We show that while the zero temperature induced fermion number in a chiral sigma model background depends only on the asymptotic values of the chiral field, at finite temperature the induced fermion number depends also on the detailed shape of the chiral background. We resum the leading low temperature terms to all orders in the derivative expansion, producing a simple result that can be interpreted physically as the different effect of the chiral background on virtual pairs of the Dirac sea and on the real particles of the thermal plasma. By contrast, for a kink background, not of sigma model form, the finite temperature induced fermion number is temperature dependent but topological.

PACS: 11.10.Wx, 11.10.Kk, 12.39.Fe

The phenomenon of induced fermion number due to the interaction of fermions with topological backgrounds (e.g., solitons, vortices, monopoles, skyrmions) has many applications ranging from polymer physics to particle physics [1–7]. The induced fermion number is related to the spectral asymmetry of the relevant Dirac operator, and mathematical results concerning index theorems [7] relate the fermion number to asymptotic topological properties of the background. At finite temperature, the situation is less clear. In several examples [8–12], the fermion number is known to be temperature dependent, but is still topological in the sense that the only dependence on the background field is through its asymptotic properties. In this Letter, we present a simple case for which this is not true: in a 1 + 1 dimensional chiral sigma model, the finite temperature induced fermion number depends on the detailed structure of the background. This contradicts a previous analysis [13] and claim [14] that the finite T fermion number is in general a topological quantity. We give a simple physical explanation of the origin of the nontopological dependence.

Consider an abelian model in 1 + 1 dimensions with fermions interacting via scalar and pseudoscalar couplings to two bosonic fields \( \phi_1 \) and \( \phi_2 \). For the purposes of this paper \( \phi_1 \) and \( \phi_2 \) can be considered as classical external fields.

The Lagrangian is

\[
\mathcal{L} = i \bar{\psi} \gamma^\mu \partial_\mu \psi - \bar{\psi} (\phi_1 + i \gamma_5 \phi_2) \psi
\]

There are two especially interesting physical cases:

(i) kink case [1]:

\[
\phi_1 = m \quad \text{and} \quad \phi_2(\pm \infty) = \pm \hat{\phi}_2
\]

(ii) sigma model case [2]:

\[
\phi_1^2 + \phi_2^2 = m^2
\]

In the sigma model case (3), the interaction term in the Lagrangian (1) can be expressed as

\[
m \bar{\psi} e^{i \gamma_5 \theta} \psi = m \bar{\psi} (\cos \theta + i \gamma_5 \sin \theta) \psi
\]

At \( T = 0 \), both these cases have an induced topological current \( J^\mu \equiv < \bar{\psi} \gamma^\mu \psi > \) given by [2]

\[
J^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu} \partial_\nu \theta + \ldots
\]

where the angular field \( \theta \) is defined by \( \theta \equiv \arctan(\phi_2/\phi_1) \). The dots in (5) refer to higher derivative terms, which are all of the form of a total derivative of \( \theta \) and its derivatives [4]. Thus, in particular, the induced fermion number, \( N \equiv \int dx J^0 \), is

\[
N = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \, \partial_x \theta = \frac{1}{\pi} \hat{\theta}
\]
where \( \hat{\theta} \) are the asymptotic values of \( \theta(x) \) at \( x = \pm \infty \). The fermion number \( N \) is topological as it depends only on \( \hat{\theta} \), not on the detailed shape of \( \theta(x) \). The conjugation symmetric case of Jackiw and Rebbi [1] is obtained by taking \( m \to 0 \) in the kink case (2), in which case \( N \to \pm \frac{1}{2} \).

At nonzero temperature, the induced fermion number for a static background is [7,12]

\[
N = -\frac{1}{2} \int_C \frac{dz}{2\pi i} \text{tr} \left( \frac{1}{H - z} \right) \tanh \left( \frac{\beta z}{2} \right)
\]  

(7)

where \( \beta = 1/T \) is the inverse temperature, and \( \text{tr} \left( \frac{1}{H - z} \right) \) is the resolvent of the Dirac Hamiltonian \( H \). The contour \( C \) is \((-\infty + i\epsilon, +\infty + i\epsilon) \) and \((+\infty - i\epsilon, -\infty - i\epsilon) \). By considering static backgrounds we avoid the well-known complications of finite temperature calculations in non-static backgrounds [15]. The technical part of the calculation of the induced fermion number (7) is the computation of the resolvent of \( H \). Once this is done, the induced fermion number may be expressed as an integral representation, or as a sum by deforming the contour in (7) around the simple poles of the \( \tanh \) function. For static backgrounds \( \phi_1(x) \) and \( \phi_2(x) \) in (1), the Dirac Hamiltonian is

\[
H = -i\gamma_0 \gamma_1 \nabla + \gamma_0 \phi_1(x) + i\gamma_0 \gamma_5 \phi_2(x)
\]

(8)

where \( \nabla \equiv \frac{d}{dx} \), and we will work with the Dirac matrices \( \gamma^0 = \sigma_3 \), \( \gamma^1 = i\sigma_2 \), and \( \gamma^5 = -\sigma_1 \). Also, note that only the even part (in terms of the argument \( z \)) of the resolvent \( \text{tr} \left( \frac{1}{H - z} \right) \) contributes to the induced fermion number \( N \) in (7). (This is most easily seen by deforming the contour around the poles of the \( \tanh \) function.)

Consider first the kink case in (2). Then the even part of the resolvent can be computed exactly using an index theorem trace identity [16,7,8] (alternatively, it can be derived in a more elementary manner as an exact resummation of a SUSY derivative expansion [17]) :

\[
\left[ \text{tr} \left( \frac{1}{H - z} \right) \right]_{\text{even}} = \text{tr} \left( \frac{m}{(\nabla + \phi_2)(\nabla - \phi_2) + m^2 - z^2} \right) - \text{tr} \left( \frac{m}{(\nabla - \phi_2)(\nabla + \phi_2) + m^2 - z^2} \right)
\]

\[
= \frac{m}{(m^2 - z^2) \sqrt{m^2 + \phi_2^2 - z^2}}
\]

(9)

Then the induced fermion number (7) for the kink case (2) is

\[
N = \frac{2}{\pi} \left( \frac{m \beta}{\pi} \right)^2 \sin \hat{\theta} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 + (m \beta/\pi)^2} \left( 2n+1 \right)^2 \cos^2 \hat{\theta} + \left( \frac{m \beta}{\pi} \right)^2
\]

(10)

where \( \hat{\theta} \equiv \arctan(\hat{\phi}_2/m) \). This result is consistent with previous analyses [8], although these were much less explicit. The induced fermion number (10) is plotted in Fig. 1 as a function of \( \hat{\theta} \) for various values of temperature. As \( T \to 0 \), this result reduces smoothly to the zero temperature result (6). Despite its complicated form, the nonzero temperature result (10) is still topological as it only refers to the background through \( \hat{\theta} \).

![FIG. 1. Plots of \( \pi N \), where \( N \) is the finite temperature fermion number (10) for the kink case (2), as a function of \( \hat{\theta} \). These plots are for \( m \beta/\pi \) taking values 0.5, 1, and 10, as labelled. As \( T \to 0 \), note that \( \pi N \to \hat{\theta} \), as in (6).](image-url)
Thus, the induced fermion number is no longer topological. This contradicts [13], where it is stated that the first
not
This is simply the zero temperature answer (6) multiplied by a smooth function of $T$
asymptotic values exponentially fast, the term
reduces to
$1$
and then expanding $\text{tr} \left( \frac{1}{H-z} \right) = \text{tr} \left( (H+z) \frac{1}{H-z} \right)$ in powers of derivatives. A simple calculation to first order yields:
$$
\left[ \text{tr} \left( \frac{1}{H-z} \right) \right]_{\text{even}} = -\frac{1}{2} \int_{-\infty}^{\infty} dx \frac{(\phi_1 \phi_2' - \phi_2 \phi_1')}{(\phi_1^2 + \phi_2^2 - z^2)^{3/2}} + \ldots
$$
where the dots refer to terms with three or more derivatives.

In the kink case (2), where $\phi_1 = m$ is constant, this first order calculation actually reproduces the exact trace identity result (9). But in the sigma model case (3,4), where $\phi_1^2 + \phi_2^2 = m^2$ is a constant, the first order derivative expansion result (12) implies that:
$$
\left[ \text{tr} \left( \frac{1}{H-z} \right) \right]_{\text{even}} = -\frac{m^2}{2(m^2 - z^2)^{3/2}} \int_{-\infty}^{\infty} dx \theta' + \ldots
$$
So, to first order in the derivative expansion, the induced fermion number for the sigma model case is
$$
N^{(1)} = \frac{1}{\pi} \left( \frac{m \beta}{\pi} \right)^2 \frac{1}{\pi} \sum_{n=0}^{\infty} \left( \frac{m^2}{(2n+1)^2 + \left( \frac{m \beta}{\pi} \right)^2} \right) \int_{-\infty}^{\infty} dx \theta'
$$
which is simply the zero temperature answer (6) multiplied by a smooth function of $T$. As $T \to 0$, this prefactor reduces to $\frac{1}{\pi}$, so the full zero temperature result (6) is regained unproblematically. But at finite temperature, the first order (in the derivative expansion) formula (14) for the sigma model case differs from the kink case formula (10), even though each of (14) and (10) reduces to (6) at $T = 0$.

This raises the question of the higher order corrections to the derivative expansion (12). In the kink case (2), there are no higher order corrections to the even part of the resolvent in (11). This is due to the special form of the Hamiltonian in the kink background, which leads to the first order formula (12) agreeing with the exact trace identity result (9). There can, of course, be higher order corrections to the induced fermion number density, but these are all total (spatial) derivatives, and do not contribute to the integrated induced fermion number, even at nonzero temperature.

But in the sigma model case (3,4), where the trace identity does not apply, the situation is very different. Going to the next order in the derivative expansion, we find
$$
\left[ \text{tr} \left( \frac{1}{H-z} \right) \right]_{\text{even}} = -\frac{m^2}{2(m^2 - z^2)^{3/2}} \int dx \theta' - \frac{m^2}{8(m^2 - z^2)^{5/2}} \int dx \theta'' \frac{-m^2(4z^2 + m^2)}{16(m^2 - z^2)^{7/2}} \int dx (\theta')^3 + \ldots
$$
where the dots refer to terms involving five or more derivatives. For a chiral background with $\theta(x)$ approaching its asymptotic values exponentially fast, the term $\int dx \theta''$ vanishes. But $\int dx (\theta')^3$ does not vanish. Thus, the first order induced fermion number (14) acquires a third order correction:
$$
N^{(3)} = \frac{\beta^2}{8\pi^3} \left( \frac{m \beta}{\pi} \right)^2 \left( \sum_{n=0}^{\infty} \frac{(-4(2n+1)^2 + (\frac{m \beta}{\pi})^2)}{(2n+1)^2 + \left( \frac{m \beta}{\pi} \right)^2} \right) \int dx (\theta')^3
$$
This is not just a function of the asymptotic value $\hat{\theta}$ of the chiral field $\theta(x)$; it also depends on the actual shape of $\theta(x)$. Thus, the induced fermion number is no longer topological. This contradicts [13], where it is stated that the first
order derivative expansion contribution (14) is the full answer. However, the energy trace prefactor in (16) vanishes at \( T = 0 \), so the nontopological third order contribution (16) vanishes at \( T = 0 \). Thus, the nontopological nature of the finite temperature induced fermion number is still consistent (at this order) with the topological nature of the zero temperature induced fermion number (6).

We now turn to a physical explanation of why, in the sigma model case, the finite temperature induced charge is more sensitive to the background field than at zero temperature. Note first of all that the chiral background acts like a static but spatially inhomogeneous electric field, as can be seen by making a local chiral rotation [3,4]: 

\[
\psi \rightarrow \tilde{\psi} = e^{i\theta_5/2} \psi.
\]

In terms of these chirally rotated fields the Lagrangian (1), with interaction (4), becomes

\[
\mathcal{L} = i\bar{\tilde{\psi}} \frac{\partial}{\partial \tilde{\psi}} - m \bar{\tilde{\psi}} \tilde{\psi} - \bar{\tilde{\psi}} \gamma_5 \theta' \tilde{\psi}.
\]

(The chiral rotation leads to an anomalous Jacobian in the path integral, but this does not affect the induced fermion number.) Thus, the chiral field acts as an inhomogeneous electric field, as we show below.

\[
E(x) = -\frac{1}{2} \theta''(x).
\]

Given that \( \theta(x) \) itself has a kink-like spatial profile, the electric field is such that it changes sign as a function of \( x \), as shown in Fig. 2 (we choose \( \theta' > 0 \)). This electric field acts on the Dirac sea to polarize the vacuum by aligning the virtual vacuum dipoles of the Dirac sea, producing a localized build-up of charge near the kink center. But at nonzero temperature, the electric field also has an effect on the thermal plasma, as we show below.

![Fig. 2. For a kink-like chiral field \( \theta(x) \), the electric field, \( E = -\theta''/2 \), has the form shown in the solid line, producing a vacuum polarization charge distribution localized near the kink center, roughly following the dotted line \( \theta'/2 \).](image-url)

First, consider the full derivative expansion (12) of the even part of the resolvent, at low but nonzero temperature. At fifth order, there are three independent terms, involving \( \theta''''(\theta'')^2 \), and \( (\theta')^5 \). The \( \theta'''' \) term vanishes when integrated over \( x \), but the other two terms are generally nonzero. However, as \( T \to 0 \) the \( (\theta')^5 \) term dominates the \( \theta''''(\theta'')^2 \) term. Indeed, for low temperature, the dominant term with \( (2l - 1) \) derivatives in the derivative expansion (12) involves \( (\theta')^{2l-1} \). Then, using the chirally rotated form (17) of the Lagrangian, the dominant term at \( (2l - 1)^{th} \) order is simply:

\[
N^{(2l-1)}_{\text{dom}} = \delta_{l,1} \int dx \frac{\theta'}{2\pi} \left( \frac{2mT}{\pi} \right)^{2l-1} \left( 1 + \sum_{n=-\infty}^{\infty} \int \frac{dk}{2\pi} \text{tr}(\gamma_0(\hat{p} + m)^{2l}) \right) \int dx \left( \frac{\theta'}{2} \right)^{2l-1}.
\]

with Euclidean \( p = (\omega_n, k) \) and \( \omega_n = (2n + 1)\pi T \) the Matsubara modes.

At zero temperature, all these terms \( N^{(2l-1)} \) vanish, except for \( l = 1 \). This fact is not obvious; it involves highly nontrivial cancellations between terms in the expansion of the trace. But at nonzero temperature, all the terms in (19) are non-vanishing. Moreover, they have a remarkably simple low temperature \( (T \ll m) \) limit:

\[
N^{(2l-1)} = \delta_{l,1} \int dx \frac{\theta'}{2\pi} - \sqrt{\frac{2mT}{\pi}} e^{-m/T} \frac{1}{(2l - 1)!} \int dx \left( \frac{\theta'}{2T} \right)^{2l-1} + \ldots
\]

Thus, in the low temperature limit, we can resum the entire derivative expansion, to obtain the induced fermion number in the sigma model case (3,4):

4
\[ N = \int dx \frac{\theta'}{2\pi} - \sqrt{\frac{2mT}{\pi}} \int dx e^{-m/T} \sinh \left( \frac{\theta'}{2T} \right) + \ldots \]  

(21)

where the dots refer to subleading terms for \( T \ll m \).

Several features of this result (21) deserve comment. First, at zero temperature, only the first term survives, producing the familiar result (6) that the induced fermion number depends on the chiral field \( \theta(x) \) only through its asymptotic value \( \bar{\theta} \equiv \theta(\infty) \). Second, although (20) contains powers of the ‘dangerous’ ratio \( \theta'/T \), the resummed expression (21) has a smooth \( T \to 0 \) limit: the resummed exponential factors \( e^{-(m^2 + \theta')/T} \) in (21) only require the derivative expansion condition \( \theta' \ll m \), precisely as at \( T = 0 \). This is consistent with the fact that for a static background, \( T \) does not enter the computation of the single particle fermion spectrum. Third, at zero temperature, one can invoke Lorentz invariance to constrain the form of higher order corrections to (5) to be total derivatives [4], but these arguments do not apply at finite temperature. We see this in (21): the temperature dependent corrections are not total derivatives of terms made from \( \theta \) and its derivatives. At nonzero temperature this shows clearly that the induced fermion number is nontopological - it depends also on the detailed shape of \( \theta(x) \). Finally, the form of the exponential factors in (21) suggests an interpretation of this result as an adiabatic change of the local Fermi level with a local chemical potential \( \mu(x) = -\theta'/2 \), which once again is only sensible in the derivative expansion regime where \( \theta' \ll m \).

To make this physical picture more precise, we can interpret the result (21) as follows. The first, topological, term refers to the induced charge coming from the polarization of the Dirac sea. This is temperature independent as the short-lived virtual “electron-positron dipoles” of the Dirac sea do not come to thermal equilibrium. The next term in (21) corresponds to the induced charge arising from the response of the real charges in the thermal plasma to the spatially inhomogeneous electric field (18). Indeed, the linear response [19] of the plasma at low temperature to such an electric field yields an induced fermion number density

\[ \rho(x) = \int \frac{dk}{2\pi} f(x, k) \]  

(22)

where the static distribution function \( f(x, k) \) satisfies the Boltzmann equation

\[ v \frac{\partial}{\partial x} f(x, k) = -E(x) \frac{\partial}{\partial k} f(x, k) \]  

(23)

with \( v = k/\sqrt{k^2 + m^2} \). Regarding \( \mu(x) = -\theta'/2 \) as a local chemical potential, (23) is satisfied by local Fermi particle and antiparticle distribution functions

\[ f_{\pm}(x, k) = \frac{1}{e^{\beta(\sqrt{k^2 + m^2} - \mu)} + 1} \]  

(24)

Inserting \( f = f_+ - f_- \) into (22), we obtain precisely the second, nontopological, term in (21) in the low temperature limit.

To conclude, we comment briefly on possible implications of these results for models in other dimensions for which there is an induced fermion number due to some nontrivial background. In 2 + 1 dimensions, fermions in a static magnetic background acquire an induced charge that is topological, expressed in terms of the net magnetic flux of the background. At finite temperature, the induced charge remains topological, but is multiplied by a smooth function of the temperature [9]. In 3 + 1 dimensions, fermions in a static Dirac monopole background acquire an induced charge that is temperature dependent at finite \( T \), but still only depends on the background through the total magnetic charge and the self-adjoint extension parameter [10,12]. A more interesting case is a static ’t Hooft-Polyakov monopole background, which has a characteristic size scale. Consider, for example, the coupling

\[ \mathcal{L}_{\text{int}} = \bar{\psi} ( A + \phi + i\gamma_5 m ) \psi \]  

(25)

where \( \psi \) is an isodoublet fermion, \( A_\mu \) is a static \( SU(2) \) ’t Hooft-Polyakov monopole, and \( \phi \) is the corresponding static Higgs field. We have computed the finite temperature induced fermion number, using the 3 + 1 trace identity used in the zero temperature case [20], and we find precisely the same expression (10) as in the 1 + 1 kink case, with the identification \( \bar{\theta} = \arctan(\hat{\phi}/m) \), where \( \hat{\phi} \) is the asymptotic value of the magnitude \( |\phi| = \sqrt{\phi^\dagger \phi} \) of the Higgs field. Given that (10) reduces to (6) at \( T = 0 \), this monopole result is consistent with the familiar zero temperature result [2,20] that the induced fermion number is proportional to \( \bar{\theta} \) [21]. The 3 + 1 dimensional analogue of the 1 + 1 sigma model case (3,4) is the sigma model with coupling
\( \mathcal{L}_{\text{int}} = m\bar{\psi}(\pi_0 + i\gamma_5 \vec{\pi} \cdot \vec{\tau}) \psi \\
= m\bar{\psi}\left(\frac{1}{2}(g + g^t) + \frac{1}{2}(g - g^t)\gamma_5\right) \psi \)  

(26)

where \( \vec{\tau} \) are \( su(2) \) generators, the fields \( \pi_0 \) and \( \vec{\pi} \) are constrained by \( \pi_0 + \vec{\pi}^2 = 1 \), and the fields \( g \) in the second line are defined by \( g = \pi_0 + i\vec{\pi} \cdot \vec{\tau} \). At zero temperature, there is an induced topological charge density \([2–7]\)

\[ J^0 = \frac{1}{24\pi^2}ijk \text{tr} (g^{-1}\partial_ig\psi g^{-1}\partial_jg\psi g^{-1}\partial_kg) \]  

(27)

The corresponding zero temperature integrated charge is given by the winding number of the background field \( g \) at zero temperature. We conjecture that at finite temperature this induced charge will acquire additional nontopological contributions similar to those found here for the \( 1+1 \) sigma model case.

Acknowledgements:
This work has been supported in part (GD) by the U.S. Department of Energy grant DE-FG02-92ER40716.00, and by PPARC grant PPA/V/S/1998/00910. GD thanks Balliol College, Oxford, for a Visiting Fellowship, and the Theoretical Physics Department at Oxford, and the CSSM at Adelaide for their hospitality and support.

[21] This is similar to, but slightly different from, the point-monopole case considered in [11] where the theta parameter is related to the boundary conditions used in the point limit.