of the parameters $\hat{\alpha}$ and $\hat{\beta}$. However this does not give any information about off-shell nucleon-nucleon amplitudes. Since the change in the Lagrangian, $\Delta_1 \mathcal{L}(\Psi) + \Delta_2 \mathcal{L}(\Psi)$, originates in a field transformation it can not affect the bremsstrahlung amplitude. This means that the amplitude using the full Lagrangian must be independent of the parameters $\hat{\alpha}$ and $\hat{\beta}$, which can be verified explicitly. As a corollary, the Lagrangian $\mathcal{L}_0(\Psi) - \Delta_2 \mathcal{L}(\Psi)$ must give exactly the same bremsstrahlung result as $\mathcal{L}_0(\Psi) + \Delta_1 \mathcal{L}(\Psi)$. But the former Lagrangian contains only contact terms and no off-shell effects while the latter has mainly off-shell effects. One could also use any linear combination of these two Lagrangians. Thus there are an infinite number of Lagrangians, with different proportions of off-shell effects and contact terms which give the same bremsstrahlung amplitude. So again we see that the concept of an off-shell amplitude as a part of a bremsstrahlung amplitude is not well defined and such off-shell amplitudes are not measurable.

Let us now summarize what has been learned. Field transformations change the coefficients of the off-shell part of the elastic amplitude without changing the bremsstrahlung amplitude. In effect such transformations simply move terms back and forth between the bremsstrahlung diagrams containing off-shell amplitudes and the contact terms. This means that the off-shell contributions are not uniquely defined and therefore that the 'off-shell amplitude' is not measurable in bremsstrahlung reactions, contrary to historical expectations. Hence
the claim that measuring bremsstrahlung will distinguish among potentials on
the basis of their off-shell behavior is just not valid.

This result means that most previous calculations and experiments dealing
with nucleon-nucleon bremsstrahlung were driven by an aim – to measure
off-shell effects – which is just not possible. Does this make nucleon-nucleon
bremsstrahlung any less interesting? The answer is clearly no. What emerges
from these results is the importance of the contact terms. These contact terms
originate from the photon probe of the currents internal to the strong interac-
tion. To understand them in detail one must understand these interactions at a
microscopic level, which is probably much more interesting and gives us much
more insight about the physical mechanisms involved, than does understanding
gross properties of a phenomenological potential.

This means however that for the future we need both comprehensive ex-
periments with enough data over a wide enough kinematic range to distinguish
details of the process and also microscopic calculations which include speci-

cally details of the contact terms.

Finally nothing here depends specifically on the bremsstrahlung process,
and so it is probable that off-shell amplitudes are unmeasurable in any process.
Thus calculations purporting to show sensitivity to off-shell amplitudes should
be viewed with suspicion.

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References

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