On the Hopf algebraic origin of Wick normal-ordering

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Abstract
A combinatorial formula of G.-C. Rota and J.A. Stein is taken to perform Wick-re-ordering in quantum field theory. Wick’s theorem becomes a Hopf algebraic identity called Cliffordization. The combinatorial method relying on Hopf algebras is highly efficient in computations and yields closed algebraic expressions.

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1 Introduction
Quantum field theory needs due to the quantization of fields an operator ordering. Using e.g. canonical quantization for bosonic fields \( b(\vec{r}, t) \) and fermionic fields \( \psi(\vec{r}, t) \) and their canonical conjugates \( \Pi_b \) and \( \Pi_\psi \) one assumes the canonical (anti) commutation relations:

\[
\begin{align*}
\left[ \Pi_b(\vec{r}, t), b(\vec{r'}, t) \right] & = \delta(\vec{r} - \vec{r'}) \\
\{ \Pi_\psi(\vec{r}, t), \psi(\vec{r'}, t) \} & = \delta(\vec{r} - \vec{r'}).
\end{align*}
\]

(1)

Hence one is forced to introduce ordered monomials in this variables to span the space of the theory. Feynmann introduced the time-ordering device in [20]. However, for practical calculations one has to change the ordering to the so-called normal-ordering [39, 10]. This transition is motivated in terms of Feynmann diagrams as passing over to one-particle irreducible graphs, see e.g. [23]. However, this should be compared with the transition in thermodynamics formulated in the BBGKY hierarchy to Ursell functions [37, 32]. From this comparison one learns that the process of transforming a hierarchy from time-ordered to normal-ordered correlation functions removes the two-particle correlations in higher correlation functions. The thermodynamic transition to Ursell functions does remove any \((n - 1)\)-point correlation from \(n\) -point correlation
functions. The well known $\tau_n$- and $\phi_n$-functions \[28\] for time- and normal-ordered correlation functions are given as:

$$\begin{align*}
\tau_n(1, \ldots, n) & := \text{Phys} \langle 0 \mid T(\psi(\vec{r}_1, t_1), \ldots, \psi(\vec{r}_n, t_n)) \mid 0 \rangle \text{Phys} \\
\phi_n(1, \ldots, n) & := \text{Phys} \langle 0 \mid N(\psi(\vec{r}_1, t_1), \ldots, \psi(\vec{r}_n, t_n)) \mid 0 \rangle \text{Phys}
\end{align*}$$

where $\text{Phys} \langle 0 \mid \ldots \mid 0 \rangle \text{Phys}$ is the expectation value w.r.t. the actual vacuum of the theory –not a Fock vacuum in general, due to Haag’s theorem \[22\]– and $T, N$ are time- and normal-ordering operators. The contraction –sometimes called covariance– is denoted as $C(\psi_1, \psi_2)$ which could be in principle any scalar valued function but will be later on assigned as propagator $F$, see \[21, 14, 18, 15\]. The hierarchy of $\tau$- and $\phi$-correlation functions is connected via Wick’s theorem as:

$$\begin{align*}
\tau_0 &= \langle 0 \mid 0 \rangle = 1 \\
\tau_1(1) &= \phi_1(1) \\
\tau_2(12) &= \langle 0 \mid T(\psi_1 \psi_2) \mid 0 \rangle = \langle 0 \mid N(\psi_1 \psi_2) \mid 0 \rangle + C(\psi_1 \psi_2) \\
&= \phi_2(12) + C(\psi_1 \psi_2) \\
\tau_3(123) &= \phi_3(123) + C(\psi_1 \psi_2)\tau_1(3) + C(\psi_2 \psi_3)\tau_1(1) + C(\psi_3 \psi_1)\tau_1(2) \\
&\vdots
\end{align*}$$

We have simplified our notation and dropped the subscript $\text{Phys}$ from the vacuum. Furthermore we use integers as variables to denote the index sets of the field operators and correlation functions, which might be algebraic or continuous. Now, irreducibility means that an $\tau_n$-function cannot be written as a product of $\tau_{n-r}$-functions ($r \geq 1$) e.g. $\tau_4(1234) = \tau_2(12)\tau_2(34)$ would be reducible.

If one is interested in composite particle calculations one has to remove lower order correlations from the desired correlation functions. This can be achieved in a non-perturbative manner as shown in \[35\]. The Wick re-ordering theorem can be easily proved by Clifford algebraic methods \[12\] unveiling its hidden geometric origin.

**Remark:** Caianiello \[4\] tried some time ago to connect time– and normal-ordered correlation functions using Clifford algebras. Our approach is different, since we do not transform the Grassmann wedge products into Clifford products as Caianiello did, but into another dotted wedge product, see below. Since the Schwinger sources force the $\tau_n$- and $\phi_n$-functions to be antisymmetric, this cannot be achieved using a Clifford basis without destroying the invariance under basis transformations. Our approach is however $\text{sl}(n)$ invariant. As long as normalized states are considered, that is quotients of expectation values, this approach is $\text{gl}(n)$ invariant.

In this paper we will show, that the transition from time- to normal-ordering is based on a Hopf algebra structure of the Schwinger sources of quantum field theory. We introduce the well known Shuffle-Hopf algebra \[38\] which could be called Grassmann-Hopf algebra also. This point of view is of particular use in our case. Then we use a combinatorial formula of Rota and Stein \[34\] to show that there are infinitely many Grassmann algebras which are not isomorphic as Hopf algebras. It is shown that the Rota-Stein Cliffordization –employed in a certain sense– is exactly a closed form of the Wick transformation. This should be compared with the cumbersome recursive
process usually performed in such reorderings, see [23]. The computational benefits of the Hopf algebraic method compared with the usually employed and non-perturbative method of Stumpf is discussed.

2 Non-perturbative normal-ordering:

2.1 Using generating functionals

Driven by needs of composite particle quantum field theory Stumpf introduced the non-perturbative normal-ordering [35]. This method is based on generating functionals —avoiding path integrals for certain reasons— to describe the Schwinger-Dyson hierarchy of quantum field theory. A closed formula is then given for the transition from time- to normal-ordering and vice versa. However, if actual computations have to be performed, a replacement mechanism is effectively used. We define the (fermionic) sources $j_K(\vec{r}, t) \cong j_I \cong j_n$ using abstract indices or even integers to denote the set of relevant quantum numbers —the bosonic case can be handled along the same lines [8, 14, 18, 15]— and their duals $\partial_K(\vec{r}, t) \cong \partial_I \cong \partial_n$ which have to fulfil

$$\{\partial_{I_1}, \partial_{I_2}\} := 0 \quad \{j_{I_1}, j_{I_2}\} := 0 \quad \{\partial_{I_1}, j_{I_2}\} := \delta_{I_1I_2}. \quad (4)$$

Furthermore we define the functional Fock state for convenience as

$$\partial_I |0\rangle_F = 0 \quad \forall I. \quad (5)$$

Then we are able to write down the generating functionals which code the hierarchy as

$$|T(j)\rangle := \sum_{n=0}^{\infty} \frac{i^n}{n!} \tau_n(1, \ldots, n)j_1 \ldots j_n |0\rangle_F \quad (6)$$

or

$$|\mathcal{N}(j)\rangle := \sum_{n=0}^{\infty} \frac{i^n}{n!} \phi_n(1, \ldots, n)j_1 \ldots j_n |0\rangle_F. \quad (7)$$

The functional form of quantum field theory may be found in [35] where it is shown that this formalism is able to replace usual methods as e.g. the pathintegral approach.

Let now $F_{K_1K_2}(\vec{r}_1, t_1, \vec{r}_2, t_2) \cong F_{I_1I_2}$ be the exact propagator of the theory. One can prove the following theorem [35, 12, 18]:

$$|T(j)\rangle = e^{-\frac{1}{2} F_{I_1I_2} j_{I_1} j_{I_2}} |\mathcal{N}(j)\rangle$$

$$|\mathcal{N}(j)\rangle = e^{\frac{1}{2} F_{I_1I_2} j_{I_1} j_{I_2}} |T(j)\rangle. \quad (8)$$

Expanding the series and comparing coefficients of the $j$-sources one arrives at Wick’s theorem. The factor $1/2$ was introduced to meet the definitions of [35] and could be absorbed in $F$. 
2.2 Using Clifford algebras

It was shown in a series of publications [14, 18, 16, 15] that quantum field theory can be re-formulated in terms of –infinite dimensional– Clifford algebras of arbitrary bilinear form –see [5, 29, 1, 17, 30, 13, 18]– now called quantum Clifford algebras [12]. Essentially one performs the following steps:

i) Let $V = \langle j_I \rangle$ be the linear space spanned by the Schwinger-sources $j_I$.

ii) Build the exterior algebra –symmetric algebra for bosons– over this space as the formal polynomial ring in the anti-commuting sources

$$\{j_{I_1}, j_{I_2}\}_+ = 0$$

$$\bigwedge V = \mathbb{C} \oplus V \oplus V \wedge V \oplus \ldots$$

$$|T(j)\rangle \in \bigwedge V, \quad |N(j)\rangle \in \bigwedge V.$$  \hspace{1cm} (9)

iii) Define the space $V^*$ of linear forms on $V$ as the span of the dual bases $\partial_I$, i.e.

$V^* = \langle \partial_I \rangle$ and build up the dual exterior algebra:

$$V^* = \langle \partial_I \rangle$$

$$\{\partial_{I_1}, \partial_{I_2}\}_+ = 0$$

$$\{\partial_{I_1}, j_{I_2}\}_+ = \delta_{I_1,I_2}.$$  \hspace{1cm} (10)

This space is defined to be reflexive since the same index set is used for $j$ and $\partial$ bases. We use also the notation $\partial_I = j_I \cdot J_I$ where we introduced the contraction. Indeed, this can be written basis free, e.g. $x_J = x_I \partial_I$ using summation convention for discrete and continuous parts of the index set.

iv) Extend this setting to an action of $\bigwedge^* V \simeq \bigwedge V^*$ on $\bigwedge V$ in the following way [5, 29, 25, 18, 13, 12]:

i) $\partial_{I_1}(j_{I_2}) = \delta_{I_1,I_2}$

ii) $\partial_I(AB) = (\partial_I A)B + \hat{A}(\partial_I B)$

iii) $A^*(BC) = A^*(B^*C).$  \hspace{1cm} (11)

The notation is as follows: $A, B, C \in V$, i.e. $A = A_1 \ldots A_n$, $A^*, B^* \in V^*$, $\hat{A} = (-1)^r A_{i_1 \ldots i_r} j_{I_1 \ldots I_r}$ and, note the reversion of indices here $A \mapsto A^* = A_{i_1 \ldots i_r} \partial_{I_1} \ldots \partial_{I_r}$. This can be recast entirely – avoiding wedge and contraction– in Clifford algebraic form by index doubling, see [12, 18].

v) Define the field operators as Clifford algebra elements obtained by the Clifford map according to Chevalley deformation:

$$\psi_K(\vec{r}, t) \equiv \psi_I := \partial_I + B_{I_1,I_2} j_{I_2}$$

$$\simeq \partial_K(\vec{r}, t) + B_{K_1,K_2}(\vec{r}_1, t_1, \vec{r}_2, t_2) j_{K_2}(\vec{r}_2, t_2).$$  \hspace{1cm} (12)

where summation and integration is once again implicit. This is a particular form of a deformation quantization.
Define now explicitly the wedge product denoted by $\wedge$ as the sign for the product of the Schwinger sources. Furthermore, we use index free notation now to shorten the formulas. Let us furthermore define the ‘bi’-vector $F := F_{I_1J_1} \wedge j_{I_2}$. Let us introduce a second wedge product, the doted-wedge [26] denoted by $\dot{\wedge}$, defining on $x, y \in V$

$$x \dot{\wedge} y := x \wedge y + F \frac{1}{\delta}(x \wedge y) = x \wedge y + F_{xy}. \quad (13)$$

Observe that $x \dot{\wedge} y = -y \dot{\wedge} x$ and $\dot{\wedge}$ is indeed antisymmetric and a proper exterior product which can be extended to the whole algebra $\dot{\wedge} V$.

In Ref. [12] the following theorem was proved:

Let $e^F$ denote the exterior exponential of $F \in \wedge^2 V$ and $A = A(j, \partial)$ an arbitrary operator in $\text{End}(\wedge V)$, then we have:

$$e^F \wedge A(j, \partial) \wedge e^{-F} = A^{\dot{\wedge}}(j, \partial) = A^{\wedge}(j, d)$$

$$d := \partial - Fj \quad (14)$$

for operators and

$$e^F | T(j) \rangle^{\dot{\wedge}} = | \mathcal{N}(j) \rangle^{\dot{\wedge}}$$

$$e^{-F} | \mathcal{N}(j) \rangle^{\dot{\wedge}} = | T(j) \rangle^{\wedge} \quad (15)$$

for functional states. The peculiar feature of this transition is, that in a first step only the product is transformed, hence we have:

$$| \mathcal{N}(j) \rangle^{\dot{\wedge}} = \sum_{i} \frac{i^n}{n!} \tau_n(1, \ldots, n) j_{I_1} \ldots \dot{\wedge} j_{I_n} | 0 \rangle_F. \quad (16)$$

The normal-ordered functional is thus written with the time-ordered correlation functions. The usually obtained normal-ordered correlation functions appear only after we have re-written the normal-ordered functional in terms of the old wedge $\wedge$. We arrive at

$$| \mathcal{N}(j) \rangle^{\wedge} = \sum_{i} \frac{i^n}{n!} \phi_n(1, \ldots, n) j_{I_1} \ldots \wedge j_{I_n} | 0 \rangle_F \quad (17)$$

where the $\phi_n$-functions are connected to the $\tau_n$-functions via the Wick theorem w.r.t. the contraction –or covariance– $F$. This transition is called Wick isomorphism $\phi$ and is a $\mathbb{Z}_2$-graded algebra homomorphism [12].

Symbolically we write this as

$$\mathcal{C}(V, Q) = \phi \circ \mathcal{C}(V, B), \quad (18)$$

where $\mathcal{C}(V, Q)$ and $\mathcal{C}(V, B)$ are Clifford algebras over the space $V$ w.r.t. the quadratic form $Q$ or the –not necessary symmetric– bilinear form $B$, see [18, 13, 12]. In the present case, we look at both ‘Clifford algebras’ as degenerated ones, i.e. as Grassmann algebras, that is we set $Q \equiv 0$ and let $B = -B^T$ be a totally antisymmetric form, which is also degenerated since $1/2(B + B^T) \equiv 0$. 5
**Example 1:** [Schwinger-Dyson hierarchy for a free Hamiltonian] Let $H$ be the functional Hamiltonian of a free fermionic field. Such an $H$ takes the form

$$H^\wedge(j,\partial) := D_{I_1, I_2} j_{I_1} \partial_{I_2},$$  \hspace{1cm} (19)$$

where $D_{I_1, I_2}$ is the kinetic operator e.g. a d’Alembertian or Laplacian, see [36, 15, 18] for a detailed model of spinor QED. The normal-ordered operator is obtained as

$$H^\dot\wedge(j,\partial) = e^F \wedge H^\wedge(j,\partial) \wedge e^{-F} = H^\wedge(j, d)$$

$$= D_{I_1, I_2} j_{I_1} (\partial_{I_2} - F_{I_2 I_3} j_{I_3})$$

$$= D_{I_1, I_2} j_{I_1} \partial_{I_2} - D_{I_1, I_2} F_{I_2 I_3} j_{I_1} j_{I_3}$$  \hspace{1cm} (20)$$

and the generating functional of the Schwinger-Dyson hierarchy transforms as follows:

$$e^F \wedge H^\wedge(j,\partial) \mid T(j)\rangle^\wedge = e^F \wedge E \mid T(j)\rangle^\wedge$$

$$H^\dot\wedge(j,\partial) \mid \mathcal{N}(j)\rangle^\wedge = E \mid \mathcal{N}(j)\rangle^\dot\wedge$$

$$H^\wedge(j, d) \mid \mathcal{N}(j)\rangle^\wedge = E \mid \mathcal{N}(j)\rangle^\wedge $$  \hspace{1cm} (21)$$

This example shows, that finally the transition from time- to normal-ordering is given by re-expressing the doted-wedge $\dot\wedge$ in terms of the undoted wedge $\wedge$ and vice versa. This can be achieved by the above given formal substitution in the operators e.g. $H^\dot\wedge(j,\partial) = H^\wedge(j, d)$, using (14) and expanding the doted wedges in the generating functional, see [14, 18]. However, the deeper origin of the need of a normal-ordering remains hidden. The root of such a re-ordering will be found in the Hopf algebraic structure of the Grassmann algebra.

### 3 Grassmann-Hopf algebra

We have already introduced the Grassmann algebra above. To turn it into a Hopf algebra, we have to add in a compatible way a co-algebra structure and an antipode. Let us denote the unit map which injects the real or complex field into the algebra with $\eta : k \mapsto \bigwedge V$, while $\wedge \equiv m_\wedge$, $m_\Lambda: \bigwedge V \otimes \bigwedge V \mapsto \bigwedge V$ is the product map. Let us furthermore denote the linear space underlying the Grassmann algebra as $W$, thus $W = \langle \bigwedge V \rangle$, then we can describe the Grassmann algebra by the triple $\bigwedge V = (W, m_\Lambda, \eta)$. The co-algebra structure is then given by a diagonalization $\Delta$—called also co-product— and a co-unit $\epsilon$ which arise naturally from ‘dualizing’ the algebra structure in a functorial sense, see [38, 27]. The compatibility of algebra and co-algebra structure requires the diagonalization and co-unit to be algebra homomorphisms. That is we require that:

$$\epsilon(Id) = 1$$

$$\epsilon(j_I) = 0$$

$$\epsilon(A \wedge B) = \epsilon(A) \wedge \epsilon(B).$$  \hspace{1cm} (22)$$
While the co-product has to fulfil:

\[
\begin{align*}
\Delta(\text{Id}) &= \text{Id} \otimes d \\
\Delta(j_I) &= j_I \otimes \text{Id} + \text{Id} \otimes j_I \\
\Delta(A \wedge B) &= \Delta(A) \wedge \Delta(B)
\end{align*}
\]  

(23)

obeying an \(\mathbb{Z}_2\)-graded tensorproduct. We introduce for further usage also the Sweedler notation of co-products as [38]

\[
\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)},
\]  

(24)

where we omit the subscript at the sum-sign or even the sum-sign itself which is them implicit. The above written wedge-product for tensors \((a \otimes \ldots \otimes b) \wedge (c \otimes \ldots \otimes d)\) needs for its evaluation the graded switch \(\tau\), which is defined as \((u \in \bigwedge^r V, v \in \bigwedge^s V)\):

\[
\tau : \bigwedge^r V \otimes \bigwedge^s V \mapsto \bigwedge^s V \otimes \bigwedge^r V
\]

\[
\tau(u \otimes v) = (-1)^{rs}(v \otimes u),
\]  

(25)

extended by linearity to all elements of \(\otimes^2 W\). Finally we define the antipode \(S^\wedge \equiv S \in \text{End}(W)\) –i.e. the convolutive inverse of \(\text{Id}\)– as:

\[
\begin{align*}
S^\wedge & : \bigwedge^r V \mapsto \bigwedge^r V \\
S^\wedge(x) &= (-1)^r x,
\end{align*}
\]  

(26)

also extended by linearity. Note that this is exactly the grade involution of the Grassmann algebra. The sextuple

\[
H^\wedge := (W, m^\wedge, \eta, \Delta, \epsilon, S^\wedge)
\]  

(27)

is the Grassmann-Hopf algebra, which is in a certain sense unique (universality property) see [38]. It can be checked that with the above given definitions all axioms for a Hopf algebra are met, especially the antipode axioms are fulfilled [see Figure 1]:

\[
\begin{align*}
i) & : \wedge \circ (S \otimes \text{Id}) \circ \Delta = \eta \circ \epsilon \\
ii) & : \wedge \circ (\text{Id} \otimes S) \circ \Delta = \eta \circ \epsilon.
\end{align*}
\]  

(28)

Observe, that in our case \(S^2 = \text{Id}\) and the Grassmann-Hopf algebra is \(\mathbb{Z}_2\)-graded co-commutative.

4 The Rota-Stein Cliffordization formula as Wick theorem

In Ref. [34] Rota and Stein introduced a deformed product on Hopf algebras. Adding the structure of a Laplace pairing –definitions see below– one can add to the universal
structure of a Grassmann-Hopf algebra a new product which relays on the Laplacian pairing. This corresponds to an algebra deformation of the Grassmann algebra into a Clifford algebra. We will, however, use this mechanism to connect Grassmann-Hopf algebras which are different, i.e. non-isomorphic, as $\mathbb{Z}_n$-graded (Hopf) algebras. This can be done using a Laplace pairing w.r.t. an antisymmetric form, which will be afterwards defined to be the propagator of the theory.

We define the Laplace pairing following Ref. [34]. Let $(w|w')$ be a bilinear mapping from elements $w, w'$ of the Grassmann Hopf algebra $H_\wedge$ into $H_\wedge$ -- we will need $\mathbb{k}$ as a target only -- which satisfies the Laplace identities:

\begin{align*}
& i) \quad (w \wedge w'|w'') = \sum \pm (w|w''_{(1)}) \wedge (w|w''_{(2)}) \\
& ii) \quad (w|w' \wedge w'') = \sum \pm (w_{(1)}|w') \wedge (w_{(2)}|w'') \\
& iii) \quad (w|w'_{(1)}) \wedge ((w|w'_{(2)}|w'') = \sum \pm (w_{(1)}|w'_{(1)}) \wedge ((w_{(2)}|w'_{(2)}|w'') \\
& iv) \quad (w|(w''_{(1)}) \wedge (w'|w'')_{(2)} = \sum \pm (w|(w'_{(1)}|w''_{(1)}) \wedge (w'_{(2)}|w''_{(2)})
\end{align*}

The signs $\pm$ have to be chosen due to the action of the graded switch to produce the correct permutations. If the the bilinear form is scalar valued, i.e. in $\mathbb{k}$ as we assume, the wedge products can be safely removed. Relations $i)$ and $ii)$ are the Hopf algebraic expression of the Laplace-expansion of determinants. Relations $iii)$ and $iv)$ state the compatibility of the bilinear form and the co-product being an algebra homomorphism.

We use tangles [40] to make some of the relations more feasible. The tangle for the scalar valued pairing is given as in Figure 2.

\begin{align*}
\text{Figure 1: Tangle definition of the antipode axioms.}
\end{align*}

\begin{align*}
\text{Figure 2: Tangle for the scalar valued pairing.}
\end{align*}

We are now ready to define with Rota and Stein the deformed product as:

\begin{align*}
w \wedge w' := \sum \pm w_{(1)} \wedge (w_{(2)}|w'_{(1)}) F \wedge w'_{(2)}
\end{align*}

or in terms of tangles [see Figure 3] as the so called ‘Rota-sausage’ [31]. Note, that the deformed product is given by a non-local graph. This process is called Cliffordiza-
tion since it is used usually to introduce a non-trivial symmetric bilinear form. The non-locality of the ‘sausage’ might have implications on Feynmann diagrams build up from such products. Indeed we found in Ref. [19] that some singularities which arise due to re-ordering in vertices of non-linear spinor field models dissappear due to this rearrangements. Hopf algebras are recently used successfully in perturbative renormalization theory [7, 24, 33].

Observe the perplexing fact, that this definition of the doted wedge product is generic for any element of the underlying space $W = \langle \bigwedge V \rangle$ of Grassmann polynomials. This contrasts strongly the recursive definition of the doted wedge product due to Chevalley deformation of the Grassmann wedge product [5, 6]. There one has $(x \in V)$

$$\gamma_x : \bigwedge V \mapsto \bigwedge V$$

$$x \mapsto \gamma_x := x \frac{d}{F} + x \wedge$$

which works only for $x \in V$ and has to be extended by the rules given above for the Schwinger sources (11). At this place we note that the grade involution appearing in (11-ii) is exactly the antipode $S^\wedge$.

The only weak point in our definition is, that we have not yet given a computational definition of the pairing. The pairing can be axiomatized [9]. However, the most important relations are the two Laplace expansions (29-i) and (29-ii) which allow us by applying the coproduct to decompose the pairing into pairings which contain only elements of the space $V$. The Laplace pairing is exactly a pairing which allows such a decomposition.

However, we will give a second definition of the pairing, which is equivalent to the above one, but might be more familiar to physicists. We want nevertheless to stress that the evaluation of this expression is done by applying the Laplace expansion rules given above, beside the fact that this time the Hopf algebraic nature of the expansion is disguised.

Let $w_r = x_1 \wedge \ldots \wedge x_r$ and $w_s' = y_s' \wedge \ldots \wedge y_1'$ with $x_i, y_j \in V$ and define

$$(w_r | w_s')_F := \begin{cases} \det(x_i | y_j')_F & r = s \\ 0 & \text{otherwise} \end{cases}$$

and extend it by linearity to the whole space $W$. This can be rewritten using the
projection onto the scalar part \( \langle \rangle_0^\wedge : W \leftrightarrow k \) and the contraction \( \downarrow_F \) w.r.t. \( F \) as:

\[
(w | w') = \langle w \downarrow_F w' \rangle_0^\wedge = \epsilon(w \downarrow_F w').
\]

(33)

We note, that the projection onto the scalars \( \langle \rangle_0^\wedge \) is exactly the co-unit \( \epsilon \) by definition.

The following example shows how this mechanism works. They have been produced for higher dimensions using the computer algebra packages CLIFFORD for Mapel, developed by Rafal Abłamowicz [2, 3] and BIGEBRA [11]. However, the below given example can still done easily by hand.

**Example 2:** [Hopf algebraic Wick re-ordering] Let \( x, y \in V \), let \( \circ \) denote the concatenation of operations and calculate the ‘sausage’, i.e. formula (30):

\[
x \hat{\wedge} y = \wedge \otimes \wedge \circ (Id \otimes (\cdot \cdot \cdot \otimes Id) \circ (\Delta \otimes \Delta))(x \otimes y)
\]

\[
= \wedge \otimes \wedge \circ (Id \otimes (\cdot \cdot \cdot \otimes Id)(x \otimes Id \otimes y \otimes Id
\]

\[
+ x \otimes Id \otimes Id \otimes y + Id \otimes x \otimes y \otimes Id + Id \otimes x \otimes Id \otimes y)
\]

\[
= \wedge \otimes \wedge (0 + (Id|Id)F x \otimes y + (x|y)F Id \otimes Id + 0)
\]

\[
= x \wedge y + F_{x,y}Id.
\]

(34)

Thus we obtain as the worked out Rota-sausage the relation (30), of the doted wedge, expressed in undoted wedges. Let us compute a product of three doted wedges \( (x_i \in V) \)

\[
x_1 \hat{\wedge} x_2 \hat{\wedge} x_3 = x_1 \hat{\wedge} (F_{23}Id + x_2 \wedge x_3)
\]

\[
= x_1 \wedge x_2 \wedge x_3 + x_1F_{23} + x_2F_{3} + x_3F_{12}.
\]

(35)

It seems to be that we have to resolve the tangle recursively, but this is for demonstration only, one can write down a closed formula using nested Rota-sausages to obtain the result directly. In fact, we see clearly how the \( \tau_n \)- and \( \phi_n \)-correlation functions are interrelated due to this expansion. As a last calculation, we find the doted wedge
product of two elements of $x_1 \wedge x_2$, and $x_3 \wedge x_4$ both $\in \wedge^2 V$

$$(x_1 \wedge x_2) \wedge (x_3 \wedge x_4)$$

$$= (\wedge \otimes \wedge) \circ (Id \otimes (\cdot | \cdot) \otimes Id) \circ (\Delta \otimes \Delta)((x_1 \wedge x_2) \otimes (x_3 \wedge x_4))$$

$$= (\wedge \otimes \wedge) \circ (Id \otimes (\cdot | \cdot) \otimes Id)((x_1 \wedge x_2) \otimes Id \otimes (x_3 \wedge x_4) \otimes Id$$

$$+ (x_1 \wedge x_2) \otimes Id \otimes x_3 \otimes x_4 - (x_1 \wedge x_2) \otimes Id \otimes x_4 \otimes x_3$$

$$+ (x_1 \wedge x_2) \otimes Id \otimes (x_3 \wedge x_4) + x_1 \otimes x_2 \otimes (x_3 \wedge x_4) \otimes Id$$

$$+ x_1 \otimes x_2 \otimes Id \otimes x_3 \otimes x_4 - x_1 \otimes x_2 \otimes x_4 \otimes x_3$$

$$+ x_1 \otimes x_2 \otimes Id \otimes (x_3 \wedge x_4) - x_2 \otimes x_1 \otimes (x_3 \wedge x_4) \otimes Id$$

$$- x_2 \otimes x_1 \otimes x_3 \otimes x_4 + x_2 \otimes x_1 \otimes x_4 \otimes x_3$$

$$- x_2 \otimes x_1 \otimes Id \otimes (x_3 \wedge x_4) + Id \otimes (x_1 \wedge x_2) \otimes (x_3 \wedge x_4) \otimes Id$$

$$+ Id \otimes (x_1 \wedge x_2) \otimes x_3 \otimes x_4 - Id \otimes (x_1 \wedge x_2) \otimes x_4 \otimes x_3$$

$$+ Id \otimes (x_1 \wedge x_2) \otimes Id \otimes (x_3 \wedge x_4))$$

$$= (Id)(Id)_{f_{x_1} \wedge x_2 \wedge x_3 \wedge x_4}$$

$$+ (x_2|x_3)_{f_{x_1} \wedge x_2} - (x_2|x_4)_{f_{x_1} \wedge x_3}$$

$$- (x_1|x_3)_{f_{x_2} \wedge x_4} + (x_1|x_4)_{f_{x_2} \wedge x_3}$$

$$(x_1 \wedge x_2|x_3 \wedge x_4)_{f_{Id}}.$$  \hspace{1cm} (36)

It is obvious, that one can define an inverse mapping in the same fashion as long as the pairing is non-degenerate. This allows one to expand the undoted wedges into the dotted ones using the Rota-Stein Cliffordization w.r.t. the bilinear form $-F$. This is equivalent to the expansion of the normal-ordered $\phi_n$-correlation functions in terms of the time-ordered $\tau_n$-correlation functions.

**Remark:** In spite of the fact, that the Grassmann algebras w.r.t. the two wedges --dotted and undotted-- are isomorphic as $\mathbb{Z}_n$-graded algebras, this is *not* the case for the Hopf algebras $H_\chi$ and $H_\chi$. This can be seen easily by evaluating the co-unit $\epsilon_\chi$ on a dotted and undotted wedge $(x, y \in V)$:

$$\epsilon_\chi(Id) = 1 = \epsilon_\chi(Id)$$

$$\epsilon_\chi(x) = 0 = \epsilon_\chi(x)$$

$$\epsilon_\chi(x \wedge y) = 0$$

$$\epsilon_\chi(x \wedge y) = \epsilon_\chi(x \wedge y + F_{x,y}Id) = F_{x,y} \neq 0.$$  \hspace{1cm} (37)

Obviously we find that $\epsilon_\chi$ is *not* the co-unit of $H_\chi$, and that we are forced to introduce a new co-unit $\epsilon_\chi$ there. An analogous situation is found for the co-products, which are also not identical. This opens the interesting possibility to have a family of co-units and co-products for an --up to isomorphy-- given Grassmann algebra to make it into a Grassmann-Hopf algebra. In Ref. [16] we showed, how such a process can be used to introduce different vacua and even condensation phenomena. We learn from our present work, that the term ‘vacuum’ is connected with the co-unit of a Hopf algebra. In fact the co-unit is a sort of expectation value of the algebra elements which constitute the operators. The Hopf algebraic non-isomorphic $\mathbb{Z}_n$-gradings are responsible for this feature.
We conjecture furthermore, that this process is involved in the recent development of A. Connes and D. Kreimer [7, 24, 33] of a theory on perturbative renomalization of quantum field theory, where the antipode generates all the counterterms in the forest theorems. However, their theory uses the $\mathbb{Z}_2$-grading only and is thus not sensible to the finer $\mathbb{Z}_n$-grading used in our work.

5 Conclusion

We showed that the process of Wick-reordering is governed by the Grassmann-Hopf algebra structure uniquely assigned to Schwinger sources and a definite wedge product. Introducing a doted wedge product, using Rota-Stein Cliffordization w.r.t. an antisymmetric bilinear form $F$ and the therefrom induced pairing, we found a closed formula for re-ordering time– into normal-ordered and normal– into time-ordered $n$-point correlation functions. The Hopf algebraic nature of this process was exhibited and its importance in quantum field theory was demonstrated. It was demonstrated, that the $\mathbb{Z}_n$-grading of a Grassmann-Hopf algebra is not preserved under such an transition. This was connected to different vacua underlying the particular theory. The Stumpf approach to non-perturbative normal-ordering was given and compared to the Hopf algebraic method.

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