Higher Derivative CP\((N)\) Model and Quantization of the Induced Chern-Simons Term

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Abstract

We consider higher derivative CP\((N)\) model in 2 + 1 dimensions with the Wess-Zumino-Witten term and the topological current density squared term. We quantize the theory by using the auxiliary gauge field formulation in the path integral method and prove that the extended model remains renormalizable in the large \(N\) limit. We also find that the Maxwell-Chern-Simons theory is dynamically induced in the large \(N\) effective action and the coefficient of the Chern-Simons term must be quantized.

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The CP($N$) model in 2+1 dimensions has many remarkable properties. In contrast to the perturbative nonrenormalizability, the theory is renormalizable in the large $N$ limit, in spite of the appearance of linear divergence [1]. It also exhibits a nontrivial fixed point structure which divides the symmetric and broken phases with a second order phase transition, and dynamical generation of a gauge boson [2].

The purpose of this Letter is to extend the CP($N$) model with higher order derivative terms, and discuss its possible consequence in the large $N$ limit. In general, adding higher derivative terms in the nonlinear sigma models would make the theory less renormalizable and the path integral more involved. However, with a specific choice of higher derivative terms to be described below, the theory admits the auxiliary gauge field formulation and an exact evaluation of the path integral can be performed in the large $N$ limit. We find that (I) the extended theory remains renormalizable and (II) the auxiliary gauge field becomes dynamical with induced Maxwell-Chern-Simons terms. In order to achieve these results, two types of higher derivative terms are required. One is the third order derivative Wess-Zumino-Witten (WZW) term [3] which lives in two types of higher derivative with induced Maxwell-Chern-Simons terms. In order to achieve these results, (I) the extended theory remains renormalizable and (II) the auxiliary gauge field becomes (perturbative nonrenormalizability, the theory is renormalizable in the large $N$ limit. We find that (I) the extended theory remains renormalizable and (II) the auxiliary gauge field becomes dynamical with induced Maxwell-Chern-Simons terms. In order to achieve these results, two types of higher derivative terms are required. One is the third order derivative Wess-Zumino-Witten (WZW) term [3] which lives in $M_4$ whose boundary is our 2+1 dimensional space-time. It is well known that the coefficient of this term must be quantized which leads to the interesting consequence that the coefficient of the Chern-Simons term [4,5] must be quantized. The other is the topological current density squared term which was considered recently in the higher derivative extension of CP($N$) model in 1+1 dimensions [6].

Let us start directly from the auxiliary gauge field formulation of the extended CP($N$) model for economic presentation. We will shortly show that this theory is equivalent to the aforementioned higher derivative CP($N$) model. The Lagrangian is given by

$$
L = \frac{N}{G} \left[ (D_\mu z)^\dagger (D^\mu z) - i\theta G A_\mu \epsilon^{\mu \nu \rho} (\partial_\nu z)^\dagger (\partial_\rho z) - \lambda (z^\dagger z - 1) \right],
$$

where $D_\mu \equiv \partial_\mu - iA_\mu$ and $z$ is an $N$ components complex scalar field which obeys a constraint $z^\dagger z = 1$. The first term is the usual CP($N$) $\equiv SU(N)/SU(N - 1) \times U(1)$ model in the auxiliary gauge field formulation. The second term will be responsible for higher derivative terms when the auxiliary gauge field $A_\mu$ are eliminated through the equations of motion. Note that $z$ with the constraint $z^\dagger z = 1$ contains $2N - 1$ real scalars, whereas the coset space is a $(2N - 2)$-dimensional manifold. This mismatch is due to the local U(1) symmetry of the model. More specifically, the U(1) gauge transformation is given by

$$
z(x) \rightarrow e^{i\alpha(x)} z(x), \quad A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \alpha(x).
$$

Note that the first term in the Lagrangian (1) is manifestly gauge invariant, whereas the second term changes by a total derivative, hence the action is gauge invariant.

The field $z \equiv (z_1, \ldots, z_N)^T$ is separated into $2N - 2$ Nambu-Goldstone bosons $\psi \equiv (z_1, \ldots, z_{N-1})^T$ associated with the spontaneously broken SU($N$) symmetry and Higgs bosons $z_N \equiv (\sigma + i\chi)/\sqrt{2}$. In general there are two possible phases [2]: (I) $\langle \sigma \rangle \neq 0$, $\langle \lambda \rangle = 0$ and (II) $\langle \sigma \rangle = 0$, $\langle \lambda \rangle \neq 0$. In phase (I) both global SU($N$) and local U(1) symmetries are broken simultaneously and $\psi$ arise as massless Goldstone bosons. Through the Higgs mechanism $\chi$ turns to a longitudinal mode of massive gauge boson $A_\mu$. On the other hand in phase (II) both global SU($N$) and local U(1) symmetries are not spontaneously broken. Instead $\psi$ and $z_N$ are combined into $z$ with a universal mass $\langle \lambda \rangle^{1/2}$. We will see later that the dimensionless coupling $u \equiv G\Lambda$ shows a nontrivial ultraviolet (UV) fixed point $u^*$ which arises as a zero
of the Callan-Symanzik $\beta$-function and separates the weak coupling broken phase (I) from the strong coupling symmetric phase (II). Since we are interested in dynamical generation of gauge bosons, we confine our computation to the symmetric phase alone.

Solving the equations of motion and eliminating the $A_\mu$ fields, we see that the Lagrangian becomes

$$\mathcal{L} = \frac{N}{G} \left[ (\partial_\mu z)^\dagger (\partial^\mu z) - J_\mu J^\mu + \theta G \epsilon^{\mu\nu\rho} J_\mu \partial_\nu J_\rho + \frac{1}{4} \theta^2 G^2 J_\mu (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) J_\nu \right],$$

where $J^\mu \equiv (1/2i) [z^\dagger \partial_\mu z - (\partial_\mu z)^\dagger z]$ with the constraint $z^\dagger z = 1$. We see that the extra third and fourth terms as well as the original second term are not renormalizable in perturbative expansion. The geometrical implication of the above Lagrangian can be seen in the coadjoint orbit approach for the nonlinear sigma model [7]. In terms of the coadjoint orbit variable

$$Q = -izz^\dagger + i \frac{I}{N}, \quad z^\dagger z = 1,$$

and the topological current density $t^\mu = \epsilon^{\mu\nu\rho} \langle Q \partial_\nu Q \partial_\rho Q \rangle = -\epsilon^{\mu\nu\rho} \partial_\nu J_\rho$, we find that the Lagrangian (1) is equivalent to

$$S_Q = -\frac{N}{G} \int d^3x \left[ \frac{1}{2} \langle \partial_\mu Q \partial^\mu Q \rangle + \frac{1}{4} \theta^2 G^2 t^\mu t_\mu \right] + S_{WZW}.$$  

Here $S_{WZW}$ descends from $M_4$ whose boundary is our 2 + 1 dimensional space-time, and is given by

$$S_{WZW} = -iN\theta \int_{M_4} e^{\mu\nu\rho\sigma} \langle Q \partial_\mu Q \partial_\nu Q \partial_\rho Q \partial_\sigma Q \rangle,$$

where $N\theta$ has to be $k/64\pi$ for integer $k$ in order that $\exp(iS_{WZW})$ to be well-defined [3]. Note that the gauge formulation requires the coefficients of the topological current squared term to be fixed in terms of $\theta$.

The original CP($N$) model is not renormalizable in larger than two dimensions. However the theory may become renormalizable through a resummation of Feynman diagrams in a different way from the coupling perturbation expansion. In fact the $1/N$ expansion provides such a resummation technique and it makes the CP($N$) model in less than four dimensions renormalizable [1,8].

We can rewrite the Lagrangian up to total derivative terms as

$$\mathcal{L} = \frac{N}{G} z^\dagger \left[ -\partial^2 - m^2 - \Gamma \right] z + \frac{N\lambda}{G},$$

where we separate the Goldstone boson mass $m^2$ from $\lambda \equiv m^2 + \tilde{\lambda}$ and $\Gamma$ stands for the interaction vertices:

$$\Gamma \equiv \tilde{\lambda} - iA_\mu (\partial^\mu - \tilde{\partial}^\mu - \theta G \epsilon^{\mu\nu\rho} \tilde{\partial}_\nu \partial_\rho) - A_\mu A^\mu,$$

where $\tilde{\partial}^\mu$ and $\tilde{\partial}_\nu$ do not operate on $A_\mu$. Path integrating $z$ and $z^\dagger$ provides the large $N$ effective action:
\[ S_{\text{eff}} = \int d^2 x \mathcal{L} + i N \text{Tr} \ln \left[ -\partial^2 - m^2 \right] - \frac{1}{N} \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \left[ \frac{1}{-\partial^2 - m^2} \Gamma \right]^n. \] (9)

We divide the effective action up to quadratic terms \((n = 1, 2)\) into two parts, \(S_{\text{eff}} = S' + S''\) where \(S'\) denotes the large \(N\) effective action in the original CP\((N)\) model and \(S''\) stands for the extra induced terms. After some straightforward calculations, we obtain

\[
S' = N \int d^2 x \left[ \frac{1}{G} z^\dagger \left[ -\partial^2 - m^2 - \Gamma^I \right] z + \left( \frac{1}{u} - \frac{1}{u^*} \right) \Lambda (m^2 + \tilde{\lambda}) 
+ \frac{1}{4\pi} m \tilde{\lambda} + \frac{1}{6\pi} m^3 + \frac{1}{2} \tilde{\lambda} \Pi_{\lambda}(i\partial) \tilde{\lambda} - \frac{1}{4} F_{\mu\nu} \Pi_1(i\partial) F^{\mu\nu} \right],
\]

where we have introduced the dimensionless coupling \(u \equiv G \Lambda\) and \(u^* \equiv 2\pi^2\). \(\Gamma^I\) is the interaction vertices in the original CP\((N)\) model without higher derivative terms, and the vacuum polarization functions are given by

\[
\Pi_{\lambda}(p) = \frac{1}{4\pi \sqrt{-p^2}} \arctan \frac{\sqrt{-p^2}}{2m},
\]

\[
\Pi_1(p) = \frac{1}{4\pi^2} \left[ m - \frac{4m^2 - p^2}{2\sqrt{-p^2}} \arctan \frac{\sqrt{-p^2}}{2m} \right],
\]

\[
\Pi_2(p) = \frac{1}{2\pi^2} \Lambda - \frac{3}{8\pi} m + \frac{3}{4} p^2 \Pi_1(p).
\]

We realize that there arise linear divergences in induced Chern-Simons and Maxwell terms which have no counter terms in the original Lagrangian.

Renormalization of the coupling \(G\) can be worked out in the same manner as in the original CP\((N)\) model. The large \(N\) effective potential is defined as the effective action divided by \(\Omega \equiv \int d^2 x\) with \(\tilde{\lambda} \equiv 0\), \(A_\mu \equiv 0\) and \(z(z^\dagger) \equiv 0\). It is given by

\[
\frac{1}{N} V_{\text{eff}} = - \left( \frac{1}{u} - \frac{1}{u^*} \right) \Lambda m^2 - \frac{m^3}{6\pi}.
\]

(15)

The Goldstone boson mass \(m\) is determined as a nontrivial solution to the gap equation \(dV_{\text{eff}}/dm^2 = 0\) and reads

\[
m = 4\pi \Lambda \left( \frac{1}{u^*} - \frac{1}{u} \right).
\]

(16)

We notice that \(m\) can be independent of the ultraviolet cutoff \(\Lambda\) by imposing \(\Lambda\) dependence on the coupling \(u\). In fact the scale invariance condition \(\Lambda dm/d\Lambda = 0\) leads us to the Callan-Symanzik \(\beta\)-function

\[
\beta(u) \equiv \Lambda \frac{du}{d\Lambda} = u \left( 1 - \frac{u}{u^*} \right),
\]

(17)
which shows a nontrivial UV fixed point at \( u = u^* \). In the original CP\((N)\) model the only divergence is the one which arises in the large \( N \) effective action through a tadpole diagram coupled with \( \tilde{\lambda} \) so that the scale invariance condition \( \Lambda d m / d \Lambda = 0 \) is enough to achieve the cutoff independent theory. Since \( m \) is scale invariant, the solution to the gap equation (16) suggests that the renormalization of coupling is given by

\[
\left( \frac{1}{u} - \frac{1}{u^*} \right) \Lambda = \left( \frac{1}{u_R} - \frac{1}{u^*_R} \right) \mu, \tag{18}
\]

where \( u_R \) is the renormalized coupling at a reference energy scale \( \mu \).

In the extended model, however, linear divergences arise in the induced Chern-Simons and Maxwell terms which do not have their counter terms in the classical action. Therefore the higher derivative theory seems to be nonrenormalizable, although the coupling \( u \) can be renormalized in the same way as in the original CP\((N)\) model. However, since the extra linear divergences are always accompanied by the coupling \( G \equiv u / \Lambda \) which cancels the linear divergences, we expect the large \( N \) effective action (11) to be scale invariant in the continuum limit \( \Lambda \to \infty \).

To study this point in more detail, we first look at how \( S_I \) can be scale invariant through the renormalization procedure. The induced kinetic terms of \( A_\mu \) and \( \tilde{\lambda} \) are UV finite in themselves so that we do not need wave function renormalization for them. Then the second term in the right hand side of Eq. (10) becomes UV finite through Eq. (18) from which the \( Z \) factor for the coupling \( G \) can be read

\[
Z^{-1} \equiv \frac{G_R}{G} = 1 - \frac{u_R}{u^*} + \frac{u_R}{u^*} \left( \frac{\Lambda}{\mu} \right). \tag{19}
\]

Here \( G_R \) is connected to the dimensionless coupling \( u_R \) through \( u_R \equiv G_R \mu \). The kinetic term of \( z \) has to be UV finite in itself so that we see

\[
\frac{1}{G} \left[ - \partial^2 - m^2 \right] z = \frac{1}{G_R} \left[ - \partial^2 - m^2 \right] z_R, \tag{20}
\]

where \( z_R \) has been introduced through \( z = Z_z^{1/2} z_R \) and \( Z_z \) is thereby determined as \( Z_z \equiv Z \) in order to cancel the \( Z \) factor from the coupling renormalization. Thus we realize that \( \Gamma' \) in \( S_I \) has to remain invariant through the renormalization procedure. This forces both of \( A_\mu, \tilde{\lambda} \) to be unchanged through renormalization. This is consistent with the UV finiteness of kinetic terms for \( A_\mu \) and \( \tilde{\lambda} \) in \( S_I \).

Now that we have explicitly shown that the original CP\((N)\) model can be renormalized through the \( 1/N \) resummation, let us look at what happens in \( S^H \) if we take the continuum limit \( \Lambda \to \infty \). The Callan-Symanzik \( \beta \)-function (17) tells us that the dimensionless coupling \( u \equiv GA \) goes to the UV fixed point \( u = u^* \) as the cutoff \( \Lambda \) goes to infinity, whereas the bare coupling \( G \equiv u / \Lambda \) reduces to zero. On the other hand, the extra vacuum polarization effect \( \Pi_2 \) in Eq. (14) can be rewritten as

\[
G \Pi_2(p) = \frac{u}{u^*} - G \left[ \frac{3}{8\pi} m - \frac{3}{4} p^2 \Pi_1(p) \right]. \tag{21}
\]

Therefore, we can obtain the UV finite result \( G \Pi_2(p) \to 1 \) as \( \Lambda \to \infty \). In the same reasoning the extra contribution to the Maxwell term which contains \( G^2 \Pi_2(p) \) vanishes in the continuum limit. Moreover, the first term in \( S^H \) has an extra \( G \) after the renormalization (20) and
is thereby suppressed by a factor $1/\Lambda$ as $\Lambda \to \infty$. Thus in the continuum limit $S_{II}^H$ becomes a UV finite Chern-Simons action $-(2N\theta/3) \int d^2x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$ without any ambiguity.

According to power counting of superficial degrees of UV divergences, the three-points function of $A_\mu$ shows a linear divergence. However the extra gauge coupling in $\Gamma$ is invariant under the charge conjugation so that the three points function and its charge conjugation cancel each other in the same way as in the original $\text{CP}(N)$ model [1]. Moreover all $n$-points functions with $n \geq 4$ are UV finite and the contributions from the extra gauge interaction are accompanied by $G$. Therefore in the continuum limit such extra effects are suppressed by a factor $1/\Lambda$ and become irrelevant. Hence we can conclude that the large $N$ effective action in our extended model is renormalizable and the gauge sector is equivalent to the Maxwell-Chern-Simons theory which couples minimally to $z$ field. The effective Lagrangian at the lowest order in the derivative expansion can be rewritten as

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \frac{N}{G_R} \left[(D_\mu z_R)^\dagger (D^\mu z_R) - m^2 z_R^\dagger z_R\right],$$

(22)

where we introduced $e^2 \equiv 24\pi m/N$ and redefined the gauge field by $A_\mu = eA_\mu$ so that the covariant derivative is $D_\mu \equiv \partial_\mu - ieA_\mu$. In this Lagrangian, the Chern-Simons coefficient becomes a dimensionful parameter $\kappa \equiv 32\pi m\theta$.

Note that the ratio of the topological gauge boson mass $\kappa$ [5] to the effective gauge coupling square $e^2$ is proportional to $N\theta$ and is quantized such as $\kappa/e^2 = k/48\pi$. Recently, some authors showed that in the Maxwell-Chern-Simons theory coupled to fermion fields, the Lorentz symmetry is spontaneously broken through a dynamically induced magnetic field when $\kappa/e^2$ is quantized in a unit of $1/2\pi$ [9,10]. Specifically, Ref. [10] introduced $N$-flavored four-components fermion fields and showed that only the Lorentz symmetry is broken in $\kappa = N e^2/2\pi$, whereas both flavor $U(2N)$ and Lorentz symmetries are broken at the same time in $\kappa = N e^2/4\pi$ through a dynamically generated fermion mass and a magnetic field. Our current results provides a geometrical origin of quantization of the Chern-Simons coefficient due to the WZW term. In fact the quantization condition $\kappa = N e^2/2\pi$, $N e^2/4\pi$ correspond to $k = 24N$, $12N$, respectively. We may also possibly prove that the induced Chern-Simons term may not receive any higher order $1/N$ corrections [11].

We have investigated the gauge formulation of higher derivative $\text{CP}(N)$ model in $2+1$ dimensions with WZW term and topological current density squared, and have proved its renormalizability in the large $N$ limit. We also have found that the Maxwell-Chern-Simons theory is dynamically generated in the effective action and the coefficient of the induced Chern-Simons term must be quantized which is a direct consequence of the quantization of the WZW term. If we couple the theory to fermions with $U(2N)$ flavors, the low energy effective theory shows spontaneous break down of the Lorentz symmetry associated with an induced magnetic field when $k = 12N$, $24N$. It would be interesting to check whether this has some physical implications, especially in condensed matter phenomena such as anyon physics and the fractional quantum Hall effect [12].

There exist related subjects to be studied further. One of them is to compute the current algebra associated with (5) and (6), and to check whether it has some special properties at the fixed point $u = u^*$. Another interesting problem is to consider possible extension to higher dimensions. The theory in $2+1$ dimensions was special in the sense that the topological current $t_\mu$ interacts with original gauge field of $\text{CP}(N)$ model. In $D$ dimensional
extension, however, one would need extra $D - 2$ rank antisymmetric tensor fields $B_{\mu_3 \cdots \mu_D}$ with interaction
\[ i\epsilon_{\mu_1 \mu_2 \mu_3 \cdots \mu_D} (\partial_{\mu_1} z) \dagger (\partial_{\mu_2} \bar{z}) B_{\mu_3 \cdots \mu_D} + \frac{1}{2M_{D-4}} B_{\mu_3 \cdots \mu_D} B^{\mu_3 \cdots \mu_D}. \] (23)

This theory which corresponds to the auxiliary field formulation of the dual variable description of SU(2) Yang-Mills theory in the infrared limit [13] for CP(2) case, is not renormalizable in $D \geq 4$ even in $1/N$ expansion. So we have to consider it as an effective theory which describes a massless gauge field interacting with massive $H = dB$ field in the $B^*F$-type interaction. It remains to investigate whether this type of Maxwell-Kalb-Ramond theory [14] has some relevance with quark confinement in 3 + 1 dimensions [15,16].

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