A method of enciphering quantum states

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Abstract

In this paper, we propose a method of enciphering quantum states of two-state
systems (qubits) for sending them in secrecy without entangled qubits shared by
two legitimate users (Alice and Bob). This method has the following two properties.
First, even if an eavesdropper (Eve) steals qubits, she can extract information from
them with certain probability at most. Second, Alice and Bob can confirm that the
qubits are transmitted between them correctly by measuring a signature. If Eve
measures \(m\) qubits one by one from \(n\) enciphered qubits and sends alternative ones
(the Intercept/Resend attack), a probability that Alice and Bob do not notice Eve’s
action is equal to \((3/4)^m\) or less. Passwords for decryption and the signature are
given by classical binary strings and they are disclosed through a public channel.
Enciphering classical information by this method is equivalent to the one-time pad
method with distributing a classical key (random binary string) by the BB84 proto-
col. If Eve takes away qubits, Alice and Bob lose the original quantum information.
We cannot examine security against the case that Eve makes an attack with using
entanglement. This remains to be solved in the future.

1 Introduction

Since considerable progress was made in quantum information and computation theory,
many researchers have been studying to realize the information processing that we have
never had by using quantum mechanics[16]. At the same time, it has been studying to

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apply the uncertainty principle, the quantum no-cloning theorem, and entanglement between quantum systems to cryptography[8]. The BB84 protocol is considered to be an effective method for the key distribution. Combining it with the one-time pad method, we obtain a highly secure cryptography[1][2][5]. On the other hand, the quantum teleportation is considered to be an excellent method for sending arbitrary quantum states between two parties[6][7].

The BB84 protocol is used for a secure distribution of a classical key (binary string) to two legitimate users (Alice and Bob). Choosing a basis vector at random from four basis vectors, the rectilinear basis \{\ket{0}, \ket{1}\} and the circular basis \{(1/\sqrt{2})(\ket{0} \pm \ket{1})\}, as a state of a photon (a two-state system or a qubit), Alice sends it to Bob. Bob measures a transmitted photon in an orthonormal basis that he chooses from two bases (rectilinear and circular) at random and independently of Alice.

Not being consistent with each other, the rectilinear basis and the circular basis are called conjugate bases. A result of measurement with a wrong basis is random. If an eavesdropper (Eve) steals a photon from the channel, measures it in a basis chosen at random, and sends an alternative one, Alice and Bob find inconsistency with a probability of 1/4 or more and notice Eve’s eavesdropping. In this way, using the uncertainty principle, the BB84 protocol reveals Eve’s illegal act.

A. K. Ekert proposed another protocol for distributing a classical key by transmitting pairs of qubits in EPR states, \(\ket{\Psi^-} = (1/\sqrt{2})(\ket{01} - \ket{10})\), apart to Alice and Bob from a source[3]. They detect Eve by Bell’s theorem. Considering a simplified protocol of A. K. Ekert’s, C. H. Bennett et al. showed it was equivalent to BB84[4]. From these successive works, it has been recognized that we need not use the entanglement to distribute a classical key by quantum mechanics. (But, combining entanglement purification protocol with Ekert’s protocol, we can distribute a classical key in high secrecy[9].)

The quantum teleportation is used for transmitting an arbitrary state from Alice to Bob. They share an EPR-pair of qubits beforehand. Alice carries out the Bell-measurement on both a one-qubit state \(\ket{\psi}\) that she wants to send and her qubit of the EPR-pair. Receiving a result of her measurement, Bob can construct \(\ket{\psi}\) from his qubit of the EPR-pair. A characteristic of this method is that classical information and non-classical information of \(\ket{\psi}\) are divided perfectly and only the classical information is sent through the public channel. If they share the EPR-pair correctly, Eve can neither eavesdrop on the state nor destroy it in principle.

These methods are related to the quantum no-cloning theorem. It tells us there is no unitary transformation that makes accurate clones of arbitrary quantum states[8]. In the BB84 protocol, it gives an effect as follows. Not knowing which basis is chosen for a qubit (photon) that she extracts from the quantum channel, rectilinear or circular, Eve cannot make a clone of the qubit and keep it. What she can do is only measuring the qubit in a proper basis and sending an alternative one that depends on a result of the measurement to Bob. In the quantum teleportation, the following thing is important. Because Alice can neither measure \(\ket{\psi}\) without disturbance nor make an accurate clone of it, she cannot extract information from \(\ket{\psi}\) at all. During the whole process, Alice and Bob have no knowledge about \(\ket{\psi}\).

In the quantum teleportation, Alice and Bob have to share an EPR-pair of qubits beforehand. After being emitted by a source, this pair flies towards them apart through
a quantum channel. Therefore, for example, if Eve takes away the qubit that Bob is supposed to have and sends an alternative one to him, she succeeds in eavesdropping. To avoid such a trouble, Alice and Bob need to share a lot of EPR-pairs and to purify them\textsuperscript{9}.

In this paper, we consider a method for enciphering arbitrary quantum states for sending them in secrecy without entangled qubits shared by Alice and Bob beforehand. In our method, there are two points as follows. (See Figure 3.)

First, even if Eve takes away qubits, she can extract quantum information from them with certain probability at most. (We assume Eve makes only the Intercept/Resend attack\textsuperscript{2}.) Alice applies a unitary operator \(U_i\) which is chosen at random from a set of operators \(\mathcal{M} = \{U_j\}\) to an arbitrary \(n\)-qubit state \(|\Psi\rangle\) that she wants to send in secrecy. The subscript \(i\) of \(U_i\) is a password for decryption. Not knowing which operator is chosen from \(\mathcal{M}\), Eve regards the enciphered state as a mixed state of \(U_j|\Psi\rangle\) for all \(U_j \in \mathcal{M}\) with equal probability. If Alice prepares \(\mathcal{M}\) so that the density operator of the mixed state may be in proportion to the identity operator \(I\), Eve cannot extract the information of \(|\Psi\rangle\) at all without the password \(i\). The reason of this is that even if Eve puts auxiliary qubits on the density operator \(\rho = \frac{1}{2^n}I\), applies unitary transformations to it, or measures it, she cannot extract \(|\Psi\rangle\). After confirming that the quantum state is transmitted correctly, Alice releases the password \(i\) in our protocol. Therefore, to extract information from \(|\Psi\rangle\), Eve has to eavesdrop without disturbing Alice and Bob’s certification process.

Second, Alice and Bob can confirm that a quantum state received by Bob is a genuine one sent by Alice. Not having knowledge about the \(n\)-qubit state \(|\Psi\rangle_Q\) at all, they do not notice Eve replace the genuine qubits with alternative ones. Therefore, they need to confirm that qubits Bob receives are genuine. (It seems like authentication of the identity of a correspondent on networks.) In our method, putting an \(n\)-qubit state \(|a\rangle_S\) (\(a \in \{0, 1\}^n\)) that represents her signature on \(U_i^Q|\Psi\rangle_Q\), Alice makes entanglement between qubits of each pair in the system \(Q\) and \(S\). Here, we call the quantum system which represents the transmitted information \(Q\) and the quantum system which represents the signature \(S\). Then, to forbid Eve to make clones of qubits, Alice applies an operator chosen at random from \(\mathcal{L} = \{I, H, \sigma_x, H\sigma_x\}\) to each qubit, where \(H\) is called the Hadamard transformation and it causes \(|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\), \(|1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\), and \(\sigma_x\) is one of the Pauli matrices and it causes \(|0\rangle \rightarrow |1\rangle\), \(|1\rangle \rightarrow |0\rangle\). Hence, quantum information of each qubit is encoded in a basis chosen at random from two conjugate bases (rectilinear and circular). Therefore, if Eve does anything on the qubits, Alice and Bob can find inconsistency and detect Eve with certain probability. This is essentially the same technique used in BB84. In our method, certification of a correspondent and detection of Eve are done at the same time. The second password for decryption is which operators are chosen from \(\mathcal{L}\).

Because the passwords and the signature represented by classical binary strings are transmitted by the public channel, Eve also knows them. If \(|\Psi\rangle_Q\) represents classical information (a product state of \(|0\rangle\) and \(|1\rangle\)), our protocol is equivalent to the one-time pad method with classical key distribution by BB84.

We cannot examine security against the case that Eve makes an attack with using entanglement. This remains to be solved in the future.

This paper is organized as follows. In Sec. 2, we explain how Alice and Bob forbid
Eve to extract the original quantum information and how they confirm the qubits are transmitted correctly. In Sec. 3, we explain the whole protocol and discuss how Bob confirms that he receives qubits. In Sec. 4, we discuss security of our protocol against Eve’s Intercept/Resend attack on each qubit. In Sec. 5, we give a brief discussion.

2 Enciphering quantum states

In this section, we explain how Alice and Bob forbid Eve to extract the original quantum information and how they confirm the qubits are transmitted correctly.

First, we consider transforming an arbitrary quantum state so that Eve cannot recover an original quantum state. For simplicity, we consider an arbitrary one-qubit state for a while and we describe its density operator as \( \rho \) defined on a two-dimensional Hilbert space \( \mathcal{H}_2 \). We assume Alice wants to send \( \rho \) to Bob in secrecy. She does not know \( \rho \) at all, because her partial measurement destroys it.

Alice prepares a set of operators,

\[
\mathcal{M} = \{ \sigma_j : j = 0, x, y, z \},
\]

where \( \sigma_0 = I \) (the identity operator) and \( \{ \sigma_x, \sigma_y, \sigma_z \} \) are Pauli matrices. Taking

\[
|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

for an orthonormal basis, we write them as

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

\( \mathcal{M} \) may be disclosed in public. Choosing an operator \( \sigma_i \) from \( \mathcal{M} \) at random, Alice carries out the following unitary transformation,

\[
\rho \rightarrow \sigma_i \rho \sigma_i^\dagger.
\]

She keeps the subscript \( i \) secret as a password and never tells it to anyone. Because the subscript takes a value from \( \{0, x, y, z\} \), the password can be represented by 2-bit classical information.

Not knowing which transformation Alice applies to the qubit, Eve has to regard the state as

\[
\rho' = \frac{1}{4} \sum_{j=0,x,y,z} \sigma_j \rho \sigma_j^\dagger.
\]

Generally, the density operator \( \rho \) satisfies \( \rho^\dagger = \rho \), \( \text{Tr} \rho = 1 \), and \( 0 \leq \lambda_1 \leq 1, 0 \leq \lambda_2 \leq 1, \lambda_1 + \lambda_2 = 1 \) where \( \lambda_1, \lambda_2 \) are eigenvalues of \( \rho \). Hence, we can describe an arbitrary \( \rho \) as

\[
\rho = \frac{1}{2} (I + a \cdot \sigma),
\]

where \( a = (a_x, a_y, a_z) \).
where \( \mathbf{a} = (a_1, a_2, a_3) \) is a three-component real vector and \( 0 \leq \sum_{k=1}^{3} a_k^2 \leq 1 \). Because

\[
\rho' = \frac{1}{2} \mathbf{I} + \frac{1}{8} \sum_{j=0,x,y,z} \mathbf{a} \cdot \sigma_j \sigma_j^\dagger, \tag{1}
\]

and

\[
\sigma_j \sigma_k \sigma_j = \begin{cases} 
\sigma_k & j = 0 \text{ or } j = k \\
-\sigma_k & j \neq k \text{ and } j, k \in \{x, y, z\} 
\end{cases}
\]

there is no contribution from the second term of Eq. (1). We obtain

\[
\rho' = \frac{1}{2} \mathbf{I}.
\]

Therefore, even if Eve takes away \( \rho' \), she cannot extract information from it at all, because she does not know the password \( i \).

An arbitrary \( n \)-qubit density operator \( \rho_n \) is given by

\[
\rho_n = \frac{1}{2^n} (\mathbf{I} + \sum_{\mathbf{k} \in \{0,x,y,z\}^n, k \neq (0,\ldots,0)} a_k U_k),
\]

where

\[
U_k = \sigma_{k_1} \otimes \cdots \otimes \sigma_{k_n},
\]

and \( a_k (k \in \{0, x, y, z\}^n, k \neq (0, \ldots, 0)) \) is real. Choosing an operator \( U_i \) from

\[
\mathcal{M}_n = \{ U_k : U_k = \sigma_{k_1} \otimes \cdots \otimes \sigma_{k_n}, k \in \{0, x, y, z\}^n \},
\]

at random, Alice applies it to \( \rho_n \) for encryption as \( \rho_n \rightarrow U_i \rho_n U_i^\dagger \). If Eve takes away the density operator given by

\[
\rho'_n = \frac{1}{4^n} \sum_{j \in \{0,x,y,z\}^n} U_j \rho_n U_j^\dagger \\
= \frac{1}{2^n} \mathbf{I} + \frac{1}{4^n} \cdot \frac{1}{2^n} \sum_{j, k \in \{0,x,y,z\}^n, k \neq (0,\ldots,0)} a_k U_j U_k U_j^\dagger \\
= \frac{1}{2^n} \mathbf{I}, \tag{2}
\]

she cannot extract information from \( \rho'_n \) at all. Even if she puts auxiliary systems on \( \rho'_n \), applies unitary transformations to it, or carries out measurements, she cannot obtain \( \rho_n \). The password of \( i \) is given by a 2\( n \)-bit string.

Next, we explain how Alice and Bob confirm that the qubits are transmitted between them correctly. If Eve takes away \( \rho'_n \) and sends an alternative state \( \tilde{\rho}_n \) to Bob, he carries out the inverse operation on the state that he receives as \( \tilde{\rho}_n \rightarrow U_i \tilde{\rho}_n U_i^\dagger \). Because not having knowledge about the original \( \rho_n \) at all, Alice and Bob do not notice what Bob gets is a fake.

To avoid this trouble, they put a signature on \( \rho_n \). Reading it, they can confirm that Bob receives the state transmitted from Alice correctly. It is important that Eve cannot change the signature. For simplicity, we describe the state as a ket vector \( \forall |\Psi\rangle_Q \in \mathcal{H}_Q^n \)
instead of the density operator $\rho_n$ for a while. If the $n$-qubit state is given by a mixed state, we can give a similar discussion. We call the system that represents the quantum information $Q$ and the system that represents the signature $S$.

Preparing an $n$-bit random string $\forall a = (a_1, \cdots, a_n) \in \{0, 1\}^n$ as her signature, Alice attaches a qubit $|a_k\rangle_S$ to the $k$th qubit of

$$U_i^Q|\Psi\rangle_Q = \sum_{x \in \{0, 1\}^n} c_x |x_1\rangle \cdots |x_n\rangle \in \mathcal{H}_2^n. \tag{3}$$

Applying the controlled-NOT (C-NOT) gate to the $k$th pair as Figure 1, she obtains

$$|x_k\rangle_Q |a_k\rangle_S \rightarrow |x_k\rangle_Q |a_k \oplus x_k \mod 2\rangle_S \quad \text{for } k = 1, \cdots, n.$$ 

Choosing $L_{k,1}^Q, L_{k,2}^S \in \mathcal{L} = \{I, H, \sigma_x, H\sigma_x\}$ at random, she applies them to the qubits of the pair, where

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$ 

Alice repeats this operation on the $k = 1, \cdots, n$th qubit, and we write the whole transformation as $V_{\alpha}^{QS}$. $\alpha$ represents which operators are chosen from $\mathcal{L}$. The second password $\alpha$ is given by a classical $4n$-bit string.
We often describe successive operations on qubits as a network like Figure 1. A horizontal line represents a qubit and time proceeds from left to right. Figure 2 shows examples of unitary transformations applied to qubits. Figure 2. i and ii. represent the C-NOT gate and $\sigma_x$ each[10].

The operations that we have discussed are summarized as follows,

$$
|\Psi\rangle_Q \rightarrow U^Q_i |\Psi\rangle_Q \quad \text{(1st password } i) \\
\rightarrow |a\rangle_S \otimes U^Q_i |\Psi\rangle_Q \quad \text{(signature } a) \\
\rightarrow V^{QS}_\alpha |a\rangle_S \otimes U^Q_i |\Psi\rangle_Q \quad \text{(2nd password } \alpha).
$$

Double encryption prevents Eve from using $|a\rangle_S$ ill. If $V^{QS}_\alpha$ is not applied to the state, Eve may take away all of the qubits, keep $U^Q_i |\Psi\rangle_Q$, and send a fake of $|a\rangle_S \otimes |\tilde{\Psi}\rangle_Q$ to Bob.

If Eve does anything on the $k$th qubit of $Q$, the signature of $|a_k\rangle_S$ is destroyed and Bob fails in the certification process with certain probability or more. This is caused by the facts that conjugate bases chosen at random represent the systems $Q$ and $S$, and there is entanglement between $Q$ and $S$. If she does anything on the $k$th qubit of $S$, we can give a similar discussion. We estimate a probability that Alice and Bob notice Eve’s illegal act in Sec. 4.

## 3 The protocol for secure transmission

We consider a protocol for transmitting an arbitrary quantum state $\forall |\Psi\rangle_Q \in H_2^n$ from Alice to Bob in secrecy against Eve’s eavesdropping by using the encryption method discussed in the previous section. Alice and Bob do not have knowledge about $|\Psi\rangle_Q$ at all. Both of them can use the following two channels.

- The classical channel: It transmits classical binary strings in public. Eve can make accurate copies of them, but she cannot alter them.

- The quantum channel: It transmits sequences of qubits (quantum information). Eve can interact with them, but she cannot make accurate copies of them.

Alice sends qubits to Bob according to the following protocol. (See Figure 3.)

1. Alice sends $|\Psi_{\text{crypt}}\rangle \equiv V^{QS}_\alpha |a\rangle_S \otimes U^Q_i |\Psi\rangle_Q$ of $2n$ qubits to Bob through the quantum channel.
2. Receiving $2n$ qubits, Bob breaks off the quantum channel and reports arrival of them to Alice through the classical channel.

3. Receiving the report from Bob, Alice tells Bob what transformation $V_{QS}^\alpha$ is through the classical channel. (She discloses the $4n$-bit password $\alpha$.)

4. Applying $V_{QS}^\alpha\dagger$ to the state that he has received and measuring the signature, Bob tells Alice a result of the measurement ($n$-bit string) through the classical channel.

5. Receiving the $n$-bit string from Bob, Alice examines whether it coincides with the signature $a$ or not. If it coincides with her original signature, she tells Bob what transformation $U_{iQ}$ is through the classical channel. (She discloses the $2n$-bit password $i$.) If it does not coincide, she concludes Eve has eavesdropped on qubits and stops the protocol.

6. Bob applies $U_{iQ}^Q\dagger$ to the state that he has and obtains the original state $|\Psi\rangle_Q$.

In this protocol, it is important that Bob confirms the arrival of $2n$ qubits ($|\Psi\text{crypt}\rangle$ or Eve’s fake) and breaks off the quantum channel at the second step. To understand the reason of this, we assume the following case. (See Figure 4.) Although $|\Psi\text{crypt}\rangle$ is still halfway on the channel, Bob reports the arrival of qubits to Alice by mistake, and she discloses $V_{QS}^\alpha$ through the classical channel. Eve may take away $|\Psi\text{crypt}\rangle$, apply $V_{QS}^\alpha\dagger$ to it, and obtain $|a\rangle_S \otimes U_i |\Psi\rangle_Q$ before Bob receives qubits. Eve can keep $U_i |\Psi\rangle_Q$, combine a false state $|\tilde{\Psi}\rangle_Q$ with $|a\rangle_S$, and send $V_{QS}^\alpha[|a\rangle_S \otimes |\tilde{\Psi}\rangle_Q]$ to Bob. If the quantum channel is still open, Bob receives $V_{QS}^\alpha[|a\rangle_S \otimes |\tilde{\Psi}\rangle_Q]$. Because the signature is correct, Alice and Bob cannot notice Eve’s illegal act. Alice discloses $U_{iQ}^Q$ in public and finally Eve gets $|\Psi\rangle_Q$.

To avoid this trouble, Bob needs to verify that a batch of qubits ($|\Psi\text{crypt}\rangle$ or Eve’s fake) has arrived. For example, it is good for Bob to take the following method.

We construct a qubit from a pair of optical paths (modes) that are represented by $x$ and $y$ in Figure 5. iii[11]. We describe a state that there is no photon on a mode as $|0\rangle$ and a state that there is a photon on a mode as $|1\rangle$. Writing a state that no photon is on the mode $x$ and one photon is on the mode $y$ as $|0\rangle_x \otimes |1\rangle_y = |01\rangle$, we regard $|01\rangle$ as logical $|\bar{0}\rangle$. We regard $|1\rangle_x |0\rangle_y$ as logical $|\bar{1}\rangle$ similarly. Hence, we can write an arbitrary state of a qubit as

$$|\psi\rangle = \alpha |01\rangle + \beta |10\rangle = \alpha |\bar{0}\rangle + \beta |\bar{1}\rangle \quad \text{for } |\alpha|^2 + |\beta|^2 = 1.$$
Figure 5: The photon counting measurement with nonlinear optical devices.

(i) 50-50 beamsplitter

\[ B : |1\rangle_a |0\rangle_b \rightarrow \frac{1}{\sqrt{2}} (|1\rangle_a |0\rangle_b - |0\rangle_a |1\rangle_b) \]
\[ |0\rangle_a |1\rangle_b \rightarrow \frac{1}{\sqrt{2}} (|1\rangle_a |0\rangle_b + |0\rangle_a |1\rangle_b) \]

(ii) Kerr-type quantum phase-gate

\[ K : |1\rangle_x |0\rangle_a \rightarrow |1\rangle_x |0\rangle_a \]
\[ |0\rangle_x |1\rangle_a \rightarrow |0\rangle_x |1\rangle_a \]
\[ |1\rangle_x |1\rangle_a \rightarrow -|1\rangle_x |1\rangle_a \]
Figure 6: A quantum network for photon counting.

We assume Eve puts a 50-50 beamsplitter halfway on the quantum channel to take away photons. A state of a photon is a superposition of a state that it is on the side of Bob and a state that it is on the side of Eve with amplitude $1/\sqrt{2}$ each,

$$\frac{1}{\sqrt{2}}|\text{photon on the side of Bob}\rangle + \frac{1}{\sqrt{2}}|\text{photon on the side of Eve}\rangle.$$

To examine whether the qubit of $|\psi\rangle$ has come on his own side or not, Bob prepares another auxiliary photon and applies nonlinear interaction between the logical photon on the mode $x$ or $y$ and the auxiliary one. On an optical system of Figure 5. iii, if the photon counter $D_a$ detects the auxiliary photon, the logical photon is projected into the state that it is on the side of Bob. On the other hand, $D_b$’s detection projects the logical photon into the state that it is on the side of Eve.

In Figure 5. iii, there are beamsplitters $B$ which apply $SU(2)$ transformations to logical kets $\{|0\rangle, |1\rangle\}$ as Figure 5. i, and Kerr-type devices $K$ which induce nonlinear interactions between two incoming photons as Figure 5. ii. The device $K$ shifts a phase of a wave function by $\pi$ only if a pair of photons comes into it. (Q. A. Turchette et al. succeeds in shifting the phase by $\Delta \sim 16^\circ[12].$)

To clarify the operation of Figure 5. iii, we describe it by a network of quantum gates in Figure 6[13]. Assuming the first and the second qubits are in an arbitrary entangled state $|\Psi\rangle_Q$, we examine whether the first qubit exists or not by measuring an auxiliary qubit system $A$. When we write the whole system as

$$|\Psi\rangle_Q|0\rangle_A = \sum_{i,j \in \{0,1\}} c_{ij}|i\rangle_2|j\rangle_1|0\rangle_A,$$

$|\Psi\rangle_Q|0\rangle_A$ is transformed as follows in Figure 6,

$$|\Psi\rangle_Q|0\rangle_A \xrightarrow{\text{step 1}} \sum_i (c_{i0}|i\rangle_2|0\rangle_1 + c_{i1}|i\rangle_2|1\rangle_1)|0\rangle_A$$

$$\xrightarrow{\text{step 2}} \sum_i (c_{i0}|i\rangle_2|0\rangle_1 + c_{i1}|i\rangle_2|1\rangle_1)|1\rangle_A = |\Psi\rangle_Q|1\rangle_A.$$

Therefore, measuring an auxiliary system as shown in Figures 5 and 6 for each channel, Bob can examine whether all of the qubits are arrived or not.
Figure 7: Eve’s Intercept/Resend attack on the system \( S \).

4 Security against eavesdropping

It is difficult to consider all strategies Eve may take. In this section, we assume Eve to make only the Intercept/Resend attack. Eve measures each transmitted qubit with a proper basis independently and sends an alternative one according to the result of the measurement[2]. We pay attention to the following fact. Eve cannot extract information about \( |\Psi\rangle_Q \) at all without getting the first password \( i \) of \( U^Q_i |\Psi\rangle_Q \), because the enciphered density operator is in proportion to \( I \) for her. Therefore, Eve needs to keep her illegal action secret from Alice and Bob during their authentication process so that Alice may disclose the first password \( i \). In this section, we estimate a probability that Alice and Bob fail to notice Eve’s illegal act.

For simplicity, we assume that \( |\Psi\rangle_Q \) is an arbitrary \( n \)-qubit product state for a while. At the first encryption, Alice applies \( \forall U^Q_i = \sigma_i \otimes \ldots \otimes \sigma_i \) to \( |\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle \). Hence, \( U^Q_i |\Psi\rangle_Q \) is also a product state and we may treat each qubit independently.

We write the \( k \)th qubit (\( \forall k \in \{1, \ldots, n\} \)) of \( U^Q_i |\Psi\rangle_Q \) as \( |\psi\rangle_Q = \alpha |0\rangle_Q + \beta |1\rangle_Q \) and the \( k \)th qubit of the signature as \( |a\rangle_S \) (\( a \in \{0, 1\} \)). A state that Alice sends at \( t = 0 \) in Figure 7 is given by

\[
\alpha L_1 |0\rangle_Q L_2 |a\rangle_S + \beta L_1 |1\rangle_Q L_2 |a \oplus 1\rangle_S.
\]

Bob measures only the system \( S \) at \( t = \tau \). If he gets \( |a\rangle_S \), Alice and Bob consider the state is transmitted correctly. Bob uses the following projection operator for the measurement,

\[
\Pi_{L_1, L_2}^{QS} = (L_1 |0\rangle \langle 0| L_1^\dagger)_{Q} \otimes (L_2 |a\rangle \langle a| L_2^\dagger)_{S} + (L_1 |1\rangle \langle 1| L_1^\dagger)_{Q} \otimes (L_2 |a \oplus 1\rangle \langle a \oplus 1| L_2^\dagger)_{S}.
\]

Here, we assume Eve makes an attack only on the qubit of the system \( S \) as Figure 7 (\( U \) and \( V \) are arbitrary unitary transformations applied to one qubit). We write the dynamical process of \( S \) as a completely positive linear map \( $ \) that represents Eve’s Intercept/Resend attack on \( S \)[14]. Hence, the density operator \( \rho^S \) at \( t = \tau \) evolves to \( \$ (\rho^S) \) at \( t = \tau \). We can write the state of \( QS \) at \( t = \tau \) as

\[
\rho_{L_1, L_2}^{QS}(\tau) = |\alpha|^2 (L_1 |0\rangle \langle 0| L_1^\dagger)_{Q} \otimes \$ (L_2 |a\rangle \langle a| L_2^\dagger)_{S},
\]
Figure 8: Eve’s Intercept/Resend attack on one qubit.

\[ |\phi_s\rangle - U - V \rightarrow \cdots \] \[ |\phi_s\rangle = L_2 |0\rangle_s \]

\[ 0 \rightarrow \square \]

\[ \alpha \beta^* (L_1 |0\rangle \langle 1| L_1^\dagger) Q \otimes S_L (L_2 a \langle a + 1 | L_2^\dagger) S \]

\[ + \beta \alpha^* (L_1 |1\rangle \langle 0| L_1^\dagger) Q \otimes S_L (L_2 a \langle a + 1 | L_2^\dagger) S \]

\[ + |\beta|^2 (L_1 |1\rangle \langle 1| L_1^\dagger) Q \otimes S_L (L_2 a \langle a + 1 | L_2^\dagger) S. \]

We can write the probability \( P \) that Bob obtains \( |a\rangle_s \) as

\[ P = \text{Tr}_{Q S} [\rho L_{2L_1, L_2}^S (\tau) \Pi_{L_1, L_2}^Q S] \]

\[ = |\alpha|^2 (a L_2^\dagger S_L (a L_2^\dagger) a) + |\beta|^2 (a + 1) L_2^\dagger S_L (a + 1) |a + 1| L_2^\dagger (a + 1). \] (4)

Seeing this, we find the following fact. Although the initial state \( |\psi\rangle_Q \) of the system \( Q \) is a superposition of \( |0\rangle \) and \( |1\rangle \), we may regard the state as a mixed state of \( |\psi\rangle_Q = |0\rangle \) and \( |\psi\rangle_Q = |1\rangle \) with classical probability for evaluating \( P \).

Therefore, the probability of Bob’s authentication is equal to an average of probabilities that a network of quantum gates in Figure 8 gives \( |\phi\rangle_s \) as an outcome from an incoming state \( |\phi\rangle_s = L_2 |0\rangle_s \) for all of \( L_2 \in L \). Here, we evaluate \( P \) as follows. \( U \) and \( V \) are arbitrary unitary transformations applied to one qubit. We assume \( U \) is defined as

\[ U |\varphi_0\rangle = |0\rangle, \quad U |\varphi_1\rangle = |1\rangle, \]

where \( \{|\varphi_0\rangle, |\varphi_1\rangle\} \) is a certain orthonormal basis of \( \mathcal{H}_2 \). Then, we write \( |\psi\rangle = c_0 |\varphi_0\rangle + c_1 |\varphi_1\rangle \). The state is transformed on the network of Figure 8 as

\[ |\psi\rangle_s |0\rangle_E = (c_0 |\varphi_0\rangle_s + c_1 |\varphi_1\rangle_s) |0\rangle_E \]

\[ \xrightarrow{U} (c_0 |0\rangle_s + c_1 |1\rangle_s) |0\rangle_E \]

\[ \xrightarrow{\text{C-NOT}} c_0 |0\rangle_s |0\rangle_E + c_1 |1\rangle_s |1\rangle_E \]

\[ \xrightarrow{V} c_0 V |0\rangle_s |0\rangle_E + c_1 V |1\rangle_s |1\rangle_E. \] (5)

Seeing Eq. (5), we find that Eve measures the enciphered qubit in the basis \( \{|\varphi_0\rangle, |\varphi_1\rangle\} \) and send a ket vector of a basis \( \{|V|0\rangle, |V|1\rangle\} \) according to a result of measurement. Hence, we can write the probability \( P_\phi \) that Bob get the correct signature for \( |\phi\rangle_s \) in spite of Eve’s illegal act as

\[ P_\phi = |c_0|^2 |\langle \phi |V|0\rangle|^2 + |c_1|^2 |\langle \phi |V|1\rangle|^2. \] (6)

From now on, for simplicity, we write equations with density operators. Defining

\[ \rho_\phi = |\phi\rangle \langle \phi|, \quad \rho_0 = |\varphi_0\rangle \langle \varphi_0|, \quad \rho_1 = |\varphi_1\rangle \langle \varphi_1|, \quad \rho_0' = V |0\rangle \langle 0| V^\dagger, \quad \rho_1' = V |1\rangle \langle 1| V^\dagger, \]

we can write Eq. (5) as

\[ \rho_\phi \rightarrow S (\rho_\phi) = (\text{Tr}_{\phi} \rho_0) \rho_0' + (\text{Tr}_{\phi} \rho_1) \rho_1', \]

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and Eq. (6) as

\[ P_\phi = \text{Tr}[\rho_\phi^\dagger \rho_\phi]. \]

Four density operators \( L_2 |0\rangle\langle 0| L_2^\dagger (L_2 \in \mathcal{L}) \), emitted as \( |\phi\rangle_S \) with equal probability, are described as

\[ \rho_i = \frac{1}{2}(I + \sigma_z), \quad \rho_{\leftrightarrow} = \frac{1}{2}(I - \sigma_z), \quad \rho_{\phi} = \frac{1}{2}(I + \sigma_x), \quad \rho_{\phi'} = \frac{1}{2}(I - \sigma_x). \]

Then we define

\[ \tilde{\rho}_i = \frac{1}{2}[I + (-1)^i X \cdot \sigma], \quad \tilde{\rho}_j' = \frac{1}{2}[I + (-1)^j X' \cdot \sigma] \quad \text{for } i, j \in \{0, 1\}, \]

where \( X = (X, Y, Z) \) and \( X' = (X', Y', Z') \) are arbitrary three-component real vectors with \( |X|^2 = |X'|^2 = 1 \). Using the following formula,

\[ \text{Tr}(I + A \cdot \sigma)(I + B \cdot \sigma) = 2(1 + A \cdot B), \]

and averaging four kinds of \( P_\phi \), we estimate \( P_B \) that Bob measures the correct signature in spite of Eve’s illegal act at

\[ P_B = \frac{1}{4}(P_i + P_{\leftrightarrow} + P_{\phi} + P_{\phi'}) = \frac{1}{4}(2 + XX' + ZZ') \leq \frac{3}{4}. \quad (7) \]

Therefore, the probability that Alice and Bob do not notice Eve make an attack by the network of Figure 8 is \( 3/4 \) or less. If Eve makes the Intercept/Resend attack on the system \( Q \), we can give a similar discussion. Consequently, the probability that Alice and Bob do not notice Eve’s attacks on \( m \) qubits (either \( Q \) or \( S \) in each pair) is \( (3/4)^m \) at most.

Next, we consider the case that Eve makes the Intercept/Resend attack on both qubits of a pair \( QS \) independently in Figure 9 (\( U_1, U_2, V_1, \) and \( V_2 \) are arbitrary unitary transformations). Measuring \( \rho^{QS} \) transmitted from Alice, Eve sends the following density operator \( \rho^{QS'} \) to Bob,

\[ \rho^{QS'} = \$ (\rho^{QS}) = \sum_{i,j \in \{0,1\}} \text{Tr}(\rho^{QS} \tilde{\rho}_{Q,i} \tilde{\rho}_{S,j}^\dagger) \tilde{\rho}_{Q,i}^\dagger \tilde{\rho}_{S,j}. \]
where

\[ \tilde{\rho}_{Q,i} = U_1\langle i|U_1 = (1/2)[I + (-1)^iX_1 \cdot \sigma], \]
\[ \tilde{\rho}_{S,j} = U_2\langle j|U_2 = (1/2)[I + (-1)^jX_2 \cdot \sigma], \]
\[ \tilde{\rho}_{Q,i} = V_1\langle i|V_1 = (1/2)[I + (-1)^iX_3 \cdot \sigma], \]
\[ \tilde{\rho}_{S,j} = V_2\langle j|V_2 = (1/2)[I + (-1)^jX_4 \cdot \sigma], \]

and \(|X_k|^2 = 1 (k = 1, \ldots, 4)\).

Here, we can assume \(|a\rangle_S\) to be \(|0\rangle_S\) without losing generality. We write a transmitted state of \(Q\) as \(|\psi\rangle_Q = \alpha|0\rangle + \beta|1\rangle\). Because Alice has sixteen kinds of ways to send the state for \(L_1, L_2 \in L\), Bob’s final probability for authentication is described as

\[ P_B = \frac{1}{16} \sum_{L_1, L_2 \in L} \text{Tr}[(\rho_{L_1, L_2}^{Q_S})\Pi_{L_1, L_2}^{Q_S}]. \]  \hspace{1cm} (9)

Writing the density operator of the state \(\alpha L_1|0\rangle_Q L_2|0\rangle_S + \beta L_1|1\rangle_Q L_2|1\rangle_S\) that Alice sends to Bob as \(\rho_{L_1, L_2}^{Q_S}\), we can describe its explicit form as

\[ \rho_{L_1, L_2}^{Q_S} = |\alpha|^2(L_1|0\rangle\langle 0|L_1^\dagger)_Q \otimes (L_2|0\rangle\langle 0|L_2^\dagger)_S \]
\[ + \alpha\beta^*(L_1|0\rangle\langle 1|L_2^\dagger)_Q \otimes (L_2|0\rangle\langle 1|L_1^\dagger)_S + \beta\alpha^*(L_1|1\rangle\langle 0|L_1^\dagger)_Q \otimes (L_2|1\rangle\langle 0|L_2^\dagger)_S \]
\[ + |\beta|^2(L_1|1\rangle\langle 1|L_1^\dagger)_Q \otimes (L_2|1\rangle\langle 1|L_2^\dagger)_S. \]  \hspace{1cm} (10)

The projection operator for Bob is given by

\[ \Pi_{L_1, L_2}^{Q_S} = (L_1|0\rangle\langle 0|L_1^\dagger)_Q \otimes (L_2|0\rangle\langle 0|L_2^\dagger)_S + (L_1|1\rangle\langle 1|L_1^\dagger)_Q \otimes (L_2|1\rangle\langle 1|L_2^\dagger)_S. \]

Eq. (9) is linear for \(\rho_{L_1, L_2}^{Q_S}\). Therefore, we can divide Eq. (10) into terms for calculation.

First, we think about the first and fourth terms of Eq. (10). We write the first term as

\[ \rho_{L_1, L_2}^{Q_S} = (L_1|0\rangle\langle 0|L_1^\dagger)_Q \otimes (L_2|0\rangle\langle 0|L_2^\dagger)_S. \]  \hspace{1cm} (11)

For example, if \(L_1 = L_2 = I\), we obtain

\[ \rho_{I,I}^{Q_S} = (1/4)[I + \sigma_z]_Q \otimes [I + \sigma_z]_S, \]
\[ \Pi_{I,I}^{Q_S} = (1/4)[(I + \sigma_z)_Q \otimes (I + \sigma_z)_S + (I - \sigma_z)_Q \otimes (I - \sigma_z)_S], \]

and

\[ \text{Tr}[(\rho_{I,I}^{Q_S})\Pi_{I,I}^{Q_S}] = (1/2)(1 + Z_1Z_2Z_3Z_4). \]

From similar calculations, we get

\[ \text{Tr}[(\rho_{L_1, L_2}^{Q_S})\Pi_{L_1, L_2}^{Q_S}] = \begin{cases} 
(1/2)(1 + Z_1Z_2Z_3Z_4) & \text{for } L_1, L_2 \in \{I, \sigma_z\}, \\
(1/2)(1 + X_1X_2X_3X_4) & \text{for } L_1, L_2 \in \{H, H\sigma_x\}, \\
(1/2)(1 + Z_1Z_2Z_3X_4) & \text{for } L_1 \in \{I, \sigma_z\}, L_2 \in \{H, H\sigma_x\}, \\
(1/2)(1 + X_1Z_2X_3Z_4) & \text{for } L_1 \in \{H, H\sigma_x\}, L_2 \in \{I, \sigma_z\}. 
\end{cases} \]

Therefore, we obtain

\[ \frac{1}{16} \sum_{L_1, L_2} \text{Tr}[(\rho_{L_1, L_2}^{Q_S})\Pi_{L_1, L_2}^{Q_S}] = \frac{1}{2} + \frac{1}{8}(X_1X_3 + Z_1Z_3)(X_2X_4 + Z_2Z_4). \]
Next, we think about the second and third terms of Eq. (10). We write the second term as
\[ \Delta \rho_{L_1,L_2}^{QS} = (L_1|0\rangle\langle 1|L_1^\dagger)_Q \otimes (L_2|0\rangle\langle 1|L_2^\dagger)_S. \]
For example, if \( L_1 = L_2 = I \), we obtain
\[ \Delta \rho_{I,I}^{QS} = (1/4)(\sigma_x + i\sigma_y)_Q \otimes (\sigma_x + i\sigma_y)_S, \]
\[ \text{Tr}[\mathbb{S}(\Delta \rho_{I,I}^{QS})\Pi_{I,I}^{QS}] = (1/2)(X_1 + iY_1)(X_2 + iY_2)Z_3Z_4. \]
From similar calculations, we get
\[
\text{Tr}[\mathbb{S}(\Delta \rho_{L_1,L_2}^{QS})\Pi_{L_1,L_2}^{QS}] = \begin{cases} 
(1/2)[X_1 + iY_1][X_2 + iY_2]Z_3Z_4 & (L_1, L_2) = (I, I), \\
-1/2)[X_1 + iY_1][X_2 - iY_2]Z_3Z_4 & (I, \sigma_x), \\
(1/2)[Z_1 - iY_1][Z_2 - iY_2]X_3X_4 & (H, H), \\
-1/2)[Z_1 - iY_1][Z_2 + iY_2]X_3X_4 & (H, H\sigma_x), \\
(1/2)[X_1 - iY_1][Z_2 - iY_2]X_3X_4 & (I, H), \\
-1/2)[X_1 + iY_1][Z_2 + iY_2]Z_3X_4 & (I, H\sigma_x), \\
(1/2)[Z_1 - iY_1][X_2 + iY_2]X_3Z_4 & (H, H), \\
-1/2)[Z_1 - iY_1][X_2 - iY_2]X_3Z_4 & (H, H\sigma_x), \\
-1/2)[Z_1 + iY_1][X_2 - iY_2]X_3Z_4 & (H, H). 
\end{cases}
\]
Consequently, we obtain
\[ \frac{1}{16} \sum_{L_1,L_2} \text{Tr}[\mathbb{S}(\Delta \rho_{L_1,L_2}^{QS})\Pi_{L_1,L_2}^{QS}] = -\frac{1}{8}Y_1Y_2(Z_3 - X_3)(Z_4 - X_4). \]
Finally, obtaining
\[ P_B = \frac{1}{2} + \frac{1}{8}(X_1X_3 + Z_1Z_3)(X_2X_4 + Z_2Z_4) - \frac{1}{8}(\alpha\beta^* + \alpha^*\beta)Y_1Y_2(Z_3 - X_3)(Z_4 - X_4), \]
we can show \( P_B \leq 3/4 \). (See Appendix A.) Therefore, if \( |\Psi\rangle_Q \) is an \( n \)-qubit product state and if Eve makes the Intercept/Resend attack on both qubits of a pair \( QS \) independently in Figure 9, the probability that Eve’s illegal acts cannot be found is equal to \( 3/4 \) or less per one qubit.

Especially, if the transmitted information is classical, \( |\Psi\rangle_Q \) is a product state of \( |0\rangle \) and \( |1\rangle \). All of the \( 2n \) qubits transmitted are in states chosen from four ket vectors of two conjugate bases at random. If we regard \( a \) as a key of an \( n \)-bit random string and \( |\Psi\rangle_Q \) as an \( n \)-bit enciphered classical message, our method is equivalent to the one-time pad method with BB84. Assuming Eve makes attacks on \( m \) pairs of qubits in \( QS \), we can estimate the probability of Eve’s success in eavesdropping at \((3/4)^m\) or less.

Then, we consider the case that \( |\Psi\rangle_Q \) is an arbitrary entangled state of \( n \) qubits. The enciphered state of \( |\Psi\rangle_Q \) with \( U_1^Q \) is also entangled and it is given by Eq. (3).

First, we consider that Eve makes the Intercept/Resend attack on either one in a pair of qubits of the system \( Q \) and \( S \) as Figure 7. If Eve makes this attack on \( m \) pairs out of \( n \) pairs of the system \( QS \), we can regard the transmission as sending an ensemble of
product states where each qubit is $|0\rangle$ or $|1\rangle$ with a classical probability, like Eq. (4). We can think in a similar way before and conclude that the probability Eve’s illegal act cannot be found is $(3/4)^m$ or less.

Next, we consider the case that Eve makes the attack on both qubits of a pair on the entangled system $QS$ as Figure 9. If Eve makes this attack on $m$ pairs out of $n$ pairs, we can write the probability Eve is not found as an equation which is similar to Eq. (12) and it is estimated at $(3/4)^m$ or less. (See Appendix B.)

5 Discussion

To understand our method more clearly, we consider a simple one and compare it with ours. For transmitting an $n$-qubit quantum state in secrecy, we can take the following method. Enciphering an $n$-qubit state $\rho_n$ as Eq. (2), Alice prepares other $n$ qubits as check ones that are given as $\{|0\rangle, |1\rangle\}$ or $(1/\sqrt{2})(|0\rangle \pm |1\rangle)$ at random respectively. Then, Alice permutes all of the $2n$ qubits at random and sends them to Bob. (Alice needs to prepare $n$ check qubits or more so that fidelity of qubits that Eve takes away is equal to $1/2$. Because states of the check qubits are disclosed later, they cannot make a contribution to fidelity.)

Here, we assume Eve tries to eavesdrop on only one qubit of $\rho_n$. Because Eve does not know which qubits are check ones, a probability that Alice and Bob notice Eve’s illegal act is $(1/2) \times (1/4)$ at most, in spite of $1/4$ for our method. This is because we use entanglement in our method.

In this paper, the security against Eve’s attack with using entanglement is not considered (for example, a case that she uses a quantum computer for eavesdropping as Figure 10). In Figure 10, it is difficult to evaluate the upper bound of the probability that Bob measures the signature correctly for arbitrary unitary transformations $U_1, U_2, V_1,$ and $V_2$. For instance, assuming

\[
U_1 = U_2 = V_1 = V_2 = U \quad \text{where} \quad U = \begin{pmatrix}
\cos(\pi/8) & \sin(\pi/8) \\
\sin(\pi/8) & -\cos(\pi/8)
\end{pmatrix},
\]
and assuming that $|Ψ⟩_Q$ represents classical information (an $n$-qubit product state of $|0⟩$ and $|1⟩$), we get $P_B = (13/16) > (3/4)$ for Figure 10. i and $P_B = (11/16) < (3/4)$ for Figure 10. ii. Eavesdropping with $U$ is equivalent to measuring and resending a qubit in the following basis,

$$|φ_i⟩⟨φ_i| = \frac{1}{2}(I + (-1)^iX \cdot σ), \quad X = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}),$$

which is called the Breidbart basis[2]. About Figure 10. i, $P_B$ may exceed $3/4$. But, the amount of information Eve can extract in Figure 10. i seems to be less than the amount of it she obtains by the Intercept/Resend attack on one qubit in Figure 7.

In our method, if Eve takes away qubits, Alice and Bob lose original information of them.

We mentioned in Sec. 1 that the classical key distribution can be done only by the uncertainty, and the entanglement is not essential for it[3][4]. On the other hand, for transmitting quantum states by the quantum teleportation, the entanglement plays an essential role[6]. Our method uses both two properties.

Recently, the method for transmitting classical binary data (not a classical random string) in secrecy with a pair of entangled photons has been proposed[15]. It is characterized by the following facts. First, Alice and Bob prepare two conjugate bases on $H^2 \otimes H^2$ each for encoding message and measuring photons. Second, they use a two-dimensional subspace of $H^2 \otimes H^2$ for encoding a binary digit.

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**A The maximum value of $P_B$ for a product state $|Ψ⟩_Q$**

Here, we show that $P_B$ defined in Eq. (12) never exceeds $3/4$.

Because of $|α|^2 + |β|^2 = 1$, we get $-1 \leq α^*β + αβ^* ≤ 1$. Hence, altering the signs of $X_i$ as $X_i → -X_i$ ($i = 1, \cdots, 4$), we can write the upper bound of $P_B$ as

$$P_B ≤ \frac{1}{2} + \frac{1}{8}f_{MAX},$$

where $f_{MAX}$ is the maximum value of

$$f = (X_1X_3 + Z_1Z_3)(X_2X_4 + Z_2Z_4) + Y_1Y_2(X_3 + Z_3)(X_4 + Z_4),$$

(13)

with $|X_i|^2 = 1$ and $X_i, Y_i, Z_i ≥ 0$ for $i = 1, \cdots, 4$.

Seeing Eq. (13), we found the following facts. First, the maximum value of the first term of $f$ is equal to 1, and the maximum value of the second term is equal to 2. Next, for instance, if

$$X_1 = X_2 = (0, 1, 0), \quad X_3 = X_4 = (1/\sqrt{2}, 0, 1/\sqrt{2}),$$

(14)
we obtain $f = 2$. Hence we can say $f_{\text{MAX}} \geq 2$. Furthermore, it is clear that assuming $Y_3 = Y_4 = 0$ does not affect the value of $f_{\text{MAX}}$.

From now on, we show $f \leq 2$ ($f_{\text{MAX}} = 2$). To do it, we show there are no local maximum values of $f$ that exceed 2 by applying the Lagrange's method of indeterminate coefficients. Defining

$$F(\{X_i\}, \{\lambda_i\}) = f + \sum_{i=1}^{4} \lambda_i(1 - |X_i|^2),$$

we obtain

$$\begin{align*}
\partial F/\partial X_i &= X_i(X_2X_4 + Z_2Z_4) - 2\lambda_1X_i = 0, \quad (15) \\
\partial F/\partial Z_1 &= Z_3(X_2X_4 + Z_2Z_4) - 2\lambda_1Z_1 = 0, \quad (16) \\
\partial F/\partial X_3 &= X_1(X_2X_4 + Z_2Z_4) + Y_1Y_2(X_4 + Z_4) - 2\lambda_3X_3 = 0, \quad (17) \\
\partial F/\partial Z_3 &= Z_1(X_2X_4 + Z_2Z_4) + Y_1Y_2(X_4 + Z_4) - 2\lambda_3Z_3 = 0, \quad (18) \\
\partial F/\partial Y_1 &= Y_2(X_3 + Z_3)(X_4 + Z_4) - 2\lambda_1Y_1 = 0. \quad (19)
\end{align*}$$

We can obtain similar equations for $\partial F/\partial X_2 = 0$, $\partial F/\partial Z_2 = 0$, $\partial F/\partial X_4 = 0$, $\partial F/\partial Z_4 = 0$, $\partial F/\partial Y_2 = 0$.

We think about Eqs. (15) and (16). Assuming $X_2X_4 + Z_2Z_4 = 0$, we can get $f \leq 2$ from Eq. (13). Hence, we assume $X_2X_4 + Z_2Z_4 \neq 0$. Then, if $\lambda_1 = 0$, we get $X_3 = Z_3 = 0$ and $f = 0$. Hence, we assume $\lambda_1 \neq 0$. As a result of these, we obtain

$$X_3 = \kappa X_1, \quad Z_3 = \kappa Z_1, \quad \kappa \neq 0. \quad (20)$$

Substituting Eq. (20) into Eqs. (17) and (18), we get

$$X_1[(X_2X_4 + Z_2Z_4) - 2\lambda_3\kappa] = Z_1[(X_2X_4 + Z_2Z_4) - 2\lambda_3\kappa] = -Y_1Y_2(X_4 + Z_4).$$

If $(X_2X_4 + Z_2Z_4) - 2\lambda_3\kappa = 0$, we obtain $Y_1Y_2(X_4 + Z_4) = 0$ and $f \leq 1$. Hence, assuming $(X_2X_4 + Z_2Z_4) - 2\lambda_3\kappa \neq 0$, we obtain $X_1 = Z_1$ and $X_3 = Z_3$. From a similar discussion, we get $X_2 = Z_2$, $X_4 = Z_4$. Furthermore, because $Y_3 = Y_4 = 0$, we obtain

$$X_3 = X_4 = (1/\sqrt{2}, 0, 1/\sqrt{2}). \quad (21)$$

Substituting Eq. (21) into Eq. (19), we obtain $Y_2 = \lambda_1Y_1$. Similarly, from $\partial F/\partial Y_2 = 0$, we obtain $Y_1 = \lambda_2Y_2$. Hence we get $Y_2 = \lambda_1\lambda_2Y_2$. Because $Y_2 = 0$ gives $f \leq 1$, we assume $Y_2 \neq 0$ and we obtain $\lambda_1\lambda_2 = 1$. Then, substituting Eq. (21) into Eq. (15), we obtain $X_2 = 2\lambda_1X_1$. Similarly, from $\partial F/\partial X_2 = 0$, we obtain $X_1 = 2\lambda_2X_2$. Therefore, obtaining $X_2 = 4\lambda_1\lambda_2X_2$, we arrive at $X_2 = X_1 = 0$. But, this result is not consistent with Eqs. (20) and (21).

From this consideration, we find there are no local maximum values of $f$ that exceed 2, and we can conclude $f_{\text{MAX}} = 2$ at Eq. (14).
\section{The maximum value of $P_B$ for an entangled state $|\Psi\rangle_Q$}

We estimate the probability that Eve's illegal act cannot be found in the case where she makes the Intercept/Resend attack on $m$ pairs of qubits on the system $QS$ for an arbitrary entangled $|\Psi\rangle_Q$ of $n$ qubits.

For simplicity, we assume $|\Psi\rangle_Q$ to be an arbitrary entangled state of two-qubit system $qq'$ at first,

$$|\Psi\rangle_Q = \sum_{i,j\in\{0,1\}} c_{ij} |i\rangle_q \otimes |j\rangle_{q'} \in \forall \mathcal{H}_2^2.$$  

Alice puts two qubits of the system $S(=ss')$ for the signature on the qubits of the system $Q(=qq')$ respectively. Then, she makes entanglement between the systems $Q$ and $S$ with C-NOT gates, applies $L_1$, $L_2$, $L'_1$, $L'_2 \in \mathcal{L}$ to 4 qubits $q$, $s$, $q'$, $s'$ respectively, and sends them to Bob (see Figure 7). Eve makes the Intercept/Resend attacks on the systems $q$, $s$, $q'$, $s'$ respectively as shown in Figure 9. We can assume the initial states of qubits $s$, $s'$ that represents the signature to be $|0\rangle_s |0\rangle_{s'}$ without losing generality.

Writing the state sent by Alice as

$$\rho_{L_1L_2L'_1L'_2}^Q = (L_1L_2L'_1L'_2) \sum_{i,j\in\{0,1\}} \{ [c_{ij}]^2 |i\rangle_q \otimes |i\rangle_s \otimes (|j\rangle_{q'} \otimes |j\rangle_{s'})+c_{ij}c_{ij}^\ast |i\rangle_q \otimes |i\rangle_s \otimes (|j\rangle_{q'} \otimes |j\rangle_{s'})+c_{ij}c_{ij}^\ast |i\rangle_q \otimes |i\rangle_s \otimes (|j\rangle_{q'} \otimes |j\rangle_{s'}) \}

\begin{equation}
\begin{aligned}
\rho_{L_1L_2L'_1L'_2}^S &= (L_1L_2L'_1L'_2) \sum_{i,j,k,l\in\{0,1\}} \text{Tr} (\rho_{L_1L_2L'_1L'_2}^Q \rho_{i,j,k,l}^S) \rho_{i,j,k,l}^S, \\
\end{aligned}
\end{equation}

where $\overline{i} = i + 1 \, (\text{mod} \, 2)$.

Eavesdropping on the state $\rho_{L_1L_2L'_1L'_2}^Q$, Eve transforms it to the following state,

$$\mathcal{E}(\rho_{L_1L_2L'_1L'_2}^Q) = \sum_{i,j,k,l\in\{0,1\}} \text{Tr} (\rho_{L_1L_2L'_1L'_2}^Q \rho_{i,j,k,l}^E) \rho_{i,j,k,l}^E, \tag{22}$$

where

\begin{align}
\overline{\rho}_{q,i} &= (1/2)[I + (-1)^i X_1 \cdot \sigma], \quad \overline{\rho}_{s,j} \quad = (1/2)[I + (-1)^j X_2 \cdot \sigma], \\
\overline{\rho}_{q',k} &= (1/2)[I + (-1)^k X_3 \cdot \sigma], \quad \overline{\rho}_{s',l} \quad = (1/2)[I + (-1)^l X_4 \cdot \sigma], \\
\overline{\rho}_{q',k} &= (1/2)[I + (-1)^k X_1 \cdot \sigma], \quad \overline{\rho}_{s',l} \quad = (1/2)[I + (-1)^l X_2 \cdot \sigma], \tag{23}
\end{align}

and $|X_k|^2 = |X_k'|^2 = 1 \, (k = 1, \cdots, 4)$. Bob measures it with the projection operator,

$$\Pi_{L_1L_2L'_1L'_2}^{QS} = (L_1L_2)(|0\rangle_q \otimes |0\rangle_s + |1\rangle_q \otimes |1\rangle_s)(L_1L_2)^\dagger \otimes (L'_1L'_2)(|0\rangle_{q'} \otimes |0\rangle_{s'} + |1\rangle_{q'} \otimes |1\rangle_{s'})(L'_1L'_2)^\dagger.$$
The probability that Bob measures the correct signature is given by

\[
P_B = \left( \frac{1}{16} \right)^2 \sum_{L_1, L_2 \in \mathcal{L}} \sum_{L'_1, L'_2 \in \mathcal{L}} \text{Tr} \left[ \rho_{L_1 L_2 L'_1 L'_2} \Pi_{L_1 L_2 L'_1 L'_2} \right]
\]

\[
= \left[ \frac{1}{2} + \frac{1}{8} (X_1 X_3 + Z_1 Z_3)(X_2 X_4 + Z_2 Z_4) \right] \left[ \frac{1}{2} + \frac{1}{8} (X'_1 X'_3 + Z'_1 Z'_3)(X'_2 X'_4 + Z'_2 Z'_4) \right]
\]

\[
+ \sum_{i,j \in \{0,1\}} \left\{ c_{ij} c^*_{ij} \left[ \frac{1}{2} + \frac{1}{8} (X_1 X_3 + Z_1 Z_3)(X_2 X_4 + Z_2 Z_4) \right] \times \left[ -\frac{1}{8} Y_1 Y_2 (Z'_3 - X'_3)(Z'_4 - X'_4) \right] \right\}
\]

\[
+ c_{ij} c^*_{ij} \left[ -\frac{1}{8} Y_1 Y_2 (Z_3 - X_3)(Z_4 - X_4) \left[ -\frac{1}{8} Y'_1 Y'_2 (Z'_3 - X'_3)(Z'_4 - X'_4) \right] \right].
\]

From \( \sum_{i,j \in \{0,1\}} |c_{ij}|^2 = 1 \), we obtain \( |\sum_{i,j \in \{0,1\}} c_{ij} c^*_{ij}| \leq 1 \), \( |\sum_{i,j \in \{0,1\}} c_{ij} c^*_{ij}| \leq 1 \), and \( |\sum_{i,j \in \{0,1\}} c_{ij} c^*_{ij}| \leq 1 \). Therefore, using the result obtained in Appendix A, we can conclude

\[
P_B \leq \left( \frac{1}{2} + \frac{1}{8} f_{\text{MAX}} \right)^2 = \left( \frac{3}{4} \right)^2.
\]

(We pay attention to a fact that each term of \( P_B \) can be gathered with a binomial coefficient.) When Eve attacks on \( m \) pairs out of enciphered qubits generated from an arbitrary \( n \)-qubit entangled state \( |\Psi\rangle_Q \), we obtain \( P_B \leq (3/4)^m \).

References


