Curvature Dependence of Peaks in the Cosmic Microwave Background Distribution

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Abstract
The widely cited formula $\ell_1 \approx 200 \Omega_0^{-1/2}$ for the multipole number of the first Doppler peak is not even a crude approximation in the case of greatest current interest, in which the cosmic mass density is less than the vacuum energy density. For instance, with $\Omega_M$ fixed at 0.3, the position of any Doppler peak varies as $\Omega_0^{-1.58}$ near $\Omega_0 = 1$.

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The precise measurement$^1$ of the multipole number $\ell_1 = 197 \pm 6$ at the first ‘Doppler’ peak has provided an invaluable constraint on cosmological parameters. In a 1994 numerical calculation, Kamionkowski, Spergel and Sugiyama$^2$ presented a formula giving $\ell_1$ as a function essentially of the curvature alone:

$$\ell_1 \sim \frac{200}{\sqrt{\Omega_0}},$$

where $\Omega_0 \equiv \Omega_M + \Omega_\Lambda$, in which $\Omega_M$ and $\Omega_\Lambda$ are the present ratios of the cosmic mass density and the vacuum energy (associated, e. g., with a cosmological constant) to the critical density. This calculation was done before supernova studies$^3$ indicated the likely presence of a relatively large cosmological constant, and therefore assumed that $\Omega_\Lambda = 0$. They also explained the $\Omega_0^{-1/2}$ behavior by noting that $\ell_1$ is approximately inversely proportional to the angle subtended at the earth by the horizon at the time of last scattering, which was known$^4$ to be proportional to $\Omega_0^{1/2}$ for $\Omega_\Lambda = 0$. The same $\Omega_0$-dependence was derived on the same grounds by Frampton et al.,$^5$ explicitly for the case $\Omega_\Lambda = 0$.

Unfortunately, despite the fact that it was derived only for the case $\Omega_\Lambda = 0$, Eq. (1) continues to be quoted$^1,6,7,8,9$ as if it were generally applicable also when $\Omega_\Lambda$ is appreciable. As far as I know, this formula has not been used by observational groups in analysis of their data, but in view of the great current interest in these matters, it seems worth warning that in fact, Eq. (1) is not valid for parameters in the range suggested by supernova observations, for
which $\Omega_\Lambda > \Omega_M$. Although it is true that when $\Omega_0$ is near unity, $\ell_1$ depends less sensitively on other parameters than on $\Omega_0$, the dependence of $\ell_1$ on $\Omega_0$ bears no resemblance whatever to Eq. (1), except for the case $\Omega_\Lambda \ll 1$. Instead, we shall see that the dependence of $\ell_1$ on $\Omega_0$ near $\Omega_0 = 1$ with $\Omega_M$ fixed at values less than 0.4 is much stronger than given by Eq. (1) (for instance, $\ell_1 \propto \Omega_M^{-1.58}$ for $\Omega_M = 0.3$), and it depends sensitively on $\Omega_M$.

To calculate the full dependence of $\ell_1$ on $\Omega_0$, $\Omega_M$, $\Omega_{\text{baryon}}$, $\Omega_{\text{radiation}}$, etc. is a complicated task, requiring the consideration of the evolution of the acoustic velocity and of the ratio of radiation and matter energies, and the consideration of Doppler shifts as well as temperature fluctuations. We can avoid all these complications by considering the dependence of $\ell_1$ on $\Omega_0$ when only $\Omega_\Lambda$ is allowed to vary, with $\Omega_M$ and all other parameters held fixed. If it were really true (as Eq. (1) says) that $\ell_1$ depends only on $\Omega_0$, then this would be all we need to calculate the full $\Omega_0$-dependence.

The advantage of letting only $\Omega_\Lambda$ vary is that the vacuum energy density is negligible compared with the densities of matter and radiation at and before the redshift $z_L \simeq 1100$ of last scattering, so the only effect of variations in $\Omega_\Lambda$ on the multipole number $\ell_n$ of the $n$th Doppler peak is to change the paths followed by light rays since the time of last scattering. The angle subtended at the earth by any feature of the cosmic microwave background of proper length $d$ is

$$\theta = d/d_A,$$

(2)
where $d_A$ is the angular diameter distance of the surface of last scattering:

$$d_A = \frac{1}{\Omega_k^{1/2} H_0 (1 + z_L)} \sinh \left[ \frac{1}{2} \int_{1/(1+z_L)}^{1} \frac{dx}{\sqrt{\Omega_A x^4 + \Omega_k x^2 + \Omega_M x}} \right],$$  \hspace{1cm} (3)

and $\Omega_k$ is a measure of curvature

$$\Omega_k \equiv 1 - \Omega_\Lambda - \Omega_M = 1 - \Omega_0.$$  \hspace{1cm} (4)

It follows that the $\Omega_\Lambda$-dependence of $\ell_n$ is given by

$$\ell_n \propto d_A.$$  \hspace{1cm} (5)

Furthermore, although the relation between the present Hubble constant $H_0$ and the proper scales of phenomena at the time of last scattering depends on $\Omega_M$ and $\Omega_{\text{radiation}}$, it does not depend on $\Omega_\Lambda$. (For instance, if we neglect radiation, then the acoustic horizon at the redshift of last scattering is \(2(1+z_L)^{-3/2}/\sqrt{3\Omega_M H_0}\).) Therefore, with $\Omega_M$ fixed, the dependence of $\ell_n$ on $\Omega_\Lambda$ is given by

$$\ell_n \propto F(\Omega_\Lambda) = \frac{1}{\Omega_k^{1/2}} \sinh \left[ \frac{1}{2} \int_{0}^{1} \frac{dx}{\sqrt{\Omega_A x^4 + \Omega_k x^2 + \Omega_M x}} \right],$$  \hspace{1cm} (6)

with $\Omega_k$ given in terms of $\Omega_\Lambda$ by Eq. (4). (The lower limit on the integral has here been set equal to zero because $z_L >> 1$.) Of course, all the detailed physics of the acoustic oscillations responsible for the Doppler peaks is contained in the constant of proportionality; all we need to know here is that it does not involve $\Omega_\Lambda$.  

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Now let us consider the variation of the quantity (6) as we make small changes in $\Omega_0$ near $\Omega_0 = 1$ with $\Omega_M$ fixed. An elementary calculation gives

$$\ell_n \propto \Omega_0^{-\nu},$$

(7)

where

$$\nu \equiv \left( \frac{\partial \ln F}{\partial \Omega_\Lambda} \right)_{\Omega_\Lambda = 1 - \Omega_M} = \frac{I_1^2}{6} - \frac{I_2}{2I_1},$$

(8)

with

$$I_1 = \int_0^1 \frac{dx}{[(1 - \Omega_M)x^4 + \Omega_Mx]^{1/2}}$$

and

$$I_2 = \int_0^1 \frac{(x^2 - x^4) dx}{[(1 - \Omega_M)x^4 + \Omega_Mx]^{3/2}}.$$

(9)

The table below gives values of these integrals, and of the resulting exponent $\nu$ in Eq. (7).

<table>
<thead>
<tr>
<th>$\Omega_M$</th>
<th>$I_1$</th>
<th>$I_2$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>3.891</td>
<td>2.546</td>
<td>2.196</td>
</tr>
<tr>
<td>0.3</td>
<td>3.305</td>
<td>1.601</td>
<td>1.578</td>
</tr>
<tr>
<td>0.4</td>
<td>2.938</td>
<td>1.145</td>
<td>1.244</td>
</tr>
<tr>
<td>1.0</td>
<td>2/21</td>
<td>8/21</td>
<td>4/7</td>
</tr>
</tbody>
</table>

The only approximation made in deriving these results is that the universe becomes transparent suddenly at a redshift $z_L \gg 1$, and has been dominated since then by non-relativistic matter and vacuum energy. Also, we are neglecting the effect of changing gravitational potentials at redshifts $z \ll z_L$, which introduce an additional $\Lambda$- dependence that is quite small at the wavelengths of the Doppler peaks. Otherwise, these results are exact.
The behavior $\ell_1 \propto \Omega_0^{-4/7}$ near $\Omega_0 = 1$ for $\Omega_M$ fixed at unity is close to the behaviour $\ell_1 \propto \Omega_0^{-1/2}$ near $\Omega_0 = 1$ found\textsuperscript{2,5} for $\Omega_A$ fixed at zero, confirming that $\ell_1$ is approximately a function of $\Omega_0$ alone for $\Omega_A = 0$ and $\Omega_M$ near unity. The fact that $\nu$ depends strongly on $\Omega_M$ for smaller values of $\Omega_M$ shows that for observationally favored parameters $\ell_1$ is not approximately a function of $\Omega_0$ alone. Indeed, there is no physical reason why $\ell_1$ should be even approximately a function of $\Omega_0$ alone. For fixed values of $\Omega_M$ less than 0.4 the $\ell_n$ fall off with increasing $\Omega_0$ much more rapidly than would be expected from Eq. (1), so the measurement of the positions of the Doppler peaks provides a more stringent constraint on $\Omega_0$ than would be the case if Eq. (1) were correct.

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**References**


5. P. Frampton, Y. J. Ng, and R. Rohm, *Mod. Phys. Lett.* A13, 2541 (1998). There are aspects of this paper with which I disagree, but they are not relevant to the present work.


9. Bahcall et al.\textsuperscript{6} cited reference 2, while Turner\textsuperscript{7} cited no reference for Eq. (1). De Bernardis et al.\textsuperscript{1} cited no references for Eq. (1), but relied on references 2, 5, and 6. Roos and Harun-or-Rashid\textsuperscript{8} also cited no references, but took this formula from reference 1.

10. This formula is given, e. g., in reference 5.