We use the Barnett-Pegg formalism of angle operators to study a rotating particle with and without a flux line. Requiring a finite dimensional version of the Wigner function to be well defined we find a natural time quantization that leads to classical maps from which the arithmetical basis of quantum revivals is seen. The flux line, that fundamentally alters the quantum statistics, forces this time quantum to be increased by a factor of a winding number and determines the homotopy class of the path. The value of the flux is restricted to the rational numbers, a feature that persists in the infinite dimensional limit.

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The multi-valued nature of angle has rendered subtle a quantum mechanical formulation of angle operators that are conjugate to angular momentum. A related problem has been the phase operator of a single mode of an electromagnetic field, conjugate to the number operator. We study the effect of one approach on an important model in physics, namely that of a charged particle rotating around a magnetic flux tube that does not penetrate the particle path and which was shown to be an anyon [1]. It is appropriate that such an attempt be made as at the heart of this system is also the multi-valued nature of a gauge field.

The treatment of the angle operators is within the framework proposed by Barnett and Pegg (BP) [2]. This relies on the construction of a finite dimensional Hilbert space in which physical expectation values are calculated and subsequently the limit of infinite dimensionality is taken. The periodic boundary condition on the angular momentum states, however, results in a global or topological effect that is reflected as flux quantization. The BP approach is not without its own issues although it has been applied in various situations. For instance one apparent difficulty that has been resolved only recently is the definition of an angular velocity operator [3]. The discussion below is not so much on the correctness of the methodology but rather observes the restrictions that are imposed due to the structure of the Hilbert space in the BP formalism and the way these are transcended in the infinite dimensional limit. For instance the angular momenta associated with the composite can take on only rational values while time takes on a dense set of real values.

We argue this, quite simply, by constructing a Wigner function in the finite Hilbert space and requiring this construct to be valid for all time. Wigner functions for finite quantum mechanics has also been constructed earlier in [4] while Wigner functions for photon number and phase have been studied by several authors [5]. Our following arguments are closely related to, and informed by, the allowed boundary conditions of quantized torus maps for which a recent treatment is found in [6], and the Wigner function we use has been used in this context previously. Maps are discrete time (in this context canonical as well) transformations, while time is continuous for the charged particle rotating the flux line. However in the BP formalism the finite dimensionality of the Hilbert space imposes a
natural quantization of time.

Both the classical and the quantum dynamics of a rotating particle is essentially an unperturbed twist map, quantization restricting the dynamics to rational sub-lattices. Phenomena peculiar to the quantum dynamics of the rotator such as revivals and fractional revivals due to constructive interference [7] are present almost identically in the corresponding classical arithmetical dynamics. This is an exact mathematical rendering of the periodic bunching of (classical) athletes observed on the racing tracks and used previously as a pedagogic device for quantum revivals [8].

We first consider a bare rotating particle with no threading magnetic flux. We briefly recapitulate the BP formalism. A finite dimensional Hilbert space \( H^{2l+1} \) is obtained on imposing a maximum (minimum) angular momentum quantum number \( l(-l) \). The angle states are the following superposition of the angular momentum states \(|m\rangle, m = -l, \ldots, l\), which are eigenstates of \( \hat{L}_z \) with eigenvalues \( m\bar{\hbar} \):

\[
|\theta_n\rangle = \frac{1}{\sqrt{2l+1}} \sum_{m=-l}^{l} \exp(-im\theta_n) |m\rangle,
\]

the angle eigenvalues are

\[
\theta_n = \frac{2\pi n}{2l+1} \quad (n = 0, 1, \ldots, 2l).
\]

The state \(|m = l\rangle\) is “shifted” by the exponential of the angle operator, defined from the angle states, to \(|m = -l\rangle\). Similarly the angle state \(|n = 2l\rangle\) is shifted to the zero angle state; thus the essential Hilbert space structure is that of a quantized torus phase space. The limiting process is \( l \to \infty, \hbar \) fixed and leads to the cylindrical phase space of a rotator. The BP formalism is in many ways similar to that developed by Schwinger [9] with different motivations. The usual limit of quantum maps on the torus connects \( \hbar \) and \( l \) and is the classical limit that leads to a toral phase space. However for any finite value of \( l \) and \( \hbar \) the Hilbert space structures are identical and we now use this.

Hannay and Berry [10] while studying chaotic Anasov mappings of the torus onto itself used Wigner functions that are “Dirac delta brushes” to describe the effects of the quantum map. Following them, we may define a Wigner function for the states in \( H^{2l+1} \) as [11]

\[
W(\psi, s, r) = \frac{1}{2(2l+1)} \sum_{k=-l}^{l} c_k^* c_{r-k} \exp \left[ i\pi s (r - 2k)/(2l+1) \right].
\]

Here \(-2l \leq r \leq 2l+1\) and \(0 \leq s \leq 4l+1\) are integers and \( c_k = \langle k|\psi \rangle \) is the momentum representation of the wavefunction. It is known that there is a price to pay for the definition of the Wigner function in finite dimensional Hilbert spaces. If \( N \) is the dimension of the space the Wigner function is defined on an \((2N)^2\) grid rather than a \(N^2\) grid, although there are only \(N^2\) independent components, sets of fours being related to each other. If \( N \) is odd, as in the case of the rotator, one representative point may be chosen of the four and the Wigner function can be displayed on an \(N^2\) grid.

Thus in the BP formalism the Wigner function is defined over \( \{2(2l+1)\}^2 \) points. The angle is related to \( s \), while the angular momentum to \( r \). For instance the angle corresponding to \( s \) is \( \theta = 2\pi s/(2(2l+1)) \). As usual a sum over \( s \) gives the probability \( c_{r/2}^* c_{r/2} \) if \( r \) is even
and zero otherwise; and a sum over \( r \) gives the probability density in the position basis only for the even lattice points and is zero otherwise. For this property to hold under time evolution the representative \( N^2 \) grid must map onto itself.

Now the Hamiltonian for a particle of mass \( M \) on a ring of radius \( R \) is

\[
\hat{H} = \frac{\hat{L}_z^2}{2MR^2}
\]

Therefore the time evolution is simply specified by a phase change in the momentum basis:

\[
c_k(t) = \exp(-i\tau k^2)c_k(0),
\]

where \( \tau = \hbar t/(2MR^2) \) is dimensionless time. We have implicitly assumed that the Hamiltonian generates time evolution in the usual manner. The Wigner function evolution is then straightforward:

\[
W(\psi(t), s, r) = \frac{1}{2(2l + 1)} \sum_{k=-l}^{l} \exp(i\tau k^2)c_k^*(0) \times \exp(-i\tau(r - k)^2)c_{r-k}(0) \exp\left[\frac{i\pi s(r - 2k)}{2(2l + 1)}\right].
\]

Simplifying which we get

\[
W(\psi(t), s, r) = W(\psi(0), s - r \frac{(2l + 1)\tau}{\pi}, r).
\]

In the context of linear torus maps such a relationship has been known for sometime [10,11]. However note that we have not used any discrete mapping as yet: time has been a continuous variable. We are now forced to “quantize” time due to the finite dimensionality of the Hilbert space. Indeed we will have to require that \((2l + 1)\tau/\pi = \text{integer} \) (say \( 2j \), \( j \) integer) for the Wigner function above to be even defined. This implies that \( t = jT_0 \), where

\[
T_0 = \frac{4\pi MR^2}{(2l + 1)\hbar}
\]

is the time unit chosen (and not half this). One reason is that \( T_0 \) will then correspond to the period of rotation with the highest allowed angular momentum \( \langle l\hbar \rangle \) in the large \( l \) limit, a natural time scale in finite dimensional spaces. The other has to do with the representative grid mapping onto itself so that the Wigner function retains its properties which disallows \( T_0/2 \) as the fundamental unit of time.

However, there does not seem to be any a-priori reason why other integer multiples of \( T_0 \) should not be the time quantum. The fundamental path corresponding to \( T_0 \) is a single encircling, while topologically distinct paths on the circle, members of the homotopy group \( \pi_1 \) with higher winding numbers, correspond to these other times. In fact this ambiguity is essential for the existence of anyonic behaviour in the flux-tube-charge-particle system. If we were to restrict, for some reason, the time quantum to \( T_0 \) even with the flux, we would end by restricting the composite to be either bosons or fermions but nothing “in between”. This is discussed further below.
We show now how this time quantization reveals the simple arithmetical basis for quantum revivals in the rotator while becoming irrelevant in the large \( l \) limit. If at time \( t = jT_0 \) the arguments of the Wigner function are \( s_j \) and \( r_j \) then at time \((j+1)T_0\) they are \( s_j - 2r_j \) and \( r_j \). Making the identification \( y_j = r_j / (2(2l+1)) \) and \( x_j = \theta_j / (2\pi) = s_j / (2(2l+1)) \) we then get the map

\[
\begin{align*}
x_{j+1} &= x_j - 2y_j \pmod{1} \\
y_{j+1} &= y_j.
\end{align*}
\]

This simple unperturbed twist map is an example of a parabolic cat map and the associated linear transformation \((1, -2), (0, 1)\) is of the quantizable checker-board form of alternating odd and even integer entries [10].

It is easy to see that this simple classical arithmetical map shows revivals and fractional revivals on rational sub-lattices and that the time for the full revival \( t_{\text{rev}} \) then simply corresponds to \( j = N/2 = (2l + 1) \). Thus \( t_{\text{rev}} = jT_0 = 4\pi MR^2/\hbar \); while at \( t_{\text{rev}}/2 \) there is a fractional revival into two subsidiary “wavepackets”. These of course coincide with the usual revival times of a quantum rotator. In the final analysis the finite dimensionality \( l \) plays no part, as it has no physical basis; and neither does the choice of \( T_0 \) over its integer multiples for the time quantum. We must note that the classical fractional revivals are generally not perfect as this depends on the divisibility properties of \( N \), the imperfections disappearing in the \( N \to \infty \) limit.

The effects of a magnetic flux line on a rotating charged particle forms the bound Aharonov-Bohm (AB) effect and the composite system is a model of an anyon [1]. As such there are no restrictions on the flux line threading the particle path. We may either think of the charged particle as having a new energy spectrum and single-valued wavefunctions or having the same energy but multi-valued wavefunctions that acquire a phase as we complete a loop. The differences between the two and their physical content has been discussed before, for example in [12]. However if we apply the BP formalism to a particle with a flux line and demand that the Wigner function remain defined at all instances of time, we would end in restricting the phase acquired to either 0 (periodic) or \( \pi \) (anti-periodic), unless the time quantum is suitably dilated.

The usual minimal substitution leads to the Hamiltonian:

\[
\hat{H} = \frac{(\hat{L}_z - q\Phi / 2\pi c)^2}{2MR^2},
\]

the charge being \( q \) and the speed of light is \( c \) while \( \Phi \) is the threading magnetic flux. We adopt the first viewpoint and consider single-valued momentum eigenstates and an altered energy spectrum. Then

\[
c_k(t) = \exp(-i\tau(k - \alpha)^2)c_k(0),
\]

where \( \alpha = q\Phi / 2\pi \hbar c = \Phi / \Phi_0 \). We then derive that

\[
W(\psi(t), s, r) = W\left(\psi(0), s - \frac{(r - 2\alpha)(2l + 1)\tau}{\pi}, r\right).
\]

We again require that the Wigner function be defined for all time. If the time is still quantized as before in Eq. (8) then we must require that \( 4\alpha \) be an integer. It must be noted
that this quantization of the flux is *independent* of the finite dimensionality $l$ and therefore survives the limit $l \to \infty$. This is unlike the quantization of time which takes on a dense set of real values and transcends to a certain extent the limitations of a finite Hilbert space. The allowed values of $\alpha$ are then 1 (or 0) and 1/2. The values 1/4 and 3/4 are ruled out as the representative Wigner lattice is not left invariant for these choices and time evolution will result in non-zero probabilities at points off the lattice. Thus the phase acquired by a $2\pi$ rotation must either be 0 or $\pi$ and the composite is no more a model of an anyon, as it can have only either bosonic or fermionic character.

These severe restrictions on the phase are linked then to the choice of $T_0$ as the time quantum. However if $\alpha = m/n, m$ and $n$ being co-prime integers, we may dilate $T_0$ to $nT_0$ and the restrictions on the flux are lifted and anyonic behaviour is recovered. Thus for any dimensionality rational flux lines will be supported. In the infinite dimensional limit, as time acquires a dense set of values, the flux line is still restricted to the rational numbers. The time $nT_0$ corresponds to $n$ rotations at the highest allowed angular momentum and appears as the time quantum that is linked to the flux. Thus the fundamental motion is not a single encircling of the flux line, but a path with a winding number $n$. Thus the homotopy class of the particle path is determined by the flux.

We could have adopted the other viewpoint of multi-valued momentum eigenfunctions and unchanged energy eigenvalues. Then the Dirac delta brush on which the Wigner function exits will be supported at points shifted away from the integers by $\alpha$ and once again will result in identical restrictions. Thus if the BP formalism of angle operators is combined with the Wigner function evolution AB magnetic flux lines in flux-tube-charge-particle composites are restricted to the rationals and can be interpreted as corresponding to homotopically distinct paths.

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