For a circular orbit at radius \( r \), we have
\[
\frac{d}{dt} \frac{\theta}{r^2 F} = \frac{\mu}{r^2}.
\]
with the \( z \) coordinate.

The potential energy \( U \) is \( E = \frac{1}{2} \phi \). Using Lagrange’s method, the equation of motion associated
\[
\frac{d^2}{dt^2} (\dot{r} z) + \frac{d}{dt} \left( \frac{\dot{z}}{r} \right) + \frac{\mu}{r^2} = 0.
\]

We work in a cylindrical coordinate system \((r, \theta, z)\) with the \( z \) axis vertical. It suffices to

\section{Solution}

Assume which all motion tends toward a singularity.

The final expression from the associated with a \( r / F \) force can lead to a derivative with an expression in the opposite direction.

The equations provide a mechanical analogy as do derivatives of similar

differentiation with respect to \( z \). Hence, the kinetic energy can be written
for motion on a surface of revolution \( r = f(z) \), we have \( \dot{r}^2 = \frac{d^2}{dz^2} (\dot{r} z) + \frac{d}{dz} \left( \frac{\dot{z}}{r} \right) + \frac{\mu}{r^2} = 0.
\]

Consider the motion of a particle that slides freely on an arbitrary surface of revolution,

\subsection{Problem 1}

Joseph Henry, founder of the "Philosophical Magazine," was a prominent physicist and mathematician of the 19th century.

 plethora of model attributes (and also a science for model attributes)
We write \( \dot{\theta}_0 = \Omega \), so that \( J = \dot{r}_0^2 \Omega \).

For a perturbation about this orbit of the form

\[ z = z_0 + \epsilon \sin \omega t, \tag{4} \]

we have, to order \( \epsilon \),

\[
\begin{align*}
    r(z) & \approx r(z_0) + r'(z_0)(z - z_0) \\
    &= r_0 + \epsilon v_0 \sin \omega t, \\
    r' & \approx v_0 + \epsilon \omega t \frac{s}{r_0}. \\
    \frac{1}{r^3} & \approx \frac{1}{r_0^3} \left( 1 - 3 \epsilon \sin \omega t \frac{v_0'}{r_0} \right). \\
\end{align*}
\]

Inserting (4-7) into (2) and keeping terms only to order \( \epsilon \), we obtain

\[
- \omega^2 \sin \omega t (1 + r_0^2) \approx - \frac{J^2}{2 r_0^4} \left( \frac{r'}{r_0} - 3 \epsilon \sin \omega t \frac{r'}{r_0} + \epsilon \sin \omega t \frac{r'}{r_0^2} \right). 	ag{8}
\]

From the zeroth-order terms we recover (3), and from the order-\( \epsilon \) terms we find that

\[
\omega^2 = \Omega^2 \frac{3 r_0^2 - r_0 r_0''}{1 + r_0^2}. \tag{9}
\]

The orbit is unstable when \( \omega^2 < 0 \), i.e., when

\[
r_0 r_0'' > 3 r_0'^2. \tag{10}
\]

This condition has the interesting geometrical interpretation (noted by a referee) that the orbit is unstable whenever

\[
(1/r^2)' < 0, \tag{11}
\]

i.e., where the function \( 1/r^2 \) is concave inwards.

For example, if \( r = -k/z \), then \( 1/r^2 = z^2/k^2 \) is concave outwards, \( \omega^2 = J^2/(k^2 + r_0^4) \), and there is no regime of instability.

We give three examples of surfaces of revolution that satisfy condition (11).

First, the hyperboloid of revolution defined by

\[
r^2 - z^2 = R^2, \tag{12}
\]

where \( R \) is a constant. Here, \( r_0' = z_0/r_0, r_0'' = R^2/r_0^3 \), and

\[
\omega^2 = \Omega^2 \frac{3 z_0^2 - R^2}{2 z_0^2 + R^2} = \Omega^2 \frac{3 r_0^2 - 4 R^2}{2 r_0^2 - R^2}. \tag{13}
\]

The orbits are unstable for

\[
z_0 < \sqrt{3} R, \tag{14}
\]

or equivalently, for

\[
r_0 < \frac{2 \sqrt{3}}{3} R = 1.1547 R \equiv r_{\text{crit}}. \tag{15}
\]
As \( r_0 \) approaches \( R \), the instability growth time approaches an orbital period.

Another example is the Gaussian surface of revolution,

\[
   r^2 = R^2 e^{-z^2}, \tag{16}
\]

which has a minimum radius \( R \), and a critical radius \( r_{\text{crit}} = R \sqrt[4]{e} = 1.28R \).

Our final example is the surface

\[
   r = \frac{k}{z\sqrt{1 - z^2}}, \quad (-1 < z < 0), \tag{17}
\]

which has a minimum radius of \( R = 2k \), approaches the surface \( r = -k/\sqrt{z} \) at large \( r \) (small \( z \)), and has a critical radius of \( r_{\text{crit}} = 6k/\sqrt{5} = 1.34R \).

These examples arise in a 2 + 1 geometry with curved space but flat time. As such, they are not fully analogous to black holes in 3 + 1 geometry with both curved space and curved time. Still, they provide a glimpse as to how a particle in curved spacetime can undergo considerably more complex motion than in flat spacetime.

3 Acknowledgement

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4 References