When a part of the environment responsible for decoherence is used to extract information about the decohering system, the preferred set of pointer states remains unchanged. The einselected pointer states of the family of conditional master equations are invariant under a partial environmental eavesdropping. We also find indications that the einselected states are easiest to infer from the measurements carried out on the environment.

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Introduction. — Open quantum systems undergo environment-induced superselection (einselection) which leads to the emergence of a preferred set of quasi-classical states. These pointer states [1] entangle with – and therefore, lose information to – the environment most slowly. They can be found using predictability sieve, which seeks states minimizing entropy production [2]. But the information lost to the environment could be, in principle, intercepted and recovered. Will the preferred states remain at least approximately the same when the environment is monitored in this fashion?

This is a serious concern, as decoherence is caused by the entanglement between the system and the environment. It is well known that a pair of entangled quantum systems suffers from the basis ambiguity: One can find out about different states of one of them (e. g., the system) by choosing to measure the other (e. g., the environment) in a different basis [1]. Thus, basis ambiguity may endanger the definiteness of the einselected pointer states.

This problem was pointed out, for example, by Carmichael [4], who used complete monitoring of the photon environment to develop a trajectory approach to quantum dynamics [5]. He and his collaborators have demonstrated that – when all of the environment can be intercepted – any basis of the system can be inferred from the appropriate measurement on the environment, so at least in that limit substantial ambiguity is inevitable. It is further underscored by the realization [6] that nearly all of our information comes not from direct observation of the system, but, rather, by intercepting a small fraction (e. g. photon) environment.

Here we use predictability sieve in combination with the conditional master equation (CME) [7] (which obtains when part of the environment is measured as opposed to the standard unconditional master equation (UME) which obtains for ignoring the environment state) to show that even when predictability is improved by the additional data, the pointer states are unchanged by such partial monitoring. We demonstrate, using fidelity to the initial state, that even when all of the environment is intercepted the pointer states are unchanged. Moreover, using specific models we argue that – for an observer who acquires the data about the system indirectly by monitoring the environment – pointer states are easiest to discover.

An example of CME.— The master equation for a driven two-level atom whose emitted radiation is measured by homodyne detection [5,7] is an example of CME,

\[ d\rho = d\rho^\text{UME} + d\rho_{\text{st}}, \]

where

\[ d\rho^\text{UME} = -i\, \Omega [\sigma_x, \rho] + dt \left( \rho c^\dagger c - \frac{1}{2} c^\dagger c\rho - \frac{1}{2} \rho c^\dagger c \right), \]

\[ d\rho_{\text{st}} = (dN - \overline{dN}) \left( \frac{(c + \gamma)\rho(c^\dagger + \gamma^*)}{\text{Tr}[(c + \gamma)\rho(c^\dagger + \gamma^*)]} - \rho \right). \]

We use the Itô version of stochastic calculus. \( \rho \) is a 2 \times 2 density matrix of the atom, \( dt \) is an infinitesimatal time increment. \( \Omega \) is a frequency of transitions between the excited and the ground state driven by a laser beam, \( \gamma = Re^{-i\phi} \) is the amplitude of the local oscillator in the homodyne detector and \( c = (\sigma_x - i\sigma_y)/2 \) is an annihilation operator. \( N_t \) is the number of photons detected until time \( t \). Its increment \( dN \in \{0, 1\} \) is a dichotomic stochastic process with the average

\[ \overline{dN} = \eta dt \text{Tr}[\rho(c^\dagger + \gamma^*)(c + \gamma)] \]

\[ = \eta dt [R^2 + \langle \sigma_x \rangle R \cos \phi + \langle \sigma_y \rangle R \sin \phi + \langle c^\dagger c \rangle] \]

and \( \overline{dN^2} = \overline{dN} \). Efficiency of the detector \( \eta \) is the fraction of photons which are detected. For example, \( \eta = 0.5 \) may correspond to detecting all photons emitted into the upper hemisphere but ignoring all photons emitted into the lower hemisphere. For \( \eta = 0, d\rho_{\text{st}} = 0, \) and the equation reduces to the usual Lindblad unconditional master equation (UME), \( d\rho = d\rho^\text{UME} \).

Moreover, the average over realisations \( \overline{d\rho} = d\rho^\text{UME} \), because \( \overline{dN - \overline{dN}} = 0 \) and \( \overline{d\rho_{\text{st}}} = 0 \). \( \overline{d\rho} = d\rho^\text{UME} \) is not a special property of Eqs(1,2) but an axiomatic property of any CME. The noise average means that we ignore any knowledge about the state of the environment so the state of the system cannot be conditioned by this knowledge.
For $R = 0$ the measurement scheme is simply a photodetection: $\overline{dN}$ is proportional to the probability that the atom is in the excited state. Every click of the photodetector ($dN = 1$) brings the atom to the ground state, from where it is excited again by the laser beam. For $R \gg 1$ the homodyne photodetector current is a linear function of $\langle \sigma_x \rangle \equiv \text{Tr}(\rho \sigma_x)$ for $\phi = 0$ ($x$-measurement) or of $\langle \phi \rangle$ for $\phi = \pi/2$ ($y$-measurement). These homodyne measurements drive the conditional state of the atom towards $\sigma_x$ and $\sigma_y$ eigenstates respectively.

**Conditional pointer states are unconditional.** — We will give a general argument why pointer states of any stochastic CME are the same as pointer states of its corresponding deterministic UME. According to the predictability sieve criterion [2], pointer states are states of a system for which the increase of von Neumann entropy, or, equivalently, decrease of purity $P = \text{Tr}(\rho^2)$ due to the interaction with an environment is the least. Suppose that we prepare a system in a pure state $\rho_0 = \rho_0^\text{UME}$. The noise-averaged initial rate of purity loss is

$$d\overline{P}_0 \equiv \text{Tr}(2\rho_0 d\rho_0) + \text{Tr}(d\rho_0 d\rho_0).$$

(4)

d$\rho_0$ is the first increment of $\rho$, which for our two-level atom would be given by Eqs.(1,2) with $\rho = \rho_0$ on their RHS.

Any CME can be written in the form of Eq.(1), where $N_t$ would represent a general stochastic process. The stochastic process feeds the information from measurements of the environment into the conditional state of the system. The noise-averaged $d\rho_0 = d\rho_0^\text{UME}$; it depends neither on the efficiency $\eta$ nor on the kind of measurement we make on the environment. For a deterministic UME the second term on the RHS of Eq.(4) would be $O(dt^2)$. For a stochastic CME this second term gives a contribution proportional to $\eta dt$ which comes from $\text{Tr}(d\rho_0 d\rho_0)$. The manifestly positive second term reduces the rate of purity destruction. This is not surprising as a measurement of environment tends to purify the conditional state. For $\eta = 1$ the observer gains full knowledge about the environmental state, the conditional state of the system remains pure all the time, and $dP_t = 0$. Thus we see that at $\eta = 1$ the first and the second term of Eq.(4) should cancel one another. Given that the first and the second term cancel for $\eta = 1$ and that the second term is linear in $\eta$, we can write the initial purity destruction rate as

$$\overline{P}_0 = (1 - \eta) \text{Tr}(2\rho_0 d\rho_0^\text{UME}).$$

(5)

Up to the prefactor of $(1 - \eta)$ this expression is the same as the corresponding one for the UME. Except for $\eta = 1$ we can conclude that pointer states are the same as those for the UME no matter what the efficiency is or what kind of measurement is being made.

The standard predictability sieve based on the purity destruction rate tells us nothing about the case of perfect efficiency ($\eta = 1$). When $\eta = 1$ we have $d\overline{P}_0 = 0$ and no preferred pointer states can be distinguished, in accordance with [4]. However, even the conditional pure state can drift away from the initial pure state. The faster it drifts away the less predictable the state of the system is. The fidelity with respect to the initial state is defined as $F(t) = \text{Tr}(\rho_0 \rho_t)$. For any $\eta$ the noise-averaged initial decrease of fidelity is

$$d\overline{F}_0 \equiv \text{Tr}(\rho_0 d\rho_0) = \text{Tr}(\rho_0 d\rho_0^\text{UME}),$$

(6)

that is proportional to $d\overline{P}_0$ in Eq.(5). The UME pointer states that minimize $d\overline{F}_0$ also minimize $d\overline{P}_0$: the UME pointer states lose fidelity most slowly.

The expressions (5,6) can be worked out for the example of the two-level atom master equation. For $\Omega \ll 1$ there is one pointer state: the ground state. This is not surprising as an atom in the ground state cannot change its state by photoemission and the external driving is slow. The limit of $\Omega \gg 1$ is much more interesting. In this limit the externally driven oscillations are much faster than photoemission. In fact it would be misleading to use expressions (5,6) as they stand. It is more accurate first to take their average over one period of oscillation:

$$\overline{d\overline{F}_0} \approx \frac{\Omega}{2\pi} \int_0^{2\pi/\Omega} dt \text{Tr}[\rho_0^\text{int} \rho_0^\text{int} - \rho_0^\text{int} \rho_0^\text{int}]$$

$$= \frac{1}{8} (-3 + x_0^2).$$

(7)

Here the density matrix is parametrized by $\rho_t = [I + x_i \sigma_x + y_i \sigma_y + z_i \sigma_z]/2$ with $x_i^2 + y_i^2 + z_i^2 \leq 1$. Given the last constraint, Eq.(7) implies that the states with $x = \pm 1$ are the pointer states [8]. They are eigenstates of the self-Hamiltonian $\Omega \sigma_x$. It should be noted that the $d\overline{P}_0$ or $d\overline{F}_0$ rate for, say, $y = \pm 1$ states ($\sigma_y$-eigenstates) is only 50% faster than for the pointers, for reasons that are specific to our small system.

**Pointer states are the easiest to find out.** — Pointer states do not depend on the kind of measurement carried out by the observer on the environment or on its efficiency. This robustness of pointer states might convey the wrong impression that all kinds of measurements are equivalent from the point of view of the observer trying to find out about the system by monitoring its environment. In what follows we give two examples which strongly suggest that the measurement of the environment states correlated with the pointer basis of the system is the most efficient one in gaining information about the state of the system.

We begin with the two-level atom. In the limit of $\Omega \gg 1$, the pointer states are eigenstates of the driving self-Hamiltonian $\Omega \sigma_x$. For $\eta = 0$ the UME has a stationary mixed state $\rho_\text{st} = I/2 + O(1/\Omega)$. Suppose that we start monitoring the environment of the atom at $t = 0$ (detectors are turned on at $t = 0$, and $\eta$ suddenly becomes nonzero). How fast do we find out about the system? This can be measured by the purity of the conditional state. For $\eta \ll 1$ the response of $\rho$ to the switching-
on of $\eta > 0$ at $t = 0$ can be described by a small perturbation of the density matrix $\delta \rho = [\delta x \sigma_x + \delta y \sigma_y + \delta z \sigma_z]/2$ such that $\rho \approx \rho_0 + \delta \rho$. The evolution of $\delta \rho$ is described by a set of stochastic differential equations

$$
d\delta x = dt(-\delta x/2) + dN \left( \frac{2R \cos \phi}{1 + 2R^2} \right),
$$

$$
d\delta y = dt(-\delta y/2 - 2\Omega \delta z) + dN \left( \frac{2R \sin \phi}{1 + 2R^2} \right),
$$

$$
d\delta z = dt(-\delta z + 2\Omega \delta y) + dN \left( -\frac{1}{1 + 2R^2} \right),
$$

where $dN = dN - \overline{dN}$ and $\overline{dN} = \eta dt(R^2 + 1/2)$. The formal solution of these stochastic differential equations leads to a noise-averaged purity

$$
\overline{\rho}(t) \approx \text{Tr}(\rho^2_0) + \text{Tr}(\delta \rho^2) = \frac{1}{2} + \frac{1}{2} \left[ \frac{\delta x^2_t + \delta y^2_t + \delta z^2_t}{1 + 2R^2} \right] = (9)
$$

$$
\frac{1}{2} + \frac{R^2 \cos^2 \phi}{1 + 2R^2} (1 - e^{-t}) + \frac{1 + 4R^2 \sin^2 \phi}{6(1 + 2R^2)} (1 - e^{-3t/2}).
$$

For any time $t > 0$ the highest purity is obtained for homodyne ($R \gg 1$) measurement of $\langle \sigma_x \rangle$ ($\phi = 0$). As anticipated, this is the measurement in the basis of environmental states correlated with the pointer states of the system. The purity saturates for $t \gg 1$, the saturated purity in the homodyne limit $R \gg 1$ being

$$
P_{\infty} = \frac{1}{2} + \frac{\eta}{6} \left( 3 \cos^2 \phi + 2 \sin^2 \phi \right).
$$

The small $\eta$ measurements in the pointer state $x$-basis ($\phi = 0$) are only 50% better than measurements in the $y$-basis ($\phi = \pi/2$), (see Fig.1). As mentioned before, in the two-level atom example pointer states are not well distinguished from the chaff by the predictability sieve.

To try with an example known for well distinguished pointer states let us pick the quantum Brownian motion at zero temperature. We can think of the environment quanta as phonons. Except for the Hamiltonian, the CME obtains from Eqs.(1,2) by a formal replacement $c \rightarrow a$,

$$
dp = -i dt \left[ [\omega a^\dagger a, \rho] + dt \left( a a^\dagger - \frac{1}{2} a^\dagger a \rho - \frac{1}{2} a a^\dagger \rho + \frac{i}{2} a a^\dagger \rho a^\dagger \right) + \eta dt \left( \frac{(a + \gamma) \rho(a^\dagger + \gamma^*)}{\text{Tr}(a + \gamma)} - \rho \right) \right].
$$

This equation is valid in the rotating wave approximation which is justified for $\omega \gg 1$. In the interaction picture, after the replacement $c \rightarrow a$ in Eq.(3), we get

$$
\overline{dN} = \eta dt[R^2 + \langle a_{\text{int}}^\dagger a_{\text{int}} \rangle Re^{-i(\phi + \omega t)} + \langle a_{\text{int}}^\dagger \dagger a_{\text{int}} \rangle Re^{-i(\phi + \omega t)} + \langle a_{\text{int}} a_{\text{int}} \rangle]
$$

For $R = 0$ the measurement drives the conditional state to the ground state of the harmonic oscillator. In the homodyne limit ($R \rightarrow \infty$) the phonodetector current gives information about the coherent amplitude $\langle a_{\text{int}} \rangle$ of the state. The second line of Eq.(11) vanishes for pure coherent states; the conditional state tends to be localized around coherent states.

Pointer states minimize the absolute value of $\overline{dF}_0 = \text{Tr}[\rho_0 d\rho_{\text{UME}}] = \text{Tr}[\rho_0 a^\dagger a_{\text{int}} - \rho_{\text{UME}} a_{\text{int}} a^\dagger]$ which vanishes if $\rho_0$ is a coherent state, $\rho_0 = |z\rangle\langle z|$, such that $a(z) = z|z\rangle$. Coherent states are perfect pointers. In contrast to other states like, say, number eigenstates initially they do not lose any purity ($\overline{dF}_0 = 0$) or any fidelity ($\overline{dF}_0 = 0$) (but see [3]).

We can expect homodyne measurement to be more efficient in gaining information than phonodetection. To support this expectation let us pick a coherent state $|z\rangle$. If we choose $r = |z\rangle \gg 1$, then $\langle z^* - z \rangle \approx 0$ and $\langle a^\dagger(z) \approx z^* |z\rangle$. A general density matrix in the subspace spanned by $|z\rangle$ is

$$
\rho = \frac{1 + A}{2} |z\rangle \langle z| + \frac{1 - A}{2} |z^*\rangle \langle z^*|.
$$

Here $A \in [-1, +1]$. For example, $A = 0$ and $C = 1/2$ correspond to the Schrödinger cat state $(|z\rangle + |z^*\rangle)/\sqrt{2}$. Substitution of the density matrix (13) into Eq.(11), and subsequent left and right projections on $|z\rangle$, give stochastic differential equations for $A$ and $C$. These equations are most interesting in the following two limits. In the phonodetection limit ($R = 0$) they are

$$
dA = 0,
$$

$$
dC = -C[2 r^2 dt + (dN - \overline{dN})].
$$

The off-diagonal $C$ decays after the decoherence time of $1/r^2 \ll 1$. $A$ does not change, phonodetection does not produce any purity. Phonodetection is a very poor
In the opposite homodyne detection limit \((R \to \infty)\) the noise \(dN - \overline{dN}\) can be replaced (up to a constant) by a white-noise \(dW\) such that \(\overline{dW} = 0\) and \(\overline{dW^2} = dt\). After taking an average over one period of oscillation with the frequency \(\omega\) one finds that, again, \(C\) decays after the decoherence time of \(1/\eta^2\). Introducing a quantity \(B\) as \(A = \tanh B\), \(A\) satisfies the “drunken sailor equation”

\[
dB = \sqrt{2\eta r} \, dW. \tag{15}
\]

Suppose that at \(t = 0\) we had \(A = B = 0\) and \(C = 0\). This is the most mixed state possible in our subspace. The probability distribution for \(A\) at time \(t > 0\) is

\[
p(t, A) = \frac{1}{1 - A^2} \exp \left( -\frac{\ln^2 \left( \frac{1 + A}{1 - A} \right)}{4\pi \eta^2 t} \right). \tag{16}
\]

This distribution is localized at \(A = 0\) for \(t = 0\) but after a time scale of \(1/\eta^2\) it becomes concentrated in two narrow peaks close to \(A = \pm 1\) (see Fig.2). By these times the conditional state almost certainly is one of the coherent states \(|\pm z\rangle\) and purity is 1. This result is in sharp contrast to the nil result for phonodetection. In Fig.3 we plot a single realization of a stochastic trajectory \(A(t)\).

For any \(\eta < 1\) purity becomes 1 after time proportional to \(1/\eta^2\). A patient observer gets full information about the system in spite of monitoring only a small part of the environment. This illustrates that information about pointer states is recorded by the environment in a redundant way [9].

In the above example we assumed that \(r = |z| \gg 1\) so that \(\langle +z | - z \rangle \approx 0\) and \(a^\dagger |z\rangle \approx z^* |z\rangle\). This convenient assumption also naturally separates the decoherence and purification timescales \((\sim 1/r^2)\) from the timescale for decay towards the ground state \((\sim 1)\). On the fast timescales \(\sim 1/r^2\) we can neglect the decay and that is why our system remains in the \(|\pm z\rangle\)-subspace. In this way our calculation is selfconsistent.

**Concluding remarks.** — In summary, we have shown that the most classical states of a system which is being monitored are independent both of the type of measurement and of the efficiency of the detectors. Furthermore, we have given indications that the best measurements of the environment for gaining information about a system are the ones extracting information about the system in the pointer basis.

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