KINEMATIC CONSTRAINTS TO THE KEY INFLATIONARY OBSERVABLES

Mark B. Hoffman$^1$ and Michael S. Turner$^{1,2,3}$

$^1$Department of Physics
The University of Chicago, Chicago, IL 60637-1433

$^2$Department of Astronomy & Astrophysics
Enrico Fermi Institute, The University of Chicago, Chicago, IL 60637-1433

$^3$NASA/Fermilab Astrophysics Center
Fermi National Accelerator Laboratory, Batavia, IL 60510-0500

ABSTRACT

The observables $T/S$ and $n-1$ are key to testing and understanding inflation. ($T$, $S$, and $n-1$ respectively quantify the gravity-wave and density-perturbation contributions to CMB anisotropy and the deviation of the density perturbations from the scale-invariant form.) Absent a standard model, there is no definite prediction for, or relation between, $T/S$ and $n-1$. By reformulating the equations governing inflation we show that models generally predict $T/S \approx -5(n-1)$ or 0, and in particular, if $n > 0.85$, $T/S$ is expected to be $> 10^{-3}$. 
Introduction. Cosmic microwave background (CMB) anisotropy measurements have begun to test inflation, the leading paradigm to extend the standard big-bang cosmology. Within a decade they should test inflation decisively and even probe the underlying physics [1, 2, 3]. Recent results from the BOOMERanG and MAXIMA CMB experiments [4, 5] (as well as results from earlier experiments [6]) confirm the flat Universe predicted by inflation and are beginning to address its second basic prediction: almost scale-invariant adiabatic, Gaussian density perturbations produced by quantum fluctuations during inflation [7]. The third prediction, a nearly scale-invariant spectrum of gravity waves, will be more difficult to confirm, but is a critical probe of inflation [8].

The key inflationary observables are: the level of anisotropy arising from density (scalar) perturbations (quantified by the contribution to the CMB quadrupole anisotropy, \(S\)), the level of anisotropy arising from gravity-wave (tensor) perturbations (\(T\)), and the power-law index \(n\) that characterizes the density perturbations (scale invariance refers to equal amplitude fluctuations in the gravitational potential on all length scales and corresponds to \(n = 1\)). If \(T\), \(S\) and \(n - 1\) can be measured, then the scalar-field potential that drove inflation can be reconstructed [10]. The most promising means of measuring \(T\) is its unique signature in the polarization of CMB anisotropy [9] (however, direct detection by a future space-based experiment should not be dismissed).

While there is no standard model of inflation, all known models can be cast in terms of the classical evolution of a new scalar field \(\phi\) (dubbed the inflaton) [13]. Predictions for \(S\), \(T\) and \(n - 1\) can be expressed in terms of the scalar-field potential \(V(\phi)\) and its first two derivatives. While there is a model-independent relation between \(T/S\) and the power-law index \(n_T\) that characterizes the gravity-wave spectrum, \(T/S = -5n_T\) [11, 12], no such relation for \(n\) and \(T/S\) exists [14].

This is unfortunate because \(n_T\) is very difficult to measure, and \(n\) will be measured to a precision of better than 1% by the MAP and PLANCK experiments (BOOMERanG and MAXIMA have already determined that \(n = 1 \pm 0.06\)). Even an approximate or generic relation between \((n - 1)\) and \(T/S\) would be valuable, both as a test of inflation and as a guide for the expected level of gravity waves when \(n\) is measured.

The formation of large-scale structure and CMB measurements already indicate that a significant part of CMB anisotropy arises from scalar perturbations (\(T/S\) cannot be \(\gg 1\)). On the other hand, nothing precludes \(T/S \ll 1\), and if \(T/S\) is much less than \(10^{-3}\), the prospects for measuring \(T\) are poor [9] (one inflation theorist has opined that \(T/S \ll 1\) for all reasonable models [15]).

The goal of this work is to provide objective theoretical guidance. By casting the equations governing inflation in a form that is essentially independent of the inflaton potential (“flow equations” for \(T/S\) and \(n - 1\)), we show that the \(T/S - (n - 1)\) plane is not uniformly populated by models of inflation: The lines \(T/S \approx 0\) and \(T/S \approx -5(n - 1)\) act as attractors for models that are consistent with the equations governing inflation. For \(n < 1\), there is an excluded region between these two attractors; for \(n > 1\), other values for \(T/S\) and \(n - 1\) are possible, but at the expense of a spectrum of density perturbations that is poorly represented by a power law. (the CMB will be able to test how well a power law describes the density perturbations.)
**Flow equations.** The kinematic equations that govern inflation are well known [16]

\[
\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{1}
\]

\[
H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3m_{\text{Pl}}^2} \left[V(\phi) + \frac{1}{2}\dot{\phi}^2\right] \tag{2}
\]

where \(a(t)\) is the cosmic scale factor, prime denotes \(d/d\phi\), and overdot denotes \(d/dt\). During inflation \(\phi\) rolls slowly and the \(\dddot{\phi}\) term in its equation of motion and its kinetic term in the Friedmann equation can be neglected [16, 17], so that

\[
\dot{\phi} \simeq -\frac{V'}{3H} \tag{3}
\]

\[
N(\phi) \equiv \int_{\phi}^{\phi_{\text{end}}} H dt \simeq -\frac{8\pi}{m_{\text{Pl}}^2} \int_{\phi}^{\phi_{\text{end}}} d\phi \frac{1}{x(\phi)} \tag{4}
\]

where \(x(\phi) \equiv V'(\phi)/V(\phi)\) measures the steepness of the potential and \(N(\phi)\), the number of e-folds before the end of inflation, is the natural time variable. Inflation ends when the slow-roll conditions,

\[
m_{\text{Pl}}V'/V = m_{\text{Pl}}x \ < \sqrt{48\pi} \tag{5}
\]

\[
m_{\text{Pl}}^2V''/V = m_{\text{Pl}}^2(x' + x^2) \ < \ 24\pi \tag{6}
\]

are violated (at \(\phi = \phi_{\text{end}}\)) [16, 17].

The inflationary observables are related to the same quantities that govern the kinematics of inflation [12]

\[
(n - 1) = \frac{m_{\text{Pl}}^2}{8\pi} \left[2x' - x^2\right] \tag{7}
\]

\[
\frac{T}{S} = \frac{5m_{\text{Pl}}^2}{8\pi} x^2 \tag{8}
\]

\[
T = 0.6V/m_{\text{Pl}}^4 \tag{9}
\]

These expressions are given to lowest order in \(x^2\) and \(x'\) (see Ref. [18] for higher-order corrections). Note, \(n - 1\) is only equal to \(n_T = -5(T/S)\) if \(x' = 0\).

By combining the slow-roll equations with those governing \((n - 1)\) and \(T/S\), we can write equations that govern the inflationary observables (almost) without reference to a model,

\[
\frac{d(T/S)}{dN} = (n - 1)\frac{T}{S} + \frac{1}{5}\left(\frac{T}{S}\right)^2 \tag{10}
\]

\[
\frac{d(n - 1)}{dN} = -\frac{1}{5}(n - 1)\frac{T}{S} - \frac{1}{25}\left(\frac{T}{S}\right)^2 \pm \frac{m_{\text{Pl}}^3}{16\pi^2} \sqrt{\frac{2\pi T}{5S}} x'' \tag{11}
\]

where the sign of the last term matches that of \(V'\).

We call these “flow equations” as they describe the trajectory in the \(T/S - (n - 1)\) plane during inflation. Because of the \(x''\) term they are not completely independent of the potential.
To “close” the flow equations we will assume that the potential is smooth enough so that we can treat $x''$ as being approximately constant. For sufficiently smooth and featureless potentials $x''$ will also be small.

Finally, one might wonder what happened to the most stringent constraint on inflation: achieving density perturbations of amplitude $10^{-5}$ or so ($S \sim 10^{-10}$). The flow equations involve the quantities $T/S$, $(n - 1)$ and $dN/d\phi$ which are unaffected by a rescaling of the potential, $V \rightarrow aV$. This rescaling changes $S$: $S \rightarrow aS$. Thus, any potential can be rescaled to give proper size density perturbations without affecting the flow equations.

*Trajectories and attractors.* The scales relevant for structure formation (1 Mpc to $10^4$ Mpc) crossed outside the horizon roughly $50 \, e$-folds before the end of inflation (i.e., when $N = 50$) [16], and so it is $T/S$ and $(n - 1)$ at this time that can be measured by CMB experiments. We find them by evolving $T/S$ and $(n - 1)$ until inflation ends and counting back 50 $e$-folds. To determine when inflation ends, we recast the slow-roll conditions (5,6):

$$\frac{T}{S} < 30$$  \hspace{1cm} (12)

$$\left| (n - 1) + \frac{3T}{5S} \right| < 6$$  \hspace{1cm} (13)

To be specific, we pick “initial” values in the range, $0 < T/S < 10$ and $-0.5 < (n - 1) < 0.5$, and then integrate with fixed $x''$ until one of the slow-roll conditions is violated, signaling the end of inflation. We then count back 50 $e$-folds to find $(T/S)_{50}$ and $(n - 1)_{50}$. Some trajectories are shown in Fig. 1.

Figs. 2 and 3 summarize the $(T/S)_{50} - (n - 1)_{50}$ phase space generated from the range of initial conditions considered. The $(T/S)_{50} - (n - 1)_{50}$ plane is not uniformly populated. For $x'' < \mathcal{O}(1)$, solutions cluster around two attractors, $(T/S)_{50} \approx 0$ and $(T/S)_{50} \approx -5(n - 1)_{50}$, and for $(n - 1)_{50} < 0$, there is an excluded region between the two attractor lines, which cannot be reached for any value of $x''$. We call the region between the excluded area and $x'' = 3$ as the favored region for the inflationary observables $n - 1$ and $T/S$.

Taking $x'' = 0$ it is simple to show how the attractors arise. In this limit, the flow equations are: $s \equiv (n - 1) - \frac{1}{5S} = \text{const}$ and $r \propto \exp(sN)$, where $r = T/S$. Unless $r$ and/or $s$ are small, corresponding to the attractor solutions, $r$ grows very rapidly and inflation does not last 50 $e$-folds.

For $n > 1$, values of $(T/S)$ and $(n - 1)$ outside the favored region are possible at the expense of large $x''$. Models with a large $x''$ have density-perturbation spectra which are not well represented by a power law: the running of the spectral index [20], $dn/d\ln k = -dn/dN$ includes the term

$$\frac{dn}{d\ln k} = \cdots \pm \frac{m_{Pl}^3}{16\pi^2} \left( \frac{2\pi T}{5S} \right)^{x''}$$  \hspace{1cm} (14)

which becomes large for large $x''$. This explains the results of a recent paper [21] in which models with $n$ as large as 2 were constructed. In particular, for the model with $n = 2$, $x'' \approx 2000$, $T/S \approx 3 \times 10^{-3}$ and $dn/d\ln k \approx 0.3$.

Fig. 3 shows the $T/S \approx 0$ attractor region in more detail. As $n$ increases toward unity, values of $T/S$ in the favored region grow, making the prospects for measuring $T$ better; in particular, for $n > 0.85$, $T/S > 10^{-3}$. This figure also confirms that almost, but not exactly,
Figure 1: Trajectories in the $T/S - (n-1)$ plane. Squares indicate the initial choices for $T/S$ and $(n-1)$; circles indicate the values 50 e-folds before the end of inflation. A trajectory ends when $T/S$ and/or $|n-1|$ become large; most of inflation occurs when $T/S$ and $|n-1|$ are small. The upper left panel shows a complete trajectory, with ticks indicating e-folds before the end of inflation (from the circle, 50, 49, $\cdots$, 1). The other three panels show trajectories in more detail. Note how $T/S$ and $(n-1)$ outside the attractor region are “pulled in” (the attractors are shown as broken lines and the boundary of the excluded region is a solid curve).
Figure 2: Summary of our model search using the flow equations. The lines $T/S = -5(n-1)$ and $T/S = 0$ act as attractors; the dotted curves correspond to $x'' = 0, 1, 2, 5$ (from left to right). We found no model in the excluded region, and we call the region between it and the curve $x'' = 3$ the favored region. Models outside the favored region (upper right part) have large $dn/d\ln k$ and density perturbations that are not well represented by a power law. Diamonds indicate various known inflationary models: chaotic, $V(\phi) = \lambda \phi^n$ for $n = 2, 3, \cdots$ (diamonds on the diagonal); new inflation ($n = 0.94$) and natural inflation (with $n = 0.84$). The ellipse is the $2\sigma$ error ellipse “forecasted” for the PLANCK satellite [19].
scale-invariant density perturbations are a generic prediction of inflation [7, 17]: the favored
region just touches $n = 1$ (for $x'' \simeq 1.5$). Finally, in the disfavored region where $T/S \ll 1$,
large $x''$ does not imply a poor power law because $dn/d\ln k$ is proportional to $T/S$. Indeed
one of the models in this region is new inflation.

So far, we have only considered one-field inflation. There are potentials that are so flat
and smooth that the slow-roll conditions never break down; the most well known of these
is power-law inflation, $V(\phi) \propto \exp(-\beta \phi/m_{Pl})$. In a “never ending” model, another field
causes the slow-roll conditions to break down (e.g., by classical evolution in hybrid inflation
or a phase transition in extended inflation). The flow equations can also be applied to such
models.

In our framework two-field models are models that would inflate forever on their own.
We find such models when the right hand sides of Eqs. (10,11) vanish prior to violating the
slow-roll conditions. When this happens, we obtain fixed points in the $T/S - (n - 1)$ plane,
which are the most likely values for $T/S$ and $(n - 1)$ 50 e-folds prior to when the second field
ends inflation. These points, shown in Fig. 4, populate the region $T/S \approx 0$ and for $n > 1$
along with the attractor line, $T/S = -5(n - 1)$.

It is also possible that a self-ending model has an auxiliary field that ends inflation “early”.
We treat this possibility by populating the $(T/S)_{50} - (n - 1)_{50}$ plane with the values of $T/S$
and $(n - 1)$ at $N > 50$ for all one-field models. We find that the two-field models behave
similarly to the one-field models. The only significant difference is that two-field models
extend the $(T/S)_{50} = 0$ attractor to $(n - 1)_{50} > 0$ (see Fig. 4).

Finally, what about our taking $x'' \simeq$ constant? It can affect the relationship between
the initial and final values of $n - 1$ and $T/S$ if $x''$ is large, since $x''$ need not be constant (as
is the case in some known models). Since we have covered a wide range of initial values we
would expect that this fact would only slightly modify the $(n - 1)_{50} - (T/S)_{50}$ phase space;
indeed, we have also formulated the flow equations assuming $V'''/V =$ constant and obtain
similar results. Further, in the favored part of the $(n - 1)_{50} - (T/S)_{50}$ plane, $x''$ is small.

Discussion. Prior to this work there was one guiding relation for the inflationary ob-
servables: $T/S = -5nT$. It has the virtue of exactitude and can test the consistency of the
scalar-field inflationary framework, but it involves the power-law index of the gravity wave
perturbations, the most difficult inflationary observable to measure. By reformulating the
equations governing inflation, we have found generic relations between $T/S$ and $(n - 1)$:
Inflationary kinematics constrain models to cluster along the lines $T/S = -5(n - 1)$ and
$T/S = 0$, with a forbidden region between these two lines for $n < 1$. Large $n - 1$ is possible,
but at the expense of a poor power-law for the density perturbations (i.e., large $dn/d\ln k$),
unless $T/S$ is very small. Further, our results support the view that inflation generically
predicts almost, but not exactly scale-invariant density perturbations [17].

These results provide practical guidance to CMB experimenters and additional tests for
inflation. For example, if $n$ is found to be significantly greater than 1, then a poor power-law
is also expected unless $T/S \ll 1$. If $n$ is found to be $> 0.85$, then $T/S > 10^{-3}$ is likely, which
would makes prospects for detecting the gravity-wave of inflation signature more favorable.
Figure 3: Same as Fig. 2, except with a logarithmic scale for $T/S$ to show more detail. As $n \to 1$, $T/S$ increases in the favored region. Below and to the left of the broken line ($dn/d\ln k = 10^{-2}$), a poor power law does not occur for large $x''$ because $dn/d\ln k \propto \sqrt{T/S x''}$ and $T/S$ is small.
Figure 4: Summary of two-field models. The filled circles represent the values of $(T/S)_{50}$ and $n_{50}$ for the corresponding one-field models, and the attached curves are the values obtained for inflation ending early due to an auxiliary field. The dashed lines represent fixed points in the $(T/S) - n$ plane that result from models that do not end without an auxiliary field. In general, two-field models populate the same region as one-field models and extend the $T/S \approx 0$ attractor to $n > 1$. 
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References


[12] The precise relationship between $S$ and $T$ and the inflaton potential depends upon the cosmological model; see M.S. Turner and M. White, Phys. Rev. D 53, 6822 (1996). The values used in this paper are for $\Omega_M = 0.35$ and $\Omega_\Lambda = 0.65$. For $\Omega_M = 1$, the relation is the more familiar $T/S = -7n_T$.


[14] It had been conjectured that for many models $T/S \approx -5(n-1)$, though there are many exceptions to this; see R. Davis, Phys. Rev. Lett. 69, 1856 (1992) and V.F. Mukhanov, L. Wang and P.J. Steinhardt, Phys. Lett. B 414, 18 (1997).


