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DIFFEOMORPHISMS ON S^1 , PROJECTIVE STRUCTURES AND INTEGRABLE SYSTEMS

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Dedicated to Professor Martin Kruskal on his 75th birthday with great respect and admiration

Abstract

If u_i s are periodic function on the line, the operator $\frac{d^n}{dx^n} + u_{n-1}\frac{d^{n-1}}{dx^{n-1}} + u_{n-2}\frac{d^{n-2}}{dx^{n-2}} + \dots + u_1\frac{d}{dx} + u_0$, acting on periodic functions, is called a Adler-Gelfand-Dikii (or AGD) operator. In this paper we consider a projective connection as defined by this n -th order operator on the circle. It is well known that the KdV equation is the avatar of a scalar Lax equation defined by a Lax pair of scalar n th order differential (AGD) operator. In this paper we derive (formally) the Korteweg-de Vries (KdV) as an evolution equation of the AGD operator (at least for $n \leq 4$) under the action of $Vect(S^1)$. The solutions of the AGD operator defines an immersion $\mathbf{R} \rightarrow \mathbb{R}P^{n-1}$ in homogeneous coordinates. In this paper we derive the Schwarzian KdV equation as an evolution of the solution curve associated to $\Delta^{(n)}$, for $n \leq 4$. We also establish a connection between the projective vector field, a vector field leaves fixed a given (extended) projective connection, and the C. Neumann system using the idea of Knörrer and Moser. We show that certain quadratic function of a projective field satisfy the C. Neumann system.

1 Introduction

The space of linear differential operators on a manifold M considered as a module over the group of diffeomorphisms is a well known classical text. This space has various algebraic structures, e.g. the structure of an associative algebra and of a Lie algebra [7]. In the one dimensional case, this was studied by E. Wilczynski [32] and E. Cartan [6].

The space of n th order differential operators or Adler-Gelfand-Dikii (AGD) space [1, 8] is connected to the $gl(n, R)$ current algebra, i.e. an algebra of loops $C^\infty(S^1, gl(n, R))$. This automatically reduces to the $sl(n, R)$ current algebra when $u_{n-1} = 0$. Gelfand and Dikii [8] established the relation between dual spaces of Kac-Moody algebras on the circle and the AGD space. The later is a Poisson subspace of the former [23]. They coincide only for $n = 2$.

This space has an interesting structure, and this has been studied in various directions. In this paper we will focus mainly from the point view of projective connections on the circle [12, 14].

Projective connections on the circle were classified from a geometric point of view by Kuiper [20]. Lazutkin and Pankratova [21] were the first to formulate this analytically. A projective connection on the circle is a linear second order differential operator, $\frac{d^2}{dx^2} + u(x)$, which acts on a periodic functions, known as Hill's operator. This is a dual space of the Virasoro algebra. It has a sequence of eigenvalues tending to infinity. If the eigenvalues are fixed, then the possible functions u form an infinite dimensional torus. One can imagine this to be a product of one circle for each pair of consecutive eigenvalues. The Korteweg-de Vries equation evolves in a straight line on one of the above tori.

The connection between the geodesic flow on the Virasoro-Bott group and the periodic KdV equation follows from the work of Arnold [4], Kirillov [15, 17, 18], Segal [30, 31], Ovsienko and Khesin [29] and Witten [35]. The connection between the C. Neumann system and the Bott-Virasoro group has been studied in [10, 11, 12].

It is well known that the KdV equation is the canonical example of a scalar Lax equation, which is an equation defined by a Lax pair of scalar differential operators

$$\frac{d\Delta}{dt} = [P, \Delta]$$

where

$$\Delta = \frac{d^n}{dx^n} + u_{n-2} \frac{d^{n-1}}{dx^{n-2}} + \cdots + u_0.$$

Here P is a differential operator whose coefficients are differential polynomial in the variables, essentially determined by the requirement that $[P, L]$ be an operator of order less than n .

Earlier, physicists have studied the AGD operators and its connection to extended classical conformal algebras [5, 24]. Pierre Mathieu [24] has listed several extended conformal operators. These are denoted by $\Delta^{(n)}$, and some of members of this family are:

$$\Delta^{(0)} = 1,$$

$$\Delta^{(1)} = \partial_x,$$

$$\Delta^{(2)} = \partial_x^2 + u,$$

$$\Delta^{(3)} = \partial_x^3 + 4u\partial_x + 2u',$$

$$\Delta^{(4)} = \partial_x^4 + 9u^2 + 3u'' + 10u'\partial_x + 10u\partial_x^2,$$

etc. etc.

They define the Poisson (Gelfand-Dikii) bracket of the spin- k field [5]. All these scalar differential operators are covariant operators, in other words, these transform covariantly under the coadjoint action of $Diff(S^1)$. As shown by Mathieu at least for $n \leq 4$, these operators $\Delta^{(n)}$ depend only on u and its derivatives.

We will take a slightly different route [12]. We will analyse the evolution equation defined by the action of a vector field $Vect(S^1)$ on a scalar differential operator. This gives rise to a KdV flow again. The solution of $\Delta^{(n)}$ defines an immersion in homogeneous coordinates [29, 9]. In this paper we establish a connection between this KdV flow and an evolution of an solution curve associated to the immersion. We know that there exists a projective action of $SL(n, \mathbf{R})$ on $\mathbb{R}P^{n-1}$, induced by the action of $SL(n, \mathbf{R})$ on $\mathbf{R}^n$. We will see that the evolution of the solution curve of Δ is invariant under the $SL(n, \mathbf{R})$ action. Earlier, Mari Beffa et. al. [9] found an explicit formula for the most general vector evolution of curves on $\mathbb{R}P^{n-1}$. Our approach is different; it gives an elegant presentation of the Schwarzian KdV [27] ($SL(2, \mathbf{R})$ invariant KdV) equation, also called Ur-KdV [25, 33, 34], as an evolution equation of the space curve connected to the operator $\Delta^{(n)}$.

Organization of the paper:

This paper is organized as follows:

In the next section, we present a brief description of the AGD space and the smooth orientation preserving immersion given by the solution of the equation involved. Section 3 is devoted to projective connection on the circle. We will describe the action of $Diff(S^1)$ on the Adler-Gelfand-Dikii manifold in section 4. In section 5 we will derive the KdV equation, which is an evolution equation defined by the action of $Vect(S^1)$ on the space of n th order linear differential operators. We derive the evolution of the solution curve in section 6. Section 7 is devoted to the stabilizer set of the action of $Vect(S^1)$ on the space of Adler-Gelfand-Dikii manifolds and its connection to C. Neumann system. In section 8 we discuss some properties of Kirillov's super algebra. This is connected to the solution of C. Neumann systems [19, 26].

Results of the paper:

In this paper we argue that the action of $Vect(S^1)$ on the AGD manifold, $[\mathcal{L}_v, \Delta^{(n)}] = \mathcal{L}_v \circ \Delta^{(n)} - \Delta^{(n)} \circ \mathcal{L}_v$, (notation will be explained later) can be considered as a coadjoint action.

The action of $Vect(S^1) = T_c Diff^+(S^1)$ on the Adler-Gelfand-Dikii operator $\Delta^{(n)}$ (for $n \leq 4$) yields an evolution equation

$$\frac{\partial \Delta^{(n)}}{\partial t} + [\mathcal{L}_v, \Delta^{(n)}] = 0.$$

This generates a nonlinear evolution equation for periodic functions u

$$\frac{\partial u}{\partial t} = \frac{1}{2}f''' + 2f'u + fu'.$$

Theorem 1.1 *The Euler-Arnold equation gives the KdV equation, it is a Hamiltonian flow on the "coadjoint orbits" in $\Delta^{(n)}$ for the Hamiltonian function $H(u) = \frac{1}{2}u^2$.*

Theorem 1.2 *The solutions of an operator $\Delta^{(n)}$ define an immersion*

$$\mathbf{R} \longrightarrow \mathbb{R}P^{n-1}$$

in homogeneous coordinates. The evolution equation of the solution curve is a Hamiltonian flow, and it is given by the Schwarzian KdV equation.

Theorem 1.3 *Let $v = f(x)\frac{d}{dx} \in Vect(S^1) = T_c Diff^+(S^1)$ be a vector field, this acts on $\Delta^{(n)}$ by*

$$[\mathcal{L}_v, \Delta^{(n)}] := \mathcal{L}_v^{-(n+1)/2} \circ \Delta^{(n)} - \Delta^{(n)} \circ \mathcal{L}_v^{(n-1)/2},$$

where $\mathcal{L}_v$ is the Lie derivative with respect to $v = f(x)\frac{d}{dx}$ vector field.

(a) The stabilizer set of the action of $f(x)\frac{d}{dx}$ on the space of Adler-Gelfand-Dikii operators $\Delta^{(n)}$ is a projective vector and satisfies

$$f''' + 4f'u + 2fu' = 0.$$

(b) If $f = \langle \chi, A^{-1}\chi \rangle$, then χ satisfies

$$\chi'' = -A\chi - u\chi$$

where $\langle \chi, \chi \rangle = 1$ and

$$u = -\langle \chi, A\chi \rangle + \langle \chi', \chi' \rangle$$

which is the system of C. Neumann equation.

(c) Let f be a projective vector field, the square root of f is the element of scalar density of weight $-\frac{1}{2}$. Then the pair satisfies Lie super algebra.

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2 AGD Space

In late seventies, Adler [1] defined a family of second Hamiltonian structures with respect to which generalized KdV equation can be written as Hamiltonian system. The Adler-Gelfand-Dikii brackets are defined on the space of n -th order scalar differential operators of the form

$$\mathcal{S} = \left\{ \frac{d^n}{dx^n} + u_{n-1} \frac{d^{n-1}}{dx^{n-1}} + u_{n-2} \frac{d^{n-2}}{dx^{n-2}} + \cdots + u_1 \frac{d}{dx} + u_0 \right\}, \quad (1)$$

where the coefficients $u_i \in C^\infty(S^1)$ are smooth and periodic. This space of differential operators are called Adler-Gelfand-Dikii (AGD) manifold. It is known that the extended classical conformal algebras, that is, $n > 2$ spin algebras are obtained from the second Hamiltonian structure of Lax equations for a Lax operators of type (1).

We realize (the regular part of) the dual space to $\mathcal{S}$ as follows:

$$\mathcal{S}^* = \left\{ l = \sum_{m=1}^n l_m \partial^{-m} \mid l_m \in C^\infty(S^1) \right\},$$

and the pairing $\mathcal{S} \otimes \mathcal{S}^* \rightarrow \mathbf{R}$ is given by the formula

$$\langle \Delta^{(n)}, l \rangle = \int_{S^1} \text{Res}(\Delta^{(n)} l) dx, \quad (2)$$

where $\text{Res}(X)$ is the coefficient to ∂^{-1} in a pseudodifferential operator X .

In the Adler-Gelfand-Dickey formalism, we assign a vector field V_X on $\mathcal{S}$ to every regular linear functional X . Its value at a point $\Delta^{(n)} \in \mathcal{S}$ is:

$$V_{X_f}(\Delta^{(n)}) = \Delta^{(n)}(X_f \Delta^{(n)})_+ - \Delta^{(n)}(X_f)_+ \Delta^{(n)}. \quad (3)$$

the positive sign denotes the projection to differential part.

Thus we define a Lie-Poisson bracket on the space of smooth functions on $\mathcal{S}$:

$$\{f, g\}_{\Delta^{(n)}} = \langle dg|_{\Delta^{(n)}}, V_{df}|_{\Delta^{(n)}}(\Delta^{(n)}) \rangle. \quad (4)$$

This bracket is called the Gelfand-Dickey bracket.

2.1 Immersion and Solution curve

At first we will consider the $n = 2$ case. In this case, $\mathcal{S}$ is the space of Hill's operators of the form

$$\frac{\partial^2}{\partial x^2} + u.$$

Lemma 2.1 *There is a one to one correspondence between (1) the Hill's equation on S^1*

$$\Delta\psi = \psi'' + u\psi = 0,$$

where $u \in C^\infty(S^1)$ and ψ is the unknown function.

(2) smooth orientation preserving immersions $g : S^1 \rightarrow \mathbb{RP}^1$, modulo the equivalence upto $PSL(2, \mathbf{R})$.

This proof is very easy, it says that if we choose two independent solutions ψ_1 and ψ_2 , then

$$x \longmapsto (\psi_1(x), \psi_2(x)) \quad (5)$$

defines an immersion $\mathbf{R} \longrightarrow \mathbb{R}P^1$ in homogeneous coordinates. This defines a curve in the projective line $\mathbb{R}P^1$. Since the Wronskian of the solution curve is constant upto multiplication by a matrix in $SL(2, \mathbf{R})$, then the Wronskian $\psi_1' \psi_2 - \psi_1 \psi_2'$ of any immersion can be written in a form (5) equals one.

□

This picture can be easily extended to the case of n -th order scalar differential operator. The Adler-Gelfand-Dickey manifold is an infinite dimensional Frechet manifold of scalar differential operators with smooth and periodic coefficients. Associating to the equation $\Delta\psi = 0$ we define n independent solutions $(\psi_1, \psi_2, \dots, \psi_n)$. The map

$$x \longmapsto (\psi_1(x), \psi_2(x), \dots, \psi_n(x)) \quad (6)$$

defines an immersion

$$g : \mathbf{R} \longrightarrow \mathbb{R}P^{n-1}$$

in homogeneous coordinates. Thus we obtain a solution curve associated to L , once again the Wronskian of the components equals one. Since coefficients are periodic, hence, if $\psi(x)$ is a solution, then $\psi(x + 2\pi)$ is also a solution. This implies

$$\psi(x + 2\pi) = M_\psi \psi(x),$$

where

$$M_\psi = \psi(2\pi)\psi(0)^{-1}$$

is a monodromy matrix. This matrix preserves the skew form given by the Wronskian, so $\det(M_\psi) = 1$, i.e. $M_\psi \in SL(n, \mathbf{R})$. If one chooses a different solution curve then the new monodromy matrix will appear, this will be the conjugate of M_ψ by an element of $SL(n, \mathbf{R})$. This means that for each Lax operator we can associate a projective curve whose monodromy will be an element of the conjugacy class $[M_\psi]$. This curve is unique up to the projective action of $SL(n, \mathbf{R})$.

3 Projective Structures

Let Ω denote the cotangent bundle of the circle, this is a trivial real line bundle on S^1 . Let Ω^{-1} and Ω^m be the tangent bundle and the m -fold tensor product of Ω . The section of Ω^m is locally given by

$$s = g(x)dx^m,$$

where $g(x) = g(x + 2\pi)$. There is a natural action of $Diff(S^1)$ on the sections of Ω^m . We will compute the infinitesimal action of $Vect(S^1)$ on Ω^m . Let $h_\epsilon(x)$ be a one-parameter family of diffeomorphisms of the circle, such that

$$h_\epsilon(x) = x + \epsilon f + O(\epsilon^2).$$

Hence,

$$\begin{aligned} g(h_\epsilon(x))(d(h_\epsilon(x)))^{\otimes m} &= g(x + \epsilon f + O(\epsilon^2))d(x + \epsilon f + O(\epsilon^2))^{\otimes m} \\ &= (g(x) + \epsilon g_x f + O(\epsilon^2))(dx + \epsilon f_x dx + O(\epsilon^2))^{\otimes m} \\ &= g(x)dx^{\otimes m} + \epsilon(g_x f + m f_x g)dx^{\otimes m} + O(\epsilon^2). \end{aligned}$$

Therefore,

$$\delta g = g_x f + m f_x g dx^{\otimes m}.$$

We will express this action by

$$\mathcal{L}_v s = (f g' + m f' g) dx^m, \quad (7)$$

where $\mathcal{L}_v$ is the Lie derivative of the vector field $v = f \frac{d}{dx}$. Suppose $w = g \frac{d}{dx} \in \Omega^{-1}$, hence we get

$$\mathcal{L}_v w = [v, w] = (g f' - g' f) \frac{d}{dx}. \quad (8)$$

3.1 Projective Connection on the Circle

Let us denote $\Omega^{\pm 1/2}$ be the square root of the tangent and cotangent bundle respectively.

Definition 3.1 (11) *A projective connection on the circle is a linear second order differential operator*

$$\Delta : \Gamma(\Omega^{-\frac{1}{2}}) \longrightarrow \Gamma(\Omega^{\frac{3}{2}})$$

such that the symbol of

(1) Δ is the identity and

$$(2) \quad \int_{S^1} (\Delta s_1) s_2 = \int_{S^1} s_1 (\Delta s_2)$$

for all $s_i \in \Gamma(\Omega^{-\frac{1}{2}})$.

Let us take $s = \psi(x) dx^{-\frac{1}{2}} \in \Gamma(\Omega^{-\frac{1}{2}})$, then $\Delta s \in \Gamma(\Omega^{\frac{3}{2}})$ is locally described by

$$\Delta s = (a\psi'' + b\psi' + c\psi) dx^{\frac{3}{2}}.$$

As discussed in [6], any differential equation of the form

$$\frac{d^2 y}{dx^2} = p_3 \left(\frac{dy}{dx} \right)^3 + p_2 \left(\frac{dy}{dx} \right)^2 + p_1 \left(\frac{dy}{dx} \right) + p_0$$

defines a projective structure.

From the definition of the projective connection condition (1) implies $a = 1$ and condition (2) implies $b = 0$, hence projective connection can be identified with the Hill operator

$$\Delta^{(2)} \equiv \Delta = \frac{d^2}{dx^2} + u(x).$$

Using the equation $\mathcal{L}_v \Delta^{(2)} = \Delta^{(2)} \mathcal{L}_v$, we obtain

Proposition 3.2 *A projective vector field $v = f \frac{d}{dx} \in \Gamma(\Omega^{-1})$ satisfies*

$$f''' + 4f'u + 2fu' = 0.$$

3.2 Extended Projective Connections

In this section we will explicitly compute the equations satisfied by projective vector fields for different values of n .

Definition 3.3 (Extended Projective Connection) *An extended projective connection on the circle is a class of differential (conformal) operators*

$$\Delta^{(n)} : \Gamma(\Omega^{-\frac{n-1}{2}}) \longrightarrow \Gamma(\Omega^{\frac{n+1}{2}})$$

such that

(1) the symbol of $\Delta^{(n)}$ is the identity,

$$(2) \int_{S^1} (\Delta^{(n)} s_1) s_2 = \int_{S^1} s_1 (\Delta^{(n)} s_2)$$

for all $s_i \in \Gamma(\Omega^{-\frac{n-1}{2}})$.

It is known that the symbol of a n -th order operator from a vector bundle U to V is a section of $\text{Hom}(U, W \otimes \text{Sym}^n T)$, where

$$U = \Omega^{-\frac{(n-1)}{2}} \quad V = \Omega^{\frac{n+1}{2}}.$$

Since $T = \Omega^{-1}$, hence we get

$$V \otimes \text{Sym}^n T \cong U,$$

giving an invariant meaning to the first condition.

If $s_2 \in \Gamma(\Omega^{-\frac{n-1}{2}})$ then $s_1 \Delta^{(n)} s_2 \in \Gamma(\Omega)$ is a one form to integrate.

The consequence of the *first condition* is that all the differential operators are monic, that is the coefficient of the highest derivative is always one, and the *second condition* says that the term $u_{n-1} = 0$.

The weights $\frac{-(n-1)}{2}$ and $\frac{(n+1)}{2}$ related to the space of operator $\Delta^{(n)}$ is known to physicists and mathematicians [13], but not from the point view of extended projective connections.

Definition 3.4 *A vector field is called projective vector field which keeps fixed a given projective connection Δ*

$$\mathcal{L}_v \Delta^{(n)} s = \Delta^{(n)} (\mathcal{L}_v s), \tag{9}$$

for all $s \in \Gamma(\Omega^{-\frac{n-1}{2}})$.

n = 3:

In this case the projective connection is defined by

$$\Delta^{(3)} = \partial_x^3 + 4u\partial_x + 2u',$$

and $s \in \Gamma(\Omega^{-1})$.

After computing

$$\mathcal{L}_v \Delta^{(3)} = \Delta^{(3)} \mathcal{L}_v, \quad (10)$$

we obtain,

$$f((\phi''' + 4\phi'u + 2\phi u')' + 2f'(\phi''' + 4\phi'u + 2\phi u')) = (\partial_x^3 + 4u\partial_x + 2u')(f\phi' - f'\phi).$$

This yields the equation for v

$$(f''' + 4f'u + 2fu')'\phi + (f''' + 4f'u + 2fu')\phi' = 0. \quad (11)$$

Therefore, we say

Proposition 3.5 *A projective vector field $v = f \frac{d}{dx} \in \Gamma(\Omega^{-1})$ corresponding to $\Delta^{(3)}$ also satisfies*

$$(f''' + 4f'u + 2fu') = 0. \quad (12)$$

This only shows that the expression (12) defines a linear skew-symmetric third order operator from $\Gamma(\Omega^{-1})$ to $\Gamma(\Omega^2)$

n = 4:

The 4-th order conformal operator is given by

$$\Delta^{(4)} = \partial_x^4 + 9u^2 + 3u'' + 10u'\partial_x + 10u\partial_x^2, \quad (13)$$

and

$$\Delta^{(4)} : \Gamma(\Omega^{-\frac{3}{2}}) \longrightarrow \Gamma(\Omega^{\frac{5}{2}}).$$

In this case,

$$\mathcal{L}_v s = (f\phi' - \frac{3}{2}f'\phi)dx^{-\frac{3}{2}}. \quad (14)$$

Thus the equation

$$\mathcal{L}_v(\Delta^{(4)}s) = \Delta^{(4)}\mathcal{L}_v s \quad (15)$$

yields the equation for v in this case

$$(f'''' + 10uf''' + 10u'f'' + 8u''f' + 24u^2f' + 12uu'f + 2fu''')\phi + 5(f''' + 4f'u + 2fu')'\phi' + (f''' + 4f'u + 2fu')\phi'' = 0.$$

Thus we obtain

Proposition 3.6 A projective vector field $v = f \frac{d}{dx}$ corresponding to 4-th order conformal operator $\Delta^{(4)}$ satisfies

$$f'''' + 10uf''' + 10u'f'' + 8u''f' + 24u^2f' + 12uu'f + 2fu''' = 0 \quad (6)$$

and

$$f''' + 4f'u + 2fu' = 0.$$

Although, apparantly it seems we get a new set equations for $\Delta^{(4)}$ operator, however the first equation can be expressed as

$$\begin{aligned} & (\partial_x^2 + 6u)(f''' + 4f'u + 2fu') = \\ & f'''' + 10uf''' + 10u'f'' + 8u''f' + 24u^2f' + 12uu'f + 2fu'''. \end{aligned}$$

Hence we obtain the same equation (7) for all the extended conformal operators $\Delta^{(n)}$ (for $n \leq 4$).

3.3 Invariant property of Wronskian

In section (2.1), we have seen that the Wronskian of the solutions of a n th order differential equation is constant, and the solution defines an immersion $\mathbf{R} \rightarrow \mathbb{R}P^{n-1}$ in homogeneous coordinates.

The invariant meaning of the Wronskian can be easily realized in terms of projective connections also.

n = 2 Case :

If $a, b \in \Gamma(\Omega^{-\frac{1}{2}})$, then we find

$$ab' - a'b \in C^\infty(S^1),$$

where $a', b' \in \Gamma(\Omega^{\frac{1}{2}})$.

If a, b satisfy $\frac{d^2}{dx^2} + u(x) = 0$, then the Wronskian is constant, since $a = \psi_1 dx^{-\frac{1}{2}}$, $b = \psi_2 dx^{-\frac{1}{2}}$ satisfy

$$(\psi_1 \psi_2' - \psi_1' \psi_2)' = \psi_1 \psi_2'' - \psi_1'' \psi_2 = 0.$$

n = 3 Case :

If $a, b, c \in \Gamma(\Omega^{-1})$, then the Wronskian

$$\mathbf{Wr}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$$

is an element of $C^\infty(S^1)$. Again, if a, b, c satisfy $\frac{d^3}{dx^3} + 4u \frac{d}{dx} + 2u' = 0$, then the Wronskian is constant, since $a = \psi_1 dx^{-1}$, $b = \psi_2 dx^{-1}$, $c = \psi_3 dx^{-1}$ satisfy

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}' = 0.$$

This can be generalized to higher values of n .

3.4 Generalised Bracket and Projective Vector Field

Let us switch back to our earlier section.

Consider a one parameter family of $Vect(S^1)$ action on the space of smooth function $a(x) \in C^\infty(S^1)$ [22]

$$\mathcal{L}_v^\lambda a(x) := f(x)a'(x) - \lambda f'(x)a(x), \quad (17)$$

where $\mathcal{L}_v^\lambda$ is the Lie derivative with respect to $v = f(x)\frac{d}{dx} \in Vect(S^1)$, given by

$$\mathcal{L}_v^\lambda := f(x)\frac{d}{dx} - \lambda f'(x). \quad (18)$$

Definition 3.7 *The action of $Vect(S^1)$ on the space of Hill's operator $\Delta \equiv \Delta^{(2)}$ is defined by the commutator with the Lie derivative*

$$[\mathcal{L}_v, \Delta] := \mathcal{L}_v^{-3/2} \circ \Delta - \Delta \circ \mathcal{L}_v^{1/2}. \quad (19)$$

This action can be identified with the coadjoint action of Virasoro algebra on its dual. Here we discuss this briefly.

Similarly we can generalize this action on $\Delta^{(n)}$

Definition 3.8 *The $Vect(S^1)$ action on Δ^n is defined by*

$$[\mathcal{L}_v, \Delta^{(n)}] := \mathcal{L}_v^{-(n+1)/2} \circ \Delta^{(n)} - \Delta^{(n)} \circ \mathcal{L}_v^{(n-1)/2}. \quad (20)$$

Hence the equation (20) can be considered as some coadjoint action of $Vect(S^1)$ on some "dual" space.

For example, $[\mathcal{L}_v, \Delta^{(3)}](s) = \mathcal{L}_v(\Delta^{(3)}s) - \Delta^{(3)}\mathcal{L}_v s$ and $[\mathcal{L}_v, \Delta^{(4)}](s) = \mathcal{L}_v(\Delta^{(4)}s) - \Delta^{(4)}\mathcal{L}_v s$ are identified with the equations (11) and (16) respectively.

4 $Diff(S^1)$ Action on AGD Manifold

In this section, we describe the transformations of the AGD operator under the action of $Diff(S^1)$. This transformation of scalar differential operators has been known since last century. The action of $Diff(S^1)$ induces a change of variable in the independent parameter x .

Suppose $\Delta^{(n)}$ be a scalar differential operator. The action of $Diff(S^1)$ transform the solutions of

$$\Delta^{(4)}\psi = 0.$$

as densities of degree $\frac{n-1}{2}$

Let us consider the action of $Diff(S^1)$:

$$Diff(S^1) \times AGD \longrightarrow AGD$$

$$(x, \Delta^{(4)}) \longmapsto (\sigma(x), \Delta^{(4)})$$

where $\sigma(x)\Delta^{(4)}$ is the unique scalar operator of the form $\Delta^{(4)}$ having u_{n-1} . If μ and ξ be the solutions of $\sigma(x)\Delta^{(4)}$ and $\Delta^{(4)}$, then their solutions are connected by

$$\mu = (x'^{\frac{n-1}{2}}) \circ x.$$

In the case for $n = 2$, this coincides with the action of Virasoro group on the space of Hill operators, dual space of Virasoro algebra.

It should be noted that the operators $\Delta^{(n)}$ do not preserve their form under the action of $Diff(S^1)$, $x \rightarrow \sigma(x)$, due to the appearance of the $(n-1)$ -th term $-\frac{1}{2}n(n-1)(\sigma''/\sigma'^{n+1})$. Hence we should think the operators are acting on *densities of weight* $-1/2(n-1)$ rather than on scalar functions, in this case we can always find $u_{n-1} = 0$ as a reparametrization invariant. Therefore, the action of $Diff(S^1)$ on Δ^n is given by

$$\partial_x^n + u_{n-2}(x)\partial_x^{n-2} + \dots + u_0(x) \longrightarrow \sigma'^{-(n+1)/2}(\partial_x^n + \tilde{u}_{n-2}\partial_x^{n-2} + \dots + \tilde{u}_0)\sigma'^{-(n-1)/2}, \quad (21)$$

where

$$\tilde{u}_{n-2} = \sigma'^2 u_{n-2}(\sigma(x)) + \frac{1}{12}n(n-1)(n+1)S'(x).$$

In particular for $n = 3$ we find

$$\tilde{u}_1(x) = \sigma'^2 u_1(\sigma(x)) + 2S(x)$$

$$\tilde{u}_0(x) = \sigma'^3 u_0(\sigma(x)) + \sigma'\sigma''u_1(\sigma(x)) + S'(x).$$

For $n = 4$ we obtain:

$$\tilde{u}_2(x) = \sigma'^2 u_2(\sigma(x)) + 5S(x)$$

$$\tilde{u}_1(x) = \sigma'^3 u_1(\sigma(x)) + 2\sigma'\sigma''u_2(\sigma(x)) + 5S'(x).$$

$$\tilde{u}_0(x) = \sigma'^4(\sigma(x)) + \frac{3}{2}\sigma'^2\sigma''u_1(\sigma(x)) + \frac{3}{2}\sigma''^2u_2(\sigma(x)) + \frac{3}{2}\sigma'^2u_2(\sigma(x))S(x) + \frac{3}{2}S''(x) + \frac{3}{2}S^2(x).$$

5 The Euler-Arnold equation and the KdV equation

In the preceding section, we have already remarked that the action $[\mathcal{L}_v, \Delta] = \mathcal{L}_v \circ \Delta - \Delta \circ \mathcal{L}_v$ exactly coincides with the coadjoint action of the Virasoro on its dual. So it is quite likely to get KdV equation from the following equation:

$$\frac{\partial \Delta}{\partial t} + [\mathcal{L}_v, \Delta] = 0. \quad (22)$$

This type of equation has been considered by M. Alber and S. Alber [2, 3] of course, in a different settings.

They considered the spectral problem for an associated Schrödinger operator

$$L = -\frac{\partial^2}{\partial x^2} + V(x, t, E),$$

where E is a parameter and

$$V(x, t, E) = \sum_{j=-l}^m V_j(x, t) E^j.$$

In the case of KdV equation, E appears as an eigenvalue and one ultimately equates the potential with a solution of the nonlinear equation itself.

We want to investigate the analogous equation for the extended projective connection on the circle, i.e.

$$\frac{\partial \Delta^{(n)}}{\partial t} + [\mathcal{L}_v, \Delta^{(n)}] = 0, \quad (23)$$

where $[\mathcal{L}_v, \Delta^{(n)}]$ is given in (20). It should be noted that one single equation appears here.

Proposition 5.1 *The equation*

$$\frac{\partial \Delta^{(n)}}{\partial t} + [\mathcal{L}_v, \Delta^{(n)}] = 0 \quad n < 5$$

generates only one single equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} f''' + 2f'u + fu'. \quad (24)$$

Proof: We prove it by direct computation.

□

For a special choice of $f = u$ we obtain KdV equation.

Geometrically this follows from the following argument:

We have argued that the action $[\mathcal{L}_v, \Delta^{(n)}]$, can be considered as a coadjoint action of $Vect(S^1)$ on $\Delta^{(n)}$. Using the equation (23) we know

$$\tilde{u} = \frac{1}{2} f''' + 2f'u + fu' = \left(\frac{1}{2} \partial_x^3 + 2u\partial_x + u_x \right) f.$$

The operator $(\frac{1}{2} \partial_x^3 + 2u\partial_x + u_x)$ is called the symplectic operator. The Euler equation is the Hamiltonian flow on the coadjoint orbits in $\widehat{\mathcal{OG}}^*$ generated by the Hamiltonian

$$H(u) = \frac{1}{2} u^2(x),$$

given by

$$\frac{d}{dt} u(t) = -ad_{u(t)}^* u(t). \quad (25)$$

Thus by applying Euler-Arnold equation we prove the first theorem.

6 Flows on the space of curve

In the section (2.1), we have seen how an immersion associated to $\Delta^{(n)}$ yields a curve

$$\psi : \mathbf{R} \longrightarrow \mathbb{R}P^{n-1}$$

in the projective space.

Let us write ψ in terms of an inhomogeneous coordinates. We lift ψ to a curve on $\mathbf{R}^n$. This we may denote by $\tilde{\psi} = \eta(x)(1, \psi)$. We choose the factor $\eta(x)$ so that the Wronskian of the components of the new curve equals 1.

It turns out that there is a unique choice of $\eta(x)$ with such a property, and this is given by

$$\eta(x) = Wr(\psi'_1, \dots, \psi'_{n-1})^{-\frac{1}{n}}. \quad (26)$$

In particular, for $n = 2$ case,

$$\psi \equiv (\psi_1, \psi_2) = (\psi'^{-\frac{1}{2}}, \psi'^{-\frac{1}{2}}\psi) \quad (27)$$

is the solution curve [9]. It retains the unitarity of the Wronskian.

Lemma 6.1 *The equation $f''' + 2u'f + 4uf' = 0$ traces out a three dimensional spaces of solutions.*

Proof: If ψ_1 and ψ_2 are the solutions of

$$\Delta\psi = \left(\frac{d^2}{dx^2} + u\right)\psi = 0, \quad (28)$$

then it is easy to see that $\psi_i\psi_j \in \Gamma(\Omega^{-1})$ satisfies the above equation. Hence the solutions space is spanned by ψ_1^2 , ψ_2^2 and $\psi_1\psi_2$.

□

Substituting (28) in equation $\left(\frac{d^2}{dx^2} + u\right)\psi_i = 0$ (for $i = 1, 2$), we obtain

$$u = \frac{1}{2}\left(\frac{\psi'''}{\psi'} - \frac{\psi''^2}{\psi'^2}\right). \quad (29)$$

The right hand side is invariant under $PSL(2, \mathbf{R}) (= SL(2, \mathbf{R}/\pm 1)$ group. If we substitute this expression in the Euler-Arnold equation, we obtain the evolution equation of the solution curve on the projective space

$$\psi_t = \psi_{xxx} - \frac{3}{2}\psi_{xx}^2\psi_x^{-1}.$$

This equation is called the Schwarzian KdV (or Ur KdV by Wilson school [25, 33, 34]). One can check directly that this equation is $SL(2, \mathbf{R})$ invariant.

Thus we obtain the theorem (1.2).

The Schwarzian KdV has a bihamiltonian structure [33]

$$\begin{aligned} H_1 &= 4u, \quad D_1 = -2\psi_x^{-1}\partial\psi_x^{-1}, \\ H_2 &= u^2, \quad D_2 = -\frac{1}{2}\psi_x^{-2}\partial^3 - 3\psi_{xx}\psi_x^{-3}\partial^2 + (3\psi_{xx}^2\psi_x^{-4} - \psi_{xxx}\psi_x^{-3})\partial. \end{aligned}$$

7 Stabilizer orbit and Neumann equation

The Neumann system deals with the motion of a particle on a sphere under the influence of a quadratic potential. This system is completely integrable and the solutions are given by hyperelliptic theta function. In fact the geometrical realization of this system is very interesting. Moser [26] observed that the integrals of the Hamiltonian system describing the motion of Neumann system have a very close similarity with the integrals of the Hamiltonian system describing the geodesics on a quadric. Later Knörrer [19] showed that the Neumann problem can be recasted through the Gauss map as the geodesic motion problem on a quadric. In fact the relationship between the C. Neumann system and the Jacobi system is clarified by making use of this Gauss map. The Jacobi system is another integrable system which describes the geodesic flow on an ellipsoid.

C. Neumann System: Here we give a very brief description of the C. Neumann system. Let us consider a point moving on the $(n - 1)$ dimensional sphere

$$\xi_1^2 + \cdots + \xi_n^2 = 1$$

under the influence of a quadratic potential $v(\xi) = \langle \xi, A\xi \rangle$ where A is a symmetric $n \times n$ matrix. The differential equation describing this motion is called the C. Neumann system, given by

$$\xi'' = -A\xi + u\xi, \quad \sum_{i=1}^n \xi_i^2 = 1 \quad u = \langle \xi, A\xi \rangle - \langle \xi', \xi' \rangle.$$

By the Sylvesters theorem, we can assume that A is a diagonal matrix. Algebraic integrals for the C. Neumann problem were first found by Karen Uhlenbeck in an unpublished note in 1975 [see for details, 24].

Now we give the proof of theorem (1.3).

This follows from the properties of the Virasoro action for $n = 2$ case. Let us confine our attention to a specific hyperplane $\mu = 1$ in the coadjoint orbit. The action in this hyperplane will be

$$\tilde{u} = fu' + 2f'u + \frac{1}{2}f''.$$

The stabilizer of the action of $(f(x)\frac{d}{dx}, 1)$ on the dual $(u(x)dx^2, 1)$, it means $(f\frac{d}{dx}, 1) \in \text{Stab}(u, 1)$ if and only if

$$f''' + 2u'f + 4uf' = 0.$$

Thus, we say if $u \in \mathcal{O}_{\text{stab}}(\Delta)$, stabilizer space of Hill's operator, then u satisfies equation (6.1).

We will go on to prove for $n > 2$ case. We will consider now the stabilizer set of the action of $\text{Vect}(S^1)$ on the space of Adler-Gelfand-Dickey operators. It is clear from the equation (23) that this set is given by

$$[\mathcal{L}_v, \Delta^{(n)}] = \mathcal{L}_v^{-(n+1)/2} \circ \Delta^{(n)} - \Delta^{(n)} \circ \mathcal{L}_v^{(n-1)/2} = 0.$$

One could readily see that the equation satisfied for $n = 3$ is the derivative of $f''' + 2u'f + 4uf' = 0$. Similarly, for $n = 4$ one can express the equation (14)

$$(\partial_x^2 + 6u)(f''' + 2u'f + 4uf') = 0$$

Therefore we conclude

Lemma 7.1 *The stabilizer space of the action of $f \frac{d}{dx} \in \text{vect}(S^1)$ on the space differential operators of type (1) (for $n < 5$) is given by*

$$f''' + 2q'f + 4qf' = 0.$$

Now we seek to characterize the stabilizer space $\mathcal{O}_{stab}$.

We wish to connect the projective vector field and the C. Neumann system. Let us define a function

$$f := \langle \chi, A^{-1}\chi \rangle,$$

where A is a constant diagonal matrix.

Proposition 7.2 *If $f = \langle \chi, A^{-1}\chi \rangle$ and satisfies*

$$f''' + 2fu' + 4f'u = 0$$

then χ satisfies

$$\chi'' = -A\chi - u\chi$$

where $\langle \chi, \chi \rangle = 1$ and $u = -\langle \chi, A\chi \rangle + \langle \chi', \chi' \rangle$ which is the system of Neumann equations.

Proof :: We know

$$f = \langle \chi, A^{-1}\chi \rangle$$

hence

$$f' = 2 \langle \chi', A^{-1}\chi \rangle$$

Then after using the condition of the Neumann equation, we obtain

$$f'' = -2 - 2uf + 2 \langle \chi', A^{-1}\chi' \rangle.$$

Taking one more derivative we get

$$f''' + 4uf' + 2u'f = 0.$$

□

Thus we obtain the theorem (1.3b) part.

8 Super Algebra

The first characteristic special property of super algebra is that all the additive groups of its basic and derived structures are $\mathbb{Z}_2$. A vector superspace is a $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ -graded vector space $V = V_0 + V_1$. An element v of V_0 (resp. V_1) is said to be even (resp. odd). The super commutator of a pair of elements $v, w \in V$ is defined to be the element

$$[v, w] = vw - (-1)^{\tilde{v}\tilde{w}}wv,$$

where $\tilde{v}$ and $\tilde{w}$ are the gradings of v and w respectively.

Let us recall that the tangent bundle over S^1 has two square roots, one is trivial and other is a Möbius strip, which is denoted by $\Omega^{-\frac{1}{2}}$.

The sections of $\Gamma(\Omega^{-\frac{1}{2}})$ which satisfy the equation (28) are not functions but the square root of a *projective* vector field, since $\psi \in \Omega^{-1/2}$, the space of scalar densities of weight $-1/2$, square root of $f \in Vect(S^1)$.

Let us define

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1,$$

where we denote $\mathcal{G}_0 \equiv Vect(S^1)$ and $\mathcal{G}_1 \equiv \Omega^{-1/2}(S^1)$. $\mathcal{G}$ forms a super Lie algebra on $S^{1,1}$ [16] and $\mathcal{G}_1$ is the super-partner of $\mathcal{G}_0$. This is asserted since $\mathcal{G}_1$ is the $\mathcal{G}_0$ module and it is compatible with the structure of $\mathcal{G}_0$ module and satisfies $\mathcal{G}_1 \times \mathcal{G}_1 \longrightarrow \mathcal{G}_0$. A typical element of $\mathcal{G}$ would be

$$f(x) \frac{d}{dx} + \psi(x) \frac{d^{1/2}}{dx},$$

and the super Lie bracket is given by

$$[(f_1, \psi_1), (f_2, \psi_2)] = ([f_1, f_2] + \psi_1 \psi_2, \{f_1, \psi_2\} + \{\psi_1, f_2\}).$$

In this realization $(f(x) \frac{d}{dx} \oplus \psi(x) \frac{d^{1/2}}{dx})$ i.e. $(f(x), \psi(x))$ forms a super Lie algebra. $(f(x), \psi(x))$ satisfies

$$\begin{aligned} f(x + 2\pi) &= f(x) \\ \psi(x + 2\pi) &= \pm \psi(x) \end{aligned}$$

When it is in the '+' sector, it is called Ramond sector super Lie algebra and '-' sector is known as Neveu-Schwarz sector.

The cocycle may be extended to this superalgebra via

$$c(\psi_1, \psi_2) = \int_{S^1} \psi_1' \psi_2' dx. \quad (30)$$

Thus we propose:

Proposition 8.1 *If f is a projective vector field, satisfies (12), then the square root of f is an element of scalar density of weight $-\frac{1}{2}$, denoted by ψ . Then (f, ψ) satisfies super algebra.*

The stabilizer of the action of this superalgebra on pairs $\Delta^{(n)}$ is given by (12) and (28). This gives us a geometrical explanation that if ψ_1, ψ_2 satisfy (28) then $\psi_1 \psi_2$ satisfies (12).

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