Decoherence-free subspaces (DFSs) shield quantum information from errors induced by the interaction with an uncontrollable environment. Here we study a model of correlated errors forming an Abelian subgroup (stabilizer) of the Pauli group (the group of tensor products of Pauli matrices). Unlike previous studies of DFSs, this type of errors does not involve any spatial symmetry assumptions on the system-environment interaction. We solve the problem of universal, fault-tolerant quantum computation on the associated class of DFSs.

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1. INTRODUCTION

Methods to protect fragile quantum superpositions are of paramount importance in the quest to construct devices that can reliably process quantum information [1,2]. Compared to their classical counterparts such devices feature spectacular advantages in both computation and communication, as discussed in a number of recent reviews [3–5]. The dominant source of the fragility of a quantum information processor (QIP) is the inevitable interaction with its environment. This coupling leads to decoherence: a process whereby coherence of the QIP-wavefunction is gradually destroyed. Formally, the evolution of an open system (coupled to an environment) such as a QIP can be described by a completely-positive map [6], which can always be written in the explicit form known as the Kraus operator sum representation [7]:

$$\rho(t) = \sum_{d} A_d(t)\rho(0)A_d^{\dagger}(t).$$  (1.1)

Here $\rho$ is the system density matrix, and the “Kraus operators” $\{A_d\}$ are time-dependent operators acting on the system Hilbert space, constrained only to sum to the identity operator: $\sum_d A_d^\dagger A_d = I$ (to preserve $\text{Tr}[\rho]$). Decoherence is the situation where there are at least two Kraus operators that are inequivalent under scalar multiplication. The Kraus operators are in that case related to the different ways in which errors can afflict the quantum information contained in $\rho$ [9]. Conversely, if there is only one Kraus operator, then from the normalization condition it must be unitary: $A = \exp(-iHt)$ with $H$ Hermitian, so that $\rho$ satisfies the closed-system Liouville equation $\dot{\rho} = -i[H,\rho]$, $H$ being the system Hamiltonian. In this case there is no decoherence.

Two principal encoding methods have been proposed to solve the decoherence problem: (i) Quantum Error Correcting Codes (QECCs) [10–16] (for a recent review see [17]), (ii) Decoherence-Free Subspaces (DFSs) [18–24], also known as “noiseless”, or “error-avoiding quantum codes”. In both methods quantum information is protected against decoherence by encoding it into “codewords” (entangled superpositions of multiple-qubit states) with special symmetry properties. To exhibit these, it is useful to expand the Kraus operators over a fixed operator basis. For qubits a particularly useful basis is formed by the elements of the Pauli group: the group of tensor products of all Pauli matrices $\{\sigma^a_k\}$, where $a = 0, x, y, z$ ($\sigma^0$ is the $2 \times 2$ identity matrix) and $k = 1..K$ is the qubit index. An element of the Pauli group can be written as $E_a = \otimes_{k=1}^K \sigma^a_k$, where $a = (\alpha_1, ..., \alpha_K)$. The $4^{K+1}$ elements $\{E_a\}$ of the Pauli
the operators \( \{E_a\} \) acquire the significance of representing the different physical errors that can corrupt the quantum information. The weight \( w(E_a) \) is the number of non-zero \( \alpha_a \) in \( a \). Let us now assume a short-time expansion of the \( c_{ad}(t) \) (relative to the bath-correlation time). The situation where only those \( E_a \) with \( w(E_a) = 1 \) have non-vanishing \( c_{ad}(t) \) is called the “independent errors” model (assuming the \( c_{ad} \), which are essentially bath correlation functions [8], are statistically independent). Correlated errors corresponds to the situation where some \( E_a \) with \( w(E_a) > 1 \) have non-vanishing \( c_{ad}(t) \): two or more qubits are acted upon non-trivially with the same coefficient \( c_{ad} \). QECCs can be classified according to the maximum weight of the errors they can still correct (this is related to the notion of a “distance” of a code [17]). QECCs can generally deal at least with errors of weight 1. Barring accidental degeneracies, non-trivial DFSs, on the other hand, generally do not exist if there are errors with weight 1 [22]. To make these ideas more precise, let us briefly recall the definitions of QECCs and DFSs.

A QECC is a subspace \( C = \text{Span}[\{|i\}] \) of the system Hilbert space with the symmetry property that different errors take orthogonal codewords \( |i\) and \( |j\) to orthogonal states [16]:

\[
(i|E_a^\dagger E_b|j) = \gamma_{ab} \delta_{ij}.
\]  

(1.3)

Here \( \gamma_{ab} \) are the elements of an Hermitian matrix \( \gamma \) and \( \delta_{ij} \) is the Kronecker delta. This property ensures that if an error \( E_a \) occurs it can be detected and subsequently reversed [16]. A large variety of QECCs have been found [17]. A particularly useful and large class, one which will occupy our attention in this paper, arises when one considers Abelian subgroups \( Q \) of the Pauli group. Given such an Abelian Pauli-subgroup, or stabilizer \( Q \) (we will use both terms interchangeably in this paper), its +1 eigenspace is a QECC known as a stabilizer code [15]. The set of errors \( \{E_a\} \) is correctable by this code if for every two errors \( E_a, E_b \) there exists some \( q \in Q \) such that

\[
\{E_a^\dagger E_b, q\} = 0.
\]  

(1.4)

This is because under the stipulated condition \( \langle i|E_a^\dagger E_b|j\rangle = \langle i|E_a^\dagger E_b q|j\rangle = -\langle i|q E_a^\dagger E_b|j\rangle = -\langle i|E_a^\dagger E_b|j\rangle \) so that \( \langle i|E_a^\dagger E_b|j\rangle \propto \delta_{ij} \) [15]: the QECC condition [Eq. (1.3)] is satisfied. To correct an error \( E_a \) one simply applies the unitary operator \( E_a^\dagger \) to the code. Note that this involves active intervention: measurements to diagnose the error, and error reversal.

DFSs can be viewed as highly “degenerate” QECCs, where degeneracy refers to the rank of \( \gamma \): DFSs are rank-1 QECCs (i.e., \( \gamma_{ab} = \gamma_{a} \gamma_{b} \)) [23,25]. Equivalently, a DFS can be defined as the simultaneous eigenspace \( \mathcal{H} = \text{Span}[\{\langle j\}] \rangle \) of all Kraus operators [23]:

\[
A_d |j\rangle = a_d |j\rangle
\]  

(1.5)

(\( \{a_d\} \) are the eigenvalues). Viewed in this way, DFSs have the remarkable property that they offer complete protection for quantum information without the need for any active intervention: \( \bar{\rho}(t) = \sum_d A_d(t) \bar{\rho}(0) A_d^\dagger(t) = \bar{\rho}(0) \sum_d |a_d|^2 = \bar{\rho}(0) \), for \( \bar{\rho} \) with support exclusively on \( \mathcal{H} \). Thus a DFS is a “quiet corner” of the system Hilbert space, which is completely immune to decoherence. Like stabilizer-QECCs, DFSs can also be characterized as the +1 eigenspace of a stabilizer, which however is generally non-Abelian over the Pauli group [26,27] (i.e., a DFS is generally a non-additive code [28]). Most work on DFSs to date has focused on a model of highly correlated errors, known as “collective decoherence”. In this model the (non-Abelian) stabilizer is composed of tensor products of identical \( SU(2) \) rotations + contractions on all qubits. Here we will not concern ourselves with the collective decoherence model, and the term stabilizer will be reserved for the Abelian subgroups of the Pauli group.

In a companion paper [29] (referred to from here on as “paper 1”) we began a study of DFSs for non-collective errors. We derived a necessary and sufficient condition for a subspace to be decoherence-free when the Kraus operators are expanded as linear combinations over the elements of an arbitrary group. The decoherence-free states were shown to be those states that transform according to the one-dimensional irreducible representations (irreps) of this group. As above, it is natural to focus on the case where this group is the Pauli group. This is so not only because of the connection to stabilizer-QECCs, but also because the Pauli group arises in the context of many-qubit systems, where it is often natural to expand the Hamiltonians in terms of tensor products of Pauli matrices. To find DFSs, therefore, we focus here on subgroups of the Pauli group. Note that the non-Abelian subgroups of the Pauli group do not have one-dimensional irreps [29], and hence in this case a DFS can be associated only with the Abelian subgroups (of course have only one-dimensional irreducible representations).
We can now define the error model that will concern us in this paper. Unlike the stabilizer-QECCs case, where the errors that the code can correct are those that anti-commute with the stabilizer, in the DFS case the errors are the elements of the stabilizer itself. We shall refer to these errors as “stabilizer-errors”. The Abelian subgroups of the Pauli group cannot contain single-qubit operators, since these would generally generate the whole Pauli group. Hence as errors the elements of the subgroup represent multiple-qubit couplings to the bath. As explained above, this is therefore a correlated-errors model, which is distinguished from previous work on DFSs in that it does not involve any spatial-symmetry assumptions. The physical relevance of this error-model was discussed in paper 1, and will be embellished here. The DFS is not affected by these stabilizer-errors, but the rest of the Hilbert space is and may decohere under their influence. Several examples of DFSs corresponding to Abelian subgroups were given in paper 1. Our purpose in this sequel paper is to complete our study of this class of DFSs by showing how to perform universal fault-tolerant quantum computation on them.

The central challenge in demonstrating universal fault-tolerant quantum computation on DFSs is to show how this can be done using only 1- and 2-body Hamiltonians, and a small number of measurements. Several previous publications have addressed the issue of universal quantum computation on DFSs, but left this challenge unanswered [22,30,31]. In Refs. [26,27] we accomplished this task for the first time, in the collective decoherence model. Collective decoherence is the situation where all qubits are coupled in an identical manner to the bath, i.e., there is a strong spatial symmetry: qubit permutation-invariance. In this case, by using exchange operations, it is possible to implement universal quantum computation without ever leaving the DFS. The procedure is therefore naturally fault-tolerant. In the present paper we will show how to implement universal fault-tolerant quantum computation on DFSs that arise from the Pauli subgroup error model, without requiring any spatial symmetry assumption. However, it will not be possible to do so without leaving the DFS, thus exposing the states to the subgroup errors. As will be shown here, fault-tolerance is obtained by using the encoded states twice: in a dual DFS-QECC mode. This duality arises from the fact that the DFS remains a perfectly valid QECC for the errors that the stabilizer anticommutes with.

There are several ways to achieve universal fault-tolerant quantum computation on stabilizer-QECCs; e.g., use of the sets of gates {Hadamard, $\sigma_z^{1/2}$, Toffoli} [32,14], or {Hadamard, $\sigma_z^{1/4}$, controlled-NOT} [33]. Additional methods were provided in [34,35]. Our construction revert to the early ideas on the implementation of universal quantum computing: we use single-qubit SU(2) operations and a controlled-NOT gate [36–39], except that these are encoded operations, acting on codewords (not on physical qubits). In general such encoded operations involve multiple qubits, and are not naturally available. The key to our construction is a method to generate many-qubit Hamiltonians by composing operations on (at most) pairs of physical qubits. This is done by selectively turning certain interactions on and off. A difficulty is that the very first such step can transform the encoded states and take them outside of the DFS. However, by carefully choosing the interactions we turn on/off and their order, we show that the transformed states become a QECC with respect to the stabilizer-errors that the DFS was immune to. This fact is responsible for the fault-tolerance of our procedure. After the final interaction is turned off, the states return to the DFS, and are once again immune to the stabilizer-errors.

The structure of the paper is as follows. In Section II we briefly review the main result of paper 1, and the connection between the DFSs considered here and stabilizer-QECCs. We then discuss in Section III the meaning of fault-tolerance in light of the error-model considered in this paper. In the following two sections we present the main new ideas and results of this paper: in IV we show how to generate many-qubit Hamiltonians by composing two- and single-qubit Hamiltonians, and in V prove the fault-tolerance of this procedure. We use it to generate encoded SU(2) operations on the DFS qubits. Section VI shows how, by using similar methods, we can fault-tolerantly perform encoded CNOT operations on the encoded qubits, thus coupling blocks of qubits and completing the set of operations needed for universal computation. The final ingredient is presented in Section VII, where we show how to fault-tolerantly measure the error syndrome throughout our gate construction. While our main motivation in this paper is to study computation on DFSs in the presence of stabilizer-errors, it is also interesting to consider the implications of the techniques we develop here to the usual model of errors that anticommute with the stabilizer. We consider this question briefly in Section VIII, and show that our methods provide a new way to implement universal quantum computation which is fault tolerant with respect to error detection. We conclude and summarize in section IX.

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2The exceptions are: (i) The subgroup operators have constant $\alpha = x, y$ or $z$ – the Pauli matrix index; (ii) The single-qubit operators act only on those qubits where all other operators act as identity.

3By “small” we mean that the measurements do not have to be fast compared to the bath correlation time. If they are then the decoherence is avoided essentially by use of the quantum Zeno effect.
In paper 1 we proved the following result:

**Theorem 1.** — Suppose that the Kraus operators belong to the group algebra of some group \( G = \{G_n\} \), i.e., \( A_{ij} = \sum_{a=1}^{N} a_{ij} G_n \). If a set of states \( \{ |j\rangle \} \) belong to a given one-dimensional irrep of \( G \), then the DFS condition \( A_{ij} = c_{ij} \) holds. If no assumptions are made on the bath coefficients \( \{a_{ij}\} \), then the DFS condition \( A_{ij} = c_{ij} \) implies that \( |j\rangle \) belongs to a one-dimensional irrep of \( G \).

This theorem provides a characterization of DFSs in terms of the group-representation properties of the basis set used to expand the Kraus operators in. There are good physical reasons to choose the Pauli group as this basis set: as argued in paper 1, the Pauli group naturally appears as a basis in Hamiltonians involving qubits. Furthermore, using the Pauli group allows us to make a connection to the theory of stabilizer-QECCs. To see this consider the identity irrep, for which each element \( G_n \) in the group \( G \) acts on a decoherence free state \( |\psi\rangle \) as

\[
G_n |\psi\rangle = |\psi\rangle.
\]

Choosing \( G \) from now on as a Pauli subgroup \( Q \), the DFS fixed by the identity irrep is a stabilizer code, where \( Q \) is the stabilizer group. As mentioned above, a stabilizer code is defined as the +1 eigenspace of the Abelian group \( Q \). It is thus clear that the states fixed by \( Q \) play a dual role: they are at once a DFS with respect to the stabilizer errors and a QECC with respect to the errors that anticommute with some element of \( Q \).

It is simple to verify that basic properties of stabilizer codes hold, e.g., that if the stabilizer group has \( K - l \) generators, then the code space (in this case the DFS) has dimension \( 2^l \) (i.e., there are \( l \) encoded qubits) [15]. Indeed, the dimension of an Abelian group with \( K - l \) generators is \( N = 2^{K-l} \), and we showed in paper 1 that the dimension of the DFS is \( 2^K/N = 2^l \).

### III. THE MEANING OF FAULT-TOLERANCE

The observation that the Pauli subgroup DFSs are stabilizer codes allows us to employ some results from stabilizer theory, and aids in the analysis of when it is possible to perform universal fault-tolerant computation on these DFSs.

Before delving into the analysis, however, we should clarify what we mean by fault-tolerance in the present context. The usual meaning of fault-tolerance, as it is used in the theory of QECC, is the following: an operation (gate \( U \)) is not fault-tolerant if an error \( E \) that the code could fix before application of the gate has become an unfixable error \( (UEU^\dagger) \) after application of the gate. For example, a single qubit phase error \( (I \otimes Z) \) becomes a two-qubit phase error \( (Z \otimes Z) \) due to the application of a CNOT gate [34]; if the code used could only correct single-qubit errors then as a result of the CNOT gate (unless it is applied transversally, i.e., not coupling physical qubits involved in representing the same encoded qubit) this code can no longer offer protection. In this scenario, therefore, the CNOT gate was not a fault-tolerant operation. Conversely, an operation is fault-tolerant if the code offers the same protection against the errors that appear after application of the operation \( (UEU^\dagger) \) as it does against the errors before the operation \( (E) \).

A complementary (“Heisenberg” [40]) picture to the (“Schrödinger”) description above is to consider the errors as unchanged and the code \( C \), as well as the stabilizer \( Q \), as transformed after the application of each gate: \( C \mapsto UC \) and \( Q \mapsto UQU^\dagger \). Then fault-tolerance can be viewed as the requirement that the new code is capable of correcting the original errors. This point of view will be particularly useful for our purposes. In our case the original errors are the elements of the Pauli subgroup \( Q \) (the stabilizer), and the gates \( U \) will turn out not to preserve the original code. Nevertheless, we will show that to the new stabilizer \( Q' = UQU^\dagger \) corresponds a QECC (the transformed code \( C' = UC \)) that can correct the original errors. In this way the fault-tolerance criterion is satisfied.
We now begin in earnest our discussion of how to implement universal, fault-tolerant quantum computation on the Pauli-subgroup DFSs. In this section we show how arbitrary single encoded-qubit operations can be implemented fault-tolerantly. We will do so by generating the entire encoded $SU(2)$ group from at most two-qubit Hamiltonians. We assume that the system Hamiltonian is of the general two-qubit form

$$H_S = \sum_{i=1}^{2} \sum_{\alpha=x,y,z} \omega_i^{x_i} \sigma_i^{x_i} + \sum_{i>j=1}^{2} \sum_{\alpha,\beta=x,y,z} J_{ij}^{\alpha\beta} \sigma_i^{\alpha\beta} \otimes \sigma_j^{\alpha\beta},$$

with controllable parameters $\{\omega_i^{x_i}\}, \{J_{ij}^{\alpha\beta}\}$.

A. Background

Suppose we are given an error subgroup $Q$ generated by the elements $\{q_i\}_{i=1}^{Q}$. By the results of paper 1 we know how to identify the corresponding DFS, which is also a stabilizer-code with respect to the errors that anti-commute with $Q$. This QECC aspect will not be needed as long as we are only interested in storing information in this DFS: then the $Q$-errors will have no effect. However, here we are interested in the more ambitious goal of computing in the presence of the $Q$-errors, which means that we must be able to implement logic gates. As discussed above, these gates will take the states out the DFS and expose them to the $Q$-errors. To be able to compute we will need some basic results from the theory of fault-tolerant quantum computation using stabilizer codes, as developed primarily in Ref. [34]. Let us briefly review these results.

The set of operators which commute with the stabilizer themselves form a group called the normalizer of the code, $N(Q)$. These elements are of interest because they are operations which preserve the DFS. Let $q \in Q$, $|\psi\rangle \in \text{DFS}(Q)$; if $n \in N(Q)$ then

$$q(n|\psi\rangle) = n|\psi\rangle,$$

so that $n|\psi\rangle$ is in the DFS as well. Clearly, the stabilizer $Q$ is in the normalizer $N(Q)$ and so the only operations which act nontrivially on the subspace are those which are in the normalizer but not in the stabilizer: $N(Q)/Q$. While this means that these operations can be used to perform useful manipulations on the DFS, it also means that if they act uncontrollably, then they appear as errors that the code cannot detect. As will be seen later on, these are both crucial aspects in our construction.

For any Pauli-subgroup stabilizer code, the normalizer is generated by the single qubit $\overline{X}_i$ and $\overline{Z}_i$ operations, where $i = 1, \ldots, l$ labels the encoded qubits [34]. The bar superscript denotes that these are “encoded operations”: they perform a bit-flip and a phase-flip on the encoded qubits. The gates $\overline{X}_i$ and $\overline{Z}_i$, however, are by themselves insufficient for universal quantum computation. The usual stabilizer-QECC construction deals with (typically uncorrelated) errors that anticommute with the stabilizer. In this case, in addition to generating the normalizer of the Pauli group $N(P_K)$, one other operation is needed, such as the Toffoli gate [32]. Such constructions have been covered in several recent publications [32–35,41]. However, as emphasized above, the errors here are qualitatively different: not only are they always correlated, rather than anticommuting with the stabilizer, the errors are the stabilizer itself. Thus the usual construction does not apply, and we introduce a different approach. We show how to perform universal fault-tolerant quantum computation using the early $SU(2) + \text{CNOT}$ construction [37,39,42], but applied to encoded (DFS) qubits.

B. A Useful Formula: Conjugation by $\pi/4$

Instead of treating $\overline{X}$ and $\overline{Z}$ as gates, as in the usual stabilizer-QECC construction, we employ them as Hamiltonians. Since $\overline{X}$ and $\overline{Z}$ are in the normalizer, so are $\exp(i\theta \overline{X})$ and $\exp(i\theta \overline{Z})$, and so are any other encoded $SU(2)$ group (denoted $SU(2)$) operations obtained from them. By applying operations from $SU(2)$ alone we ensure that the code is preserved. To obtain other $SU(2)$ operations from $\overline{X}$ and $\overline{Z}$, we use the Euler angle construction [43], which shows that any rotation can be composed out of rotations about only two orthogonal axes:

$$\exp[-i\omega (\mathbf{n} \cdot \sigma)/2] = \exp(-i\beta \sigma_z/2) \exp(-i\theta \sigma_y/2) \exp(-i\alpha \sigma_z/2).$$


Here the resulting rotation is by an angle $\omega$ about the direction specified by the unit vector $n$, both of which are functions of $\alpha, \beta$, and $\theta$. Using Eq. (4.3) and the mapping $\{\sigma_x, \sigma_y, \sigma_z\} \mapsto \{\bar{X}, \bar{Y}, \bar{Z}\}$, we can construct any element of $SU(2)$. To do so, we now derive a form of the Euler angle construction which is particularly relevant to operations with Pauli matrices. Assume that $A$ and $B$ are both tensor products of Pauli matrices (and thus square to identity). Then:

$$\exp(-i\varphi A)B\exp(+i\varphi A) = (I \cos \varphi - Ai \sin \varphi)B(I \cos \varphi + Ai \sin \varphi)$$

$$= B \cos^2 \varphi + ABA \sin^2 \varphi - i \sin \varphi \cos \varphi[A, B]$$

$$= \begin{cases} 
B & \text{if } [A, B] = 0 \\
B \cos 2\varphi + iBA \sin 2\varphi & \text{if } [A, B] = 0
\end{cases} \quad (4.4)$$

For the special case of $\varphi = \pi/4$ we define the conjugation with $A$ by

$$T_A \circ \exp(i\theta B) \equiv \exp(-i\frac{\pi}{4}A) \exp(i\theta B) \exp(+i\frac{\pi}{4}A)$$

$$= \begin{cases} 
\exp(i\theta B) & \text{if } [A, B] = 0 \\
\exp[i\theta(iAB)] & \text{if } [A, B] = 0
\end{cases} \quad (4.5)$$

This can be understood geometrically as a rotation by $\varphi = \pi/4$ about the “axis” $A$, followed by a rotation by $\theta$ about $B$, followed finally by a $\beta = -\pi/4$ rotation about $A$, resulting overall in rotation by $\theta$ about the “axis” $iAB$. All $\varphi = \pi/4$ rotations about a Pauli group member are elements of the normalizer of the Pauli group: they take elements in the Pauli group under conjugation to other elements of the Pauli group.

Note that the “conjugation-by-$\frac{\pi}{4}A$” operation $T_A \circ \exp(i\theta B)$ is equivalent to multiplication of $B$ to the left by $iA$ inside the exponent. This is very useful, since the elements of the normalizer of any stabilizer can always be written as a tensor product of single qubit Pauli matrices, i.e., as a tensor product of single-body gates. This is exactly the structure that is suggested by Eq. (4.5), and thus it should allow us to construct $\exp(i\theta \bar{X})$ and $\exp(i\theta \bar{Z})$ for any Pauli subgroup using at most two-body interactions. The caveat, however, is that while $\exp(i\theta \bar{X})$ and $\exp(i\theta \bar{Z})$ always preserve the code (since they are in the normalizer), the operations that generate them from Hamiltonians involving at most two-body interactions may corrupt the code, as explained in Sec. III above.

Let us then state the challenges ahead: *To show how the Hamiltonians $\bar{X}$ and $\bar{Z}$ can be generated using (i) at most two-body interactions, (ii) fault-tolerantly.*

**C. Simple Example: The Subgroup $Q_4$**

Let us pause by introducing a simple example illustrating the notion of universal computation using normalizer elements which are two-body Hamiltonians. Our example uses a group whose natural structure is such that the two-body restriction is automatically satisfied. To this end consider the subgroup $Q_4 = \{I^\otimes 4, X^\otimes 4, Y^\otimes 4, Z^\otimes 4\}$, which we studied in detail in paper 1. It is generated by $K - l = 4 - l = 2$ elements ($X^\otimes 4, Z^\otimes 4$), and therefore encodes $l = 2$ qubits, with states given by

$$|00\rangle_L = \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle) \quad |01\rangle_L = \frac{1}{\sqrt{2}} (|1001\rangle + |0110\rangle)$$

$$|10\rangle_L = \frac{1}{\sqrt{2}} (|1100\rangle + |0011\rangle) \quad |11\rangle_L = \frac{1}{\sqrt{2}} (|0101\rangle + |1010\rangle).\quad (4.6)$$

These states are easily seen to be +1 eigenstates of $Q_4$. The normalizer in this case contains two $\bar{X}_i$ and $\bar{Z}_i$ operations, one for each encoded qubit:

$$\bar{X}_1 = IXXI \quad \bar{Z}_1 = ZZII$$

$$\bar{X}_2 = XXII \quad \bar{Z}_2 = IZZI. \quad (4.7)$$

Indeed, we have, for example $\bar{X}_1|a, b\rangle_L = |1 - a, b\rangle_L$ and $\bar{Z}_1|a, b\rangle_L = (-1)^a|a, b\rangle_L$, so $\bar{X}_1$ and $\bar{Z}_1$ act, respectively, as a bit flip and a phase flip on the first encoded qubit. As easily checked, $\bar{X}_i$ and $\bar{Z}_i$ commute with $Q_4$, so that they keep states within the DFS, as should be the case for normalizer elements. As Hamiltonians $\bar{X}_i$ and $\bar{Z}_i$ are valid two-body interactions and hence can be used directly to generate the encoded $SU(2)$ group on each encoded qubit. That is, $\exp(i\alpha \bar{X}_i)$ and $\exp(i\beta \bar{Z}_i)$ can be combined directly, with arbitrary values for the angles $\alpha$ and $\beta$, to produce any
Let $J$ be a Calderbank-Shor-Steane (CSS)-code, whose clusters can be coupled by performing bit-wise parallel controlled-

not set of two-qubit Hamiltonians $A$, which are two-body on the encoded qubits, can be implemented directly since they are also two-body on the physical qubits (this is not a generic feature, however, as discussed in Section V D below). It is a fundamental theorem of universal quantum computation [37,39,42] that the ability to perform $SU(2)$ on two qubits plus the ability to perform any nontrivial two-body Hamiltonian between these qubits is universal over the combined $SU(4)$ of these two qubits. Thus we can perform universal computation on the $Q_4$-DFS. In this case the normalizer elements which perform the $SU(4)$ are all two-body Hamiltonians, and there is no need to apply any new methods in order to perform fault-tolerant computation which preserves this DFS.

Anticipating the discussion in Section VI, note that while we have demonstrated universal computation on a single DFS-block, we have not yet addressed how to accomplish this when we have clusters of the $Q_4$-DFS. This, of course, is necessary to scale-up the quantum computer under the model of decoherence. In order to perform universal fault-tolerant computation with clusters, we must show that these can be coupled in a non-trivial manner. Methods for performing non-trivial couplings between clusters exist for any stabilizer code [34]. In particular, the $Q_4$-DFS is a Calderbank-Shor-Steane (CSS)-code, whose clusters can be coupled by performing bit-wise parallel controlled-NOT gates between two clusters of qubits. This implements as desired an encoded controlled-NOT between these clusters. In Section VI we will discuss what is needed to make this procedure fault-tolerant.

$Q_4$ is a special case because of the fact that the normalizer elements are all two-body interactions. In general the normalizer elements will be many-body interactions and more general techniques are needed, to which we turn next.

### D. Generating $\overline{X}$ and $\overline{Z}$ Using At Most Two-Body Interactions

We now move on to the general case where the normalizer elements are possibly many-body Pauli operators. Our first task is to show that the “conjugation-by-$\frac{\pi}{4}A$” operation $T_A \circ \exp(i\theta B)$ can be used to generate any many-body Hamiltonian inside the exponent using at most two-qubit Hamiltonians. In Section V we show that this is a fault-tolerant procedure if applied correctly to a DFS.

Suppose the many-body Pauli Hamiltonian $H$ we want to generate is of the following general form:

$$H = \sigma^\alpha_b \overline{\otimes} \sigma^\gamma_j,$$

where $J$ is some index set and $b \notin J$. From Eq. (4.1) we have at our disposal a single-qubit Hamiltonian $\sigma^\beta_b$, and a set of two-qubit Hamiltonians $A_j = \sigma^\alpha_j \otimes \sigma^\gamma_j$ with $j \in J$ and $\gamma_j \neq \beta$. We call the $b^{th}$ qubit the “base qubit”. $A_j$ and $\sigma^\beta_b$ agree on one qubit index but differ on the Pauli matrix applied to that qubit, so they anticommute: $\{A_j, \sigma^\beta_b\} = 0$. Let $J(i)$ denote the $i^{th}$ element in the index set $J$. If we use the “conjugation-by-$\frac{\pi}{4}A_{J(1)}$” operation about $\exp(i\theta \sigma^\beta_b)$ [recall Eq. (4.5)] we obtain:

$$T_{A_{J(1)}} \circ \exp(i\theta \sigma^\beta_b) = \exp[i\theta (i\sigma^\gamma_{J(1)} \overline{\otimes} \sigma^\alpha_{J(1)}) \sigma^\beta_b] = \exp[\pm i\theta \sigma^\gamma_{J(1)} \overline{\otimes} \sigma^\alpha_{J(1)}],$$

where the sign is determined by that of $\varepsilon_{\gamma_{J(1)}, \beta_{J(1)}}$, according to the usual rule of multiplication Pauli matrices:

$$\sigma^\alpha \sigma^\beta = -i \varepsilon_{\alpha, \beta, \gamma} \sigma^\gamma.$$

Applying all other “conjugation-by-$\frac{\pi}{4}A_{J(i)}$” operations, $i = 1..|J|$, we obtain

$$T_{A_{J(J)}} \circ \cdots \circ T_{A_{J(1)}} \circ \exp(i\theta \sigma^\gamma_{J(1)}) = \exp \left( \pm i\theta \overline{\otimes}_{j \in J} \sigma^\gamma_j \overline{\otimes} \sigma^\alpha_j \right).$$

It is clear that by appropriately choosing the sequence of Pauli matrices, i.e., the $A_{J(i)}$, we can obtain $\eta = \beta$. Further, conjugating by $-\frac{\pi}{4}$ (instead of $+\frac{\pi}{4}$) allows us to always adjust the sign in the exponent to +. Thus the action of this gate sequence is to generate the Hamiltonian $H$, as desired:

$$T_{A_{J(J)}} \circ \cdots \circ T_{A_{J(1)}} \circ \exp(i\theta \sigma^\gamma_{J(1)}) = \exp (i\theta H).$$


V. GENERATING ENCODED SU(2) FAULT-TOLERANTLY FOR ANY ABELIAN PAULI SUBGROUP

We are now ready to show how to generate encoded SU(2) operations fault-tolerantly for any Pauli error-subgroup. Let Q be such a subgroup, generated by the elements \( \{ q_j \}_{j=1}^n \), \( |Q| = 2^n \). Recall that here these elements play the dual role of errors and of defining the DFS by fixing its elements. A new (transformed) stabilizer is obtained after each application of a gate \( \exp(i\varphi_j A_j) \). To this sequence of stabilizers corresponds a sequence of stabilizer-QECCs \( \mathcal{C}_j \). Our strategy will be to find conditions on the Hamiltonians \( \{ A_j \} \) such that after each gate application, the then current QECC is able to correct the original Q-errors.

Let \( Q_j = [N(Q_j)] \) denote the stabilizer [normalizer] obtained after application of the gate \( U_j = \exp(i\varphi_j A_j) \). If \( \varphi_j \) is an integer multiple of \( \pi/4 \) (as we will always assume) then there are only three mutually exclusive possibilities for the errors \( e \in Q \) (we use the notations \( e \) and \( q \) for members of \( Q \) to emphasize the error and stabilizer element aspects, respectively):

1. \( e \in Q_j \): The error is part of the transformed stabilizer. In this case the transformed code is immune to \( e \) (i.e., the transformed code is a DFS with respect to \( e \)), and there is no problem.

2. \( e \) anticommutes with some element of \( Q_j \): The error is detectable by the transformed code.

3. \( e \in N(Q_j)/Q_j \) (i.e., \( e \) commutes with \( Q_j \) but is not in it): The error infiltrated the transformed normalizer.

This is a problem since the error is undetectable by the transformed code, and acts on it in a non-trivial manner.

Suppose the errors \( e \in Q \) are exclusively of type 1 or 2. Then those that are of type 2 are not only detectable but also correctable. This is so because they form a group (\( Q \)), and therefore any product of two errors is again either of type 1 or 2, which is exactly the error correction criterion.\(^6\) Thus the problematic case is 3., and this is the case we focus on in order to make a prudent choice of Hamiltonians \( A_j \). To simplify the notation, from now on we shall denote \( N(Q_j)/Q_j \) simply by \( N_j \) (and by \( N \) when \( Q_j = Q \)), and refer to this as the normalizer (without risk of confusion).

Is there a simple criterion to check whether \( e \in N_j \)? The answer is contained in the following theorem:

**Theorem 2.**—Given are a Pauli-subgroup of errors \( Q \), its normalizer \( N \), and a sequence of their images \( \{ Q_j \} \) and \( \{ N_j \} \) under conjugation by unitaries \( \{ U_j \} \). Corresponding to \( Q \) is a DFS (code) \( \mathcal{C} \). A sufficient condition so that no \( e \in Q \) is ever in \( N_j/Q_j \) is that either (i) each \( n_j \in N_j \) equals its source in \( N \), or (ii) for each \( n_j \in N_j \) there exists \( m \in N \) such that \( \{ n_j, m \} = 0 \). Then the transformed codes \( \mathcal{C}_j = U_j \mathcal{C}_{j-1} (\mathcal{C}_1 = U_1 \mathcal{C}) \) can always correct the original Q-errors.

**Proof.**—The normalizer is, by definition, the set of operations that commute with the stabilizer. Let us denote this by \( N = Q' \). What is \( N' \) (the set of operations that commute with the normalizer)? We have \( N' = (Q')' \), and claim that \( (Q')' = Q \).\(^7\) In other words, the only operations that commute with the normalizer are those in the stabilizer. Now let \( n_j \) be the image of \( n \in N \) after the \( j \)th transformation. The observation \( N' = Q \) allows us to exclude case 3. by checking if, for every \( n_j \in N_j \) (where \( n_j \neq n \)), there exists \( m \in N \) that \( n_j \) anti-commutes with. To see this, note first that if \( n_j = n \) then by definition \( n_j \) cannot be in \( Q \). Secondly, for an \( n_j \in N_j \) that differs from its source in \( N \), assume that it anti-commutes with some \( m \in N \). This implies that \( n_j \) is not in the commutant of \( N \), and is therefore

---

\(^6\)Note that this is not true for errors in the usual stabilizer-QECC case, where the errors do not close as a group under multiplication.

\(^7\)This is due to a fundamental theorem of \( \mathcal{C}^* \)-algebras, which states that if \( \mathcal{A} \) is a \( \mathcal{J} \)-closed algebra (i.e., \( a \in \mathcal{A} \implies \mathcal{A}^\dagger \in \mathcal{A} \)) which includes the identity, then it is the commutant of its commutant: \( \mathcal{A}'' = \mathcal{A} \mathcal{[45]} \). These conditions apply to the stabilizer.
A Hamiltonian is correctable by a group, so that all stabilizer errors are both detectable and correctable. Therefore also all products of stabilizer elements are excluded (since the stabilizer is a group), so that all stabilizer-errors are both detectable and correctable.

Below we make repeated use of the result of Theorem 2. The first application is to show how to construct two-body Hamiltonians $\{A_j\}$ which can be applied in succession to produce arbitrary normalizer elements, such that at every point the theorem is satisfied. To this end we need a basic result from the theory of stabilizer codes, regarding a standard form for the normalizer. We then illustrate the general construction with the relatively simple case of CSS codes, and finally move on to general stabilizer errors.

### A. Standard Form of the Normalizer for Stabilizer Codes

It is shown in [46] that, due to the fact that the normalizer is invariant under multiplication by stabilizer elements, the normalizer of every stabilizer code can be brought into the following standard form:

$$Z_j = (I \otimes \cdots \otimes I \otimes Z_j \otimes I \otimes \cdots \otimes I) \otimes (M^Z_j) \otimes (I \otimes \cdots \otimes I)_{K-l-r}$$  \hspace{1cm} (5.1)

$$X_j = (I \otimes \cdots \otimes I \otimes X_j \otimes I \otimes \cdots \otimes I) \otimes (N^Z_j) \otimes (M^X_j) \otimes (I \otimes \cdots \otimes I)_{K-l-r}$$  \hspace{1cm} (5.2)

Here $M^Z_j = \otimes_{n \in Z_j} Z_n$, $N^Z_j = \otimes_{n' \in Z_j'} Z_{n'}$ and $M^X_j = \otimes_{i \in X_j} X_i$, where $Z_j$, $Z_j'$ and $X_j$ are (possibly empty) index sets of lengths $r$, $r$ and $K-l-r$ respectively (i.e., $M^Z_j$, $N^Z_j$ and $M^X_j$ are tensor products of $I$’s and single qubit Pauli $Z$ and $X$ matrices, respectively). Recall that $K$ is the number of physical qubits; $l$ is the number of encoded qubits. The exact form of $M^Z_j$, $N^Z_j$ and $M^X_j$, as well as the value of the integer $r$, can be found from the stabilizer [46], but is unimportant for our purposes. We only need the result that for every pair of encoded $Z$ and $X$ operations, acting on the $j$-th encoded qubit, it is possible to express the operations in the blockwise product shown in Eqs. (5.1),(5.2).

### B. CSS-Stabilizer Errors on One Encoded Qubit

For simplicity, let us now restrict attention to the case of a single encoded qubit in CSS codes, i.e., those codes where every $Z$ and $X$ can be written as a product of only $Z$’s and only $X$’s, respectively. Then, from Eqs. (5.1),(5.2) the standard form is (dropping the $j$ index):

$$Z = Z_1 \otimes M_Z \otimes I^{\otimes K-l-r}$$  \hspace{1cm} (5.3)

$$X = X_1 \otimes I^{\otimes r} \otimes M_X,$$  \hspace{1cm} (5.4)

i.e., $N_Z = I^{\otimes r}$. Our goal is to construct such $Z$ and $X$ from single- and two-body Hamiltonians. We shall do this by starting from the single body Hamiltonians $Z_1$ and $X_1$, and conjugating by certain two-body Hamiltonians. The idea is to successively construct the $Z$’s in $M_Z$ and the $X$’s in $M_X$. We claim that the required two-body Hamiltonians have the natural form

$$A_n = X_1 Z_{x_n} \quad z_n \in Z$$  \hspace{1cm} (5.5)

$$B_i = Z_1 X_{x_i} \quad x_i \in X,$$  \hspace{1cm} (5.6)

where $n = 1..|Z|$ and $i = 1..|X|$, i.e., $A_n$ ($B_i$) has a $Z$ ($X$) in the $n$-th ($i$-th) position of the index set $Z$ ($X$). If there is an even number of $Z$’s in $Z$ then the last Hamiltonian should be taken as $A_{|Z|} = X_1$ [since as we show below in that case in the penultimate step we have $Y_1 \otimes M_Z \otimes I^{\otimes K-l-r}$ for $Z$], and similarly for the last $B_i$.\(^8\)

\(^8\)For example, suppose $Z = Z(ZZZ)(II)$ and $X = X(III)(XX)$; then $A_1 = X_1 Z_2$, $A_2 = X_1 Z_3$, $A_3 = X_1 Z_4$, $A_4 = X_1$ and $B_1 = Z_1 X_2$, $B_2 = Z_1 X_3$. Then we have: $Z(III)(II) \xrightarrow{A_1} Y(ZII)(II) \xrightarrow{A_2} Z(ZZI)(II) \xrightarrow{A_3} Y(ZZZ)(II) \xrightarrow{A_4} Z$, and $X(III)(II) \xrightarrow{B_1} Y(III)(XI) \xrightarrow{B_2} X$. 

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[A_n, \overline{X}] = [B_j, Z] = 0$, so that transforming $\overline{Z}$ does not affect $\overline{X}$, and vice versa. There are now two ways to construct $\overline{Z}$ and $\overline{X}$ fault-tolerantly: in parallel or in series. The parallel implementation has the advantage that it requires only three basic steps and thus is very efficient. Its disadvantage is that it may be hard to implement in practice because it requires simultaneous control over many qubits.

1. Series Construction

We assume throughout this discussion that we wish to generate $\exp(i\theta \overline{Z})$. The symmetry between $\overline{Z}$ and $\overline{X}$ in the CSS case implies that our arguments hold for $\exp(i\theta \overline{X})$ as well, with obvious modifications.

The series construction consists of applying first the sequence of gates $\{\exp(i\frac{\pi}{4} A_n)\}_{n=1}^{\lfloor |Z| \rfloor}$, then the gate $\exp(i\theta Z_1)$, and then the reverse sequence of gates $\{\exp(-i\frac{\pi}{4} A_n)\}_{n=1}^{\lfloor |Z| \rfloor}$. An example is shown in Fig. 1. First, as an application of the general Eq. (4.12), let us prove that this procedure indeed generates $\exp(\pm i\theta \overline{Z})$:

$$T_{A_1} \circ T_{A_2} \circ \cdots \circ T_{A_{|Z|}} \circ \exp(i\theta Z_1) = \left[ \bigotimes_{n=1}^{\lfloor |Z| \rfloor} \exp \left( -i \frac{\pi}{4} A_n \right) \right] \exp(i\theta Z_1) \left[ \bigotimes_{n=1}^{\lfloor |Z| \rfloor} \exp \left( +i \frac{\pi}{4} A_n \right) \right]$$

$$= \exp \left( i\theta (\bigotimes_{n=1}^{\lfloor |Z| \rfloor} A_n Z_1) \right)$$

$$= \exp \left( (-)^{|Z|} i\theta \overline{Z} \right),$$

where in the first line we used the definition of the “conjugation-by-$\frac{\pi}{4} A_n$” operation, in the second the result that this operation corresponds to multiplication inside the exponent, and in the third the form in Eq. (5.3) for $\overline{Z}$. Note that the reason we have a series of conjugation-by-$\frac{\pi}{4} A_n$ operations (as opposed to trivial identity operations) is that $\{\prod_{n=1}^{k-1} A_n Z_1, A_k\} = 0 \forall k \leq |Z|$. Finally, we can eliminate the minus sign [if $|Z|$ is odd] by changing one of the $\frac{\pi}{4}$’s to $-\frac{\pi}{4}$.

Next we must demonstrate that the conditions of Theorem 2 are satisfied at each point in the corresponding circuit in order to guarantee the fault-tolerance of this implementation. Let us divide the proof into three parts, by following the transformations of the normalizer elements before and after the central $\exp(i\theta Z_1)$ gate, and showing that either $\overline{Z}$ or $\overline{X}$ anticommutes with the transformed normalizer at each step along the way.

Errors before the central gate:

After application of the first $k$ gates $\{\exp(i\frac{\pi}{4} A_n)\}_{n=1}^{k}$, $\overline{Z}$ is transformed to $\overline{Z}^{(k)} \equiv \prod_{n=1}^{k} A_n \overline{Z}$ (we neglect the unimportant factors of $i$ from now on). From Eq. (5.5) the product is:

$$\prod_{n=1}^{k} A_n = \begin{cases} \prod_{n=1}^{k} Z_{z_n} & \text{if } k = 2l \\ X_1 \prod_{n=1}^{k} Z_{z_n} & \text{if } k = 2l + 1 \end{cases}.$$  

Therefore, using the standard form:

$$\left( \prod_{n=1}^{2l} A_n \overline{Z} \right) \overline{X} = \prod_{n=1}^{2l} Z_{z_n} \overline{Z} \overline{X} = -\overline{X} \prod_{n=1}^{2l} Z_{z_n} \overline{Z} = -\overline{X} \left( \prod_{n=1}^{2l} A_n \overline{Z} \right),$$

so that $\{\overline{Z}^{(2l)}, \overline{X}\} = 0$. On the other hand

$$\left( \prod_{n=1}^{2l+1} A_n \overline{Z} \right) \overline{Z} = X_1 \prod_{n=1}^{2l+1} Z_{z_n} \overline{Z} \overline{Z} = -\overline{Z} X_1 \prod_{n=1}^{2l+1} Z_{z_n} \overline{Z} = -\overline{Z} \left( \prod_{n=1}^{2l+1} A_n \overline{Z} \right),$$

so that $\{\overline{Z}^{(2l+1)}, \overline{Z}\} = 0$. Thus Theorem 2 is satisfied after each gate application, with $\overline{Z}$ and $\overline{X}$ alternating in the role of the anticommuting original-normalizer element.

Error immediately after the central gate:

At the end of step (i) $\overline{Z}$ has been transformed to $Z_1$. Since the central gate ($\theta$-rotation) uses only $Z_1$, the transformed $\overline{Z}$ does not change. Therefore it is still anticommutes with the original $\overline{X}$ and satisfies the criterion of Theorem 2. For the same reason, however, $\overline{X}$ is transformed by the central gate:
Thus it suffices to show that $\mathbf{X}_0$ anticommutes with $\mathbf{Z}$, which is true since $[\mathbf{Z}, \mathbf{X}_1] = 0$:

$$\mathbf{X}_0\mathbf{Z} = \mathbf{X}\cos(2\theta) + i\mathbf{X}\mathbf{Z}_1\sin(2\theta).$$

(5.11)

Errors after the central gate:

After application of the first $k'$ inverse gates $\{\exp(-i\frac{\pi}{4}A_n)\}_{n=|z|-k'+1}$, $Z_1$ is transformed to $\mathbf{Z}^{(k)} \equiv \prod_{n=1}^{k'} A_n\mathbf{Z}$.

Therefore the same reasoning as in (i) applies to $\mathbf{Z}^{(k)}$. For $\mathbf{X}$ (which is now $\mathbf{X}_0$), the $\mathbf{X}\cos(2\theta)$ component commutes with the inverse gates $\exp(-i\frac{\pi}{4}A_n)$ so that it does not change. The $i\mathbf{X}\mathbf{Z}_1\sin(2\theta)$ component, however, anticommutes with the inverse gates $\exp(-i\frac{\pi}{4}A_n)$. Therefore it flips back and forth between $i\mathbf{X}\mathbf{Z}_1\sin(2\theta)$ and $i\mathbf{X}\mathbf{Y}_1\sin(2\theta)$. These terms anticommute with the original $\mathbf{Z}$ and $\mathbf{Y}$, respectively. But so does the $\mathbf{X}\cos(2\theta)$ component, so their sum anticommutes alternately with the original $\mathbf{Z}$ and $\mathbf{Y}$.

We conclude that Theorem 2 is satisfied at each stage of the circuit. Therefore the series construction is indeed fault-tolerant. Of course, this fault-tolerance is achieved in practice by supplementing the circuit with error-detection and correction procedures after each gate (the parallel construction discussed next is much more economical for this reason). We discuss this issue in Section VII.

2. Parallel Construction

Since the $A_n$ ($B_i$) all commute, the corresponding gates can also be implemented in parallel. That is,

$$U_A \equiv \bigotimes_{n \in \mathbb{Z}} \exp \left( i \frac{\pi}{4} A_n \right) = \exp \left( i \frac{\pi}{4} \sum_{n \in \mathbb{Z}} A_n \right)$$

and

$$U_B \equiv \bigotimes_{i \in \mathbb{X}} \exp \left( i \frac{\pi}{4} B_i \right) = \exp \left( i \frac{\pi}{4} \sum_{i \in \mathbb{X}} B_i \right)$$

(5.13)

can be used as parallel gates in our circuit (see Fig. 2 for an example). To see directly that this circuit really does implement the normalizer gate $\exp(i\theta\mathbf{Z})$ [or $\exp(i\theta\mathbf{X})$], observe that, by definition $\{A_n, Z_1\} = \{B_i, X_1\} = 0$ for all $n$ and $i$. This means that conjugation of $Z_1$ by $U_A$ will act as multiplication by $\prod_{n \in \mathbb{Z}} A_n$ and thus transform $Z_1$ to $\mathbf{Z}$ (without changing $X_1$). The same is true for $X_1$ by changing $Z$’s to $X$’s and $U_A$ to $U_B$. Therefore $U_A Z_1 U_A^\dagger = \mathbf{Z}$ and $U_B X_1 U_B^\dagger = \mathbf{X}$, from which follows immediately by Taylor expansion that:

$$U_A \exp(i\theta Z_1) U_A^\dagger = \exp(i\theta \mathbf{Z})$$

$$U_B \exp(i\theta X_1) U_B^\dagger = \exp(i\theta \mathbf{X}).$$

(5.14)

This too is a fault-tolerant construction. The reason is that it corresponds to looking at the series construction just at the following three points: right before the central gate, right after the central gate, and the end.

C. Example: The Subgroup $Q_{2X}$

As an example with a many-body normalizer element, consider the Pauli subgroup/stabilizer generated by the errors $XXII, IXXI, IIXX$:

$$Q_{2X} = \{IIII, XXII, XIXI, IIXX, XIIX, IXIX, IIIX, IXIX, XXXX\}.$$  

(5.15)

It describes a physically interesting error-model, of bit-flip errors which act on all pairs of nearest neighbor qubits. This situation is of interest, e.g., when decoherence results from spin-rotation coupling in a dipolar Hamiltonian, typical in NMR [47]:

$$H_I = \sum_{j,k} \gamma_j \gamma_k \left[ (\sigma_j \cdot \sigma_k) - 3 (\sigma_j \cdot \mathbf{r}_{jk}) (\sigma_k \cdot \mathbf{r}_{jk}) \right].$$

(5.16)
where $Y^{\alpha \beta}$ are the spherical harmonics and $g_{jk}^{\alpha \beta}$ is the anisotropy tensor. When $g_{jk}^{\alpha \beta} = \delta_\alpha \delta_\beta g_{jk}$ only the $\sigma_j^\alpha \otimes \sigma_k^\beta$ terms remain (coupled to $Y_2^0$), which leads to decoherence described by the subgroup $Q_{2Z}$ (defined similarly to $Q_{2X}$), analyzed in paper 1.

To find the DFS under $Q_{2X}$, we construct in accordance with the techniques of paper 1 the projector $P = \frac{1}{8} \sum_i q_i$ (corresponding to the identity irrep of $Q_{2X}$), where the sum is over all $q_i \in Q_{2X}$. Applying this projector to an arbitrary initial state we find a 2-dimensional DFS, spanned by the states

$$|0_L\rangle = (|0000\rangle + |0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle) / \sqrt{8},$$

$$|1_L\rangle = (|0001\rangle + |0010\rangle + |0100\rangle + |0111\rangle + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle) / \sqrt{8}.$$

(5.18)

This DFS thus encodes a full qubit.

Since for $Q_{2X}$ there is just one encoded qubit, we expect to find just one $X$ and one $Z$. In the case of $Q_{2X}$ it is easily verified that the normalizer is generated by

$$X = XIII,$$

$$Z = ZZZZ.$$

(5.19)

$X$ is already a single-body Hamiltonian and therefore can be implemented directly. Let us show how $Z$ can be implemented as a Hamiltonian using at most two-body interactions.

Note that $Q_{2X}$ supports a CSS code. Comparing the above expressions for $Z$ to the standard form for CSS normalizer elements [Eq. (5.3)], we have $M_Z = Z_2Z_3Z_4$ and $M_X = \emptyset$. Therefore, from the recipe of Eq. (5.5): $A_n = X_1Z_{n+1}$ for $n = 1..3$, while $A_4 = X_1$. The series-circuit implementing $\exp(i\theta Z)$ thus has the form depicted in Fig. 1. The parallel version of the same circuit is shown in Fig. 2. To verify directly that these circuits indeed implement $\exp(i\theta Z)$ use Eq. (4.12) and choose the base qubit to be the first qubit. Then:

$$T_{XZI} \circ T_{XIZ} \circ T_{XIIZ} \circ T_{XI1} \circ \exp(i\theta ZIII) = \exp(i\theta ZZZZ).$$

(5.20)

As required, this is an implementation that uses at most two-body interactions.

Fig. 1 also shows the transformed $Z$ at each point, and directly below the original normalizer element (X or Z) that this transformed normalizer element anticommutes with. This verifies that the circuit is indeed a fault-tolerant implementation of $\exp(i\theta Z)$ for $Q_{2X}$.

D. CSS-Stabilizer Errors on Multiple Encoded Qubits

The CSS case of more than one encoded qubit is a simple extension of the single encoded qubit case discussed above. From Eqs. (5.1),(5.2) the standard form for a CSS code is now:

$$Z_j = Z_j \otimes M_Z^j \otimes I_{\otimes k-1-r}$$

(5.21)

$$X_j = X_j \otimes I_{\otimes r} \otimes M_X^j.$$  

(5.22)

Operations on different encoded qubits $j,j'$ commute. Therefore the single encoded qubit constructions still holds when the Hamiltonians are modified to read

$$A_n^{(j)} = X_jZ_n, \quad z_n \in Z_j$$

(5.23)

$$B_1^{(j)} = Z_jX_i, \quad x_i \in X_j.$$  

(5.24)

As is easily checked, the entire proof for the single encoded qubit case carries through when the base qubit becomes physical qubit number $j$ instead of number 1. This thus allows us to fault-tolerantly implement $SU(2)^l$ on all $l$ encoded qubits.

To couple encoded qubits within the same block [thus generating $SU(2)^l$], one could use a standard trick from stabilizer theory [34], using an auxiliary block to swap information into and out of. This transversal operation involves applying encoded controlled-NOT operations, which we treat in Section VI below. In that Section we also show how coupling multiple encoded qubits can be achieved directly, without resorting to an auxiliary block.
The entire analysis for the CSS case carries through in the general stabilizer case for the implementation of \( \exp(i\theta Z) \), since \( Z \) remains unchanged [recall Eq. (5.1)]. However, the encoded \( X \) operation now includes the additional block \( N_Z : X = X_1 \otimes N_Z \otimes M_X \) [Eq. (5.2)]. Therefore to generate this operation we must include a new set of Hamiltonians:

\[
C_{n'} = Z_1 Z_{n'} \quad \text{if } n' \in Z'.
\]  

(5.25)

If there is an even number of \( Z \)'s in \( Z \) then the last Hamiltonian should be taken as \( C_{|Z'|} = Z_1 \). We now need to repeat the analysis for the generation of \( \exp(i\theta X) \). Again, there is a series and a parallel construction. Since the \( C_{n'} \) and \( B_i \) all commute, the gate

\[
U_{BC} \equiv U_B \otimes U_C = \left[ \bigotimes_{i \in X} \exp \left( \frac{i\pi}{4} B_i \right) \right] \otimes \left[ \bigotimes_{n' \in Z'} \exp \left( \frac{i\pi}{4} C_{n'} \right) \right] = \exp \left[ \frac{i\pi}{4} \left( \sum_{i \in X} B_i + \sum_{n' \in Z'} C_{n'} \right) \right]
\]  

(5.26)

can be implemented in parallel. Conjugation of \( \exp(i\theta X_1) \) by \( U_{BC} \) will yield \( \exp(i\theta X) \) by Eq. (4.12), since \( \{X_1, B_i\} = \{X_1, C_{n'}\} = 0 \). It is further straightforward to check that this is a fault-tolerant implementation, since the arguments used in the case of a single encoded CSS qubit are still valid here.

We are thus left to check only the series construction. Here the only new element is that we must make sure that the application of the \( C_{n'} \) Hamiltonians does not allow for undetectable errors to take place. Apart from this everything is the same as in the CSS case. Now, after application of the first \( k \) gates \( \{\exp(i\frac{\pi}{4} C_{n'})\}_{n' = 1}^{k} \), \( X \) is transformed to \( \overline{X}^{(k)} \equiv \prod_{n' = 1}^{k} C_{n'} X \). This product is:

\[
\prod_{n' = 1}^{k} C_{n'} = \begin{cases} 
\prod_{n' \in Z'} Z_{n'} & \text{if } k = 2l \\
Z_1 \prod_{n' \in Z'} Z_{n'} & \text{if } k = 2l + 1
\end{cases}
\]  

(5.27)

where \( Z_{k'} \) are the first \( k \) elements of the index set \( Z' \). Therefore

\[
\{\overline{X}^{(k)}, Z\} = \left[ (Z_1)^k \prod_{n' \in Z_k'} Z_{n'} X \right] Z + Z \left[ (Z_1)^k \prod_{n' \in Z_k'} Z_{n'} X \right] = \left[ (Z_1)^k \prod_{n' \in Z_k'} Z_{n'} \right] \{X, Z\} = 0.
\]  

(5.28)

Thus Theorem 2 is satisfied after each \( C_{n'} \)-gate application, with \( Z \) playing the role of the anticommuting original-normalizer element. This means that use of the Hamiltonians \( C_{n'} \) does not spoil the fault tolerance of the circuit. We know from the calculations in the single encoded qubit case that the rest of the circuit is also fault tolerant. Hence we can conclude at this point that our method of constructing normalizer elements is fault tolerant for any stabilizer code.

F. Summary

Let us recapitulate the main result of this section. Given a set of errors corresponding to some Abelian subgroup of the Pauli group (i.e., a stabilizer), there is a DFS which is immune against these errors. We have shown how to implement arbitrary encoded \( SU(2) \) operations on this class of DFSs. To do so, we gave an explicit construction of encoded \( \sigma_x \) and \( \sigma_z \) operations, which together span encoded \( SU(2) \)'s for each DFS-qubit. The construction involves turning on and off a series of one- and two-body Hamiltonians for a specific durations. Each such operation takes the encoded states outside of the DFS. However, our construction guarantees that the errors always remain correctable by the code formed by the transformed states. That is, these states form a QECC with respect to the Pauli subgroup errors. Therefore, our construction works by supplementing the unitary gates executing the encoded \( \sigma_x \) and \( \sigma_z \) operations by appropriate error correction procedures. To complete the construction, we still need to show how to execute encoded two-body gates, and how to fault-tolerantly measure the error syndrome. This is the subject of the next two sections.
The unitary controlled-NOT (CNOT) operation from the first qubit (“control qubit”) to the second qubit (“target qubit”) can be written in the basis of $\sigma_z$ eigenstates as:

$$U_{\text{CNOT}} = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \quad (6.1)$$

(where each entry is a $2 \times 2$ matrix). Since we are working in the Heisenberg picture it is useful to consider how two-qubit operators transform under CNOT. For example,

$$X \otimes I \longmapsto U_{\text{CNOT}} (X \otimes I) U_{\text{CNOT}}^\dagger = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}. \quad (6.2)$$

As is simple to verify, the full transformation table is:

- $X \otimes I \longmapsto X \otimes X$
- $I \otimes X \longmapsto I \otimes X$
- $Z \otimes I \longmapsto Z \otimes I$
- $I \otimes Z \longmapsto Z \otimes Z. \quad (6.3)$

Since $U(A \otimes B)U^\dagger = U(A \otimes I)U(I \otimes B)U^\dagger$, the rest of the transformations under CNOT follow simply by taking appropriate products of the above, e.g., $X \otimes Z = (X \otimes I)(I \otimes Z) \longmapsto (X \otimes X)(Z \otimes Z) = -Y \otimes Y$. We need to show how to fault-tolerantly construct an encoded CNOT operation for the DFS corresponding to a given Pauli subgroup of errors.

### A. CSS-Stabilizer Errors

It is well known that a bitwise CNOT gate between physical qubits in different blocks is an operation that preserves any CSS code, and acts as the encoded CNOT gate between the blocks encoding different qubits [34]. However, this is true only at the conclusion of the operation, i.e., after all the bitwise operations have been applied. During the execution of the bitwise operations the codewords are exposed to errors. To demonstrate this, consider the transformation of the normalizer elements of a CSS code. Let $\text{CNOT}_{j_A,j_B}$ denote the CNOT operation from control qubit $j$ (in the first block $A$) to target qubit $j$ (in the second block $B$). For definiteness let us consider the transformation of $\overline{X}_j \otimes I^{\otimes K}$ under bitwise CNOT’s. Then, because of the standard form for $\overline{X}_j$, the first CNOT operation is applied from control qubit $j$, and subsequent CNOT’s from control qubits determined by the index set $X$, i.e., acting on pairs of physical qubits at positions $\{(i_A,i_B)\}_{i \in X}$. Using Eqs. (5.21),(5.22) for $\overline{Z}_j$ and $\overline{X}_j$, and the transformation table of Eq. (6.3), we find:

$$\overline{X}_j \otimes I^{\otimes K} \longmapsto [X_j \otimes I^{\otimes r} \otimes M_X] \otimes [I^{\otimes l} \otimes I^{\otimes r} \otimes I^{\otimes K-l-r}]$$

$$\overset{\text{CNOT}_{j_A,j_B}}{\longmapsto} [X_j \otimes I^{\otimes r} \otimes M_X] \otimes [X_j \otimes I^{\otimes r} \otimes I^{\otimes K-l-r}]$$

$$\overset{\text{CNOT}_{i_A,i_B}}{\longmapsto} [X_j \otimes I^{\otimes r} \otimes M_X] \otimes [X_j \otimes I^{\otimes r} \otimes X_{i_1}]$$

$$\longmapsto \ldots \longmapsto [X_j \otimes I^{\otimes r} \otimes M_X] \otimes [X_j \otimes I^{\otimes r} \otimes M_X] = \overline{X}_j \otimes \overline{X}_j. \quad (6.4)$$

Similarly one can check that the rest of the transformations of Eq.(6.3) are satisfied at the encoded level. Therefore this calculation demonstrates that the full bitwise CNOT gate indeed acts as an encoded CNOT operation, since it transforms encoded normalizer operations according to the transformation rules of CNOT, as per Eq. (6.3). In our context this implies that given a certain Pauli subgroup of errors, application of the full bitwise CNOT gate will implement the $\overline{\text{CNOT}}$ gate on the DFS in a way which keeps the codewords inside the DFS at the end of the operation. However, as in the $SU(2)$ case, this is not true at intermediate steps, meaning that the code leaves the DFS. As in the $SU(2)$ case, we must check that the original errors are still correctable at each intermediate step. Theory

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Note that this is equally true for stabilizer-QECCs, which are thus exposed to errors during gate execution.
Suppose one can also implement \( \exp(i \theta U) \) with the difference that the single-body central gates have been replaced with a two-body gate, generated by the transformed normalizer. It is easy to see from Eq. (6.4) that \( \overline{X}_j \) does not change in the first block, and the sequence of transformed \( \overline{X}_j \)'s in the second block anticommutes with \( \overline{Z}_j \) at every step. Therefore error correction is possible at each intermediate step. 

To complete the construction it is necessary to check that the remaining normalizer elements are appropriately transformed. Repeating the calculation of Eq. (6.4) it is straightforward to check that this is true, namely:

\[
I \otimes K \otimes \overline{X}_j \mapsto I \otimes K \otimes \overline{X}_j \\
\overline{Z}_j \otimes I \otimes K \mapsto \overline{Z}_j \otimes I \otimes K \\
I \otimes K \otimes \overline{Z}_j \mapsto \overline{Z}_j \otimes \overline{Z}_j,
\]

with \( I \otimes K \otimes \overline{X}_j \) and \( \overline{Z}_j \otimes I \otimes K \) invariant under the bitwise CNOT’s (thus requiring no error correction), and the transformed \( I \otimes K \otimes \overline{Z}_j \) anticommuting at each step with the original \( \overline{X}_j \).

This completes our demonstration that a \( \text{CNOT} \) gate can be implemented fault-tolerantly using bitwise CNOT’s in the CSS case.

### B. General Stabilizer Errors

In the non-CSS case the bitwise CNOT does not act as a \( \text{CNOT} \). One quick way to realize this is to note that since \( X \otimes I \mapsto X \otimes X \), by unitarity \( X \otimes X \mapsto X \otimes I \), but this is not the case at the encoded level:

\[
\overline{X} \otimes \overline{X} = [X_1 \otimes N_Z \otimes M_X] \otimes [X_{K+1} \otimes N_Z \otimes M_X] \\
\mapsto [X_1 \otimes I \otimes M_X] \otimes [I_{K+1} \otimes N_Z \otimes I^{K-i-r}] \neq \overline{X} \otimes I^{\otimes K}.
\]

Thus a different implementation of the \( \text{CNOT} \) is needed. Now, it is clear that if the product of stabilizers for different blocks (each encoding one qubit or more) is mapped to itself at the end of the \( \text{CNOT} \) implementation, then the stabilizer errors will not have changed, the DFS qubits will not have changed, and thus the DFS-code still remains valid for arbitrary operators \( O \) and \( \theta \) such that \( \exp(i \theta I_A \otimes X_B) \).

Consider two blocks \( A \) and \( B \) encoding one DFS qubit each. We already know how to implement \( \exp(i \theta I_A \otimes \overline{X}_B) \). Suppose one can also implement \( \exp(i \theta Z_A \otimes \overline{X}_B) \). Then by use of the Trotter formula \( \lim_{n \to \infty} \exp(i \theta O_1/n) \approx \exp \left( i \frac{\theta}{n} O_1 \right) \) [48], or its short-time approximation

\[
\exp[i t (O_1 + O_2)/n] = \exp[i t O_1/n] \exp[i t O_2/n] + O(n^{-2})
\]

valid for arbitrary operators \( O_1 \) and \( O_2 \), we can form, to any desired accuracy

\[
\exp[i \theta (I_A \otimes \overline{X}_B - Z_A \otimes \overline{X}_B)/2] = \begin{pmatrix} I & 0 \\ 0 & \exp(i \theta \overline{X}_B) \end{pmatrix}.
\]

For \( \theta = \pi/2 \) this is the \( \text{CNOT} \) operation between the two blocks. Thus our problem reduces to showing how \( \exp(i \theta Z_A \otimes \overline{X}_B) \) can be implemented fault-tolerantly for arbitrary stabilizer DFSs.

Consider the circuit shown in Fig 3. It describes the implementation of \( Z \) and \( \overline{X} \) operations, as in the \( \text{SU}(2) \) case, with the difference that the single-body central gates have been replaced with a two-body gate, generated by the Hamiltonian \( H_{AB} = Z_A^1 \otimes X_B^1 \) (here \( A \) and \( B \) are the two blocks and the subscript 1 indicates the first physical qubit in each block). By the \( \text{SU}(2) \) construction we have that \( U_A Z_A^1 U_A^\dagger = \overline{Z}_A \) and \( U_B X_B^1 U_B^\dagger = \overline{X}_B \) (recall Sec. VB2).
Therefore, using the fact that for any non-singular matrix $M$ the equality $M \exp(H)M^{-1} = \exp(MHM^{-1})$ holds, the gates in Fig. 3 yield:

$$(U_A \otimes U_B) \exp(i\theta H_{AB}) \left( U_A^\dagger \otimes U_B^\dagger \right) = \exp \left[ i\theta (U_A \otimes U_B) H_{AB} \left( U_A^\dagger \otimes U_B^\dagger \right) \right]$$

$$= \exp \left[ i\theta \left( U_A Z_A^1 U_A^\dagger \right) \otimes \left( U_B X_B^1 U_B^\dagger \right) \right]$$

$$= \exp \left( i\theta Z_A \otimes X_B \right),$$

(6.8)

as desired.

It remains to verify that this is a fault-tolerant construction. The only difference compared to the $SU(2)$ construction above is the fact that we are now using a two-body central Hamiltonian. It is reasonable to assume that if the system can couple the two blocks connected by this Hamiltonian, then so can the environment. Therefore instead of considering the error subgroups $Q_A$ and $Q_B$ separately, we must now consider the new error subgroup $Q_A \times Q_B$. But then the appropriate normalizer is $N_{AB} = N_A \times N_B$, and the sequence of transformed normalizers satisfy $N_{AB,j} = N_{A,j} \times N_{B,j}$. This makes the fault-tolerance verification task very simple: We already checked in our $SU(2)$ discussion that Theorem 2 is satisfied for each block separately. Now, clearly both $N_A \otimes I_B, I_A \otimes N_B \in N_{AB}$. Therefore, since for every transformed normalizer element in $N_{A,j} \ [N_{B,j}]$ there is an anticommuting element in the original normalizer $N_A \ [N_B]$, it follows that $N_A \otimes I_B \ [I_A \otimes N_B]$ will correspondingly anticommute with the elements of $N_{AB,j}$. This means that Theorem 2 is satisfied also for the combination of blocks $A$ and $B$, and fault-tolerance is guaranteed as in the $SU(2)$ case.

As promised in Section VD, the construction presented here also applies to multiple qubits encoded into a single block. To see this, consider the case of two encoded qubits in the same block, and let us show that we can generate $SU(2^l)$ (for $l$ encoded-qubits in a block). Now, from the standard form we have:

$$Z_1 = Z_1 \otimes M_{12}^1 \otimes I^\otimes K-l-r$$

$$Z_2 = Z_2 \otimes M_{22}^1 \otimes I^\otimes K-l-r.$$  

(6.9)  

(6.10)

Let $Z_1 = U_1 Z_1 U_1^\dagger$ and $Z_2 = U_2 Z_1 U_2^\dagger$. Note that $[U_1, Z_2] = [U_2, Z_1] = 0$ since $U_{1(2)}$ contains $X_{2(1)}$. For the same reason also $[U_1, U_2] = 0$. Therefore:

$$(U_1 \otimes U_2) \exp(i\theta Z_1 \otimes Z_2) \left( U_1^\dagger \otimes U_2^\dagger \right) = \exp \left[ i\theta (U_1 \otimes U_2) Z_1 \otimes Z_2 \left( U_1^\dagger \otimes U_2^\dagger \right) \right]$$

$$= \exp \left[ i\theta \left( U_1 Z_1 U_1^\dagger \right) \otimes \left( U_2 Z_2 U_2^\dagger \right) \right]$$

$$= \exp \left( i\theta Z_1 \otimes Z_2 \right).$$

(6.11)

The same idea can be used to implement $\text{CNOT}$ between multiple qubits encoded into a single block. We have thus provided a fault-tolerant implementation of $\text{CNOT}$ for any stabilizer DFS.

**VII. FAULT TOLERANT MEASUREMENT OF THE ERROR SYNDROME**

So far we have taken for granted that error detection and correction is possible in between gate applications. We now complete our discussion by showing that it is indeed possible to do so fault-tolerantly. This requires the ability to measure the sequence of transformed stabilizer generators in a manner that does not introduce new errors in a catastrophic way. To accomplish this fault-tolerant measurement we follow, with some modifications, the usual stabilizer construction [46].

Let us recall the basics of measurement within stabilizer theory. A DFS state $|\psi\rangle$ in the stabilizer $Q$ is a +1 eigenstate of all elements of $Q$. An error $e$ is an operator that anticommutes with at least one element of the stabilizer $Q$, say $q$. If $|\psi\rangle \in Q$ then $q e |\psi\rangle = -e q |\psi\rangle = -e |\psi\rangle$, so that $e |\psi\rangle$ is an eigenstate of $q$ with eigenvalue $-1$. Therefore each generator measurement that returns the eigenvalue $+1$ indicates that no error has occurred, while each $-1$ result indicates an error, which can be fixed by applying the error $e$ to the state. The sequence of ±1’s that results from measuring all stabilizer generators is called the “error syndrome”. The identity of $e$ is uniquely determined by this “syndrome”, since the measurement process projects any linear combination of errors to an error in the Pauli group.
In this case the stabilizer generators contain either products only of $Z$‘s (“$Z$-type”) or products only of $X$‘s (“$X$-type”). Suppose we wish to measure a $Z$-type stabilizer generator. The +1 eigenstates of such a generator are the “even parity states”, i.e., those states containing an even number of $|1\rangle$’s. Prepare an ancilla in the encoded $|0_L\rangle$ state (below we discuss how). Then for each data qubit where the given stabilizer generator has a $Z$ (not an $I$) apply a controlled-$X$ from this qubit to the ancilla. The ancilla will flip every time the data qubit was a $|1\rangle$, so measuring the ancilla at the end and finding it in $|0_L\rangle$ will indicate no error (even number of flips), whereas $|1_L\rangle$ will indicate an error (odd number of flips). Distinguishing between $|0_L\rangle$ and $|1_L\rangle$ amounts to measuring $Z$ on the ancilla, which we can do directly by measuring $Z$ on all those ancilla qubits whose $Z$ has a $Z$.

Now suppose we wish to measure an $X$-type stabilizer element. The same procedure as for $Z$-type generators can be applied, with one modification: a Hadamard transform

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(7.1)
must be applied before and after the controlled-$X$ operation. This DFS-construction is different than in the usual QECC-stabilizer construction, where multiple control operations to the same ancilla-qubit are not fault-tolerant because they are not transversal. There multiple CNOT’s from different data qubits to the same ancilla qubit can cause errors to spread catastrophically if the ancilla qubit undergoes a phase error (recall that under CNOT, $I \otimes Z \rightarrow Z \otimes Z$).

Note that since $X$ is in the normalizer, it commutes with all stabilizer errors. This means that any such error occuring on the ancilla before the $X$ is equivalent to the same error after the $X$, and therefore the error has no effect. In other words, neither does the ancilla ever leave the DFS under the application of $X$, nor can an error on the ancilla propagate back to the data qubits. Note further that since the ancilla is at all times unentangled from the data qubits the measurement is non-destructive on the data qubits.

What if a stabilizer error occurs on the data qubits right after the application of the Hadamard gate? This can clearly present a problem, since it may for example flip the data qubit controlling the $X$ applied to the ancilla. One (standard) way of dealing with such errors is to repeat the measurement several times in order to improve our confidence in the result. An alternative is to use concatenated codes [35,49–51]. This will be of use if the stabilizer error is correctable by the transformed code, i.e., if we can verify that the conditions of Theorem 2 are satisfied. Then we can use the DFS at the lowest level, and concatenate it with the QECC it transforms into under the stabilizer errors (see Ref. [23] for concatenated DFS-QECC in the collective decoherence model). Now, recall the CSS form of the normalizer elements, Eq. (5.3). For every Hadamard transform in the first set (i.e., before the controlled-$X$ operations) on a qubit in a position corresponding to an $X$ in an $X$-type stabilizer generator, the normalizer elements transform by having $X$ and $Z$ interchange in this position. In the standard form of Eq. (5.3), if this happens to be the first qubit then $Z \rightarrow X \otimes M_Z \otimes I$, which anticommutes with the original $Z$, and $X \rightarrow Z \otimes I \otimes M_X$, which in turn anticommutes with the original $X$. If the position of the $X$ in the $X$-type stabilizer generator is where $M_Z$ has a $Z$, then $Z \rightarrow Z \otimes M_Z' \otimes I$, where $M_Z'$ has that $Z$ changed into an $X$. This transformed $Z$ anticommutes with the original $X$. Similarly, $X \rightarrow X \otimes I \otimes M_X'$ with an $X$ changed into a $Z$, and this transformed $X$ anticommutes with the original $Z$. Thus the conditions of Theorem 2 are again satisfied.

The second set of Hadamard transforms restores the original normalizer. One then proceeds to measure the next stabilizer generator. We thus see that this measurement procedure is fault-tolerant of stabilizer errors.

### B. General Stabilizer Errors

In the non-CSS case the stabilizer generators may contain $Y$’s as well, so our analysis above requires some modifications. The unitary operation that transforms $Y$ to $Z$ is

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$  

(7.2)

\[10\]This DFS-construction is different than in the usual QECC-stabilizer construction, where multiple control operations to the same ancilla-qubit are not fault-tolerant because they are not transversal. There multiple CNOT’s from different data qubits to the same ancilla qubit can cause errors to spread catastrophically if the ancilla qubit undergoes a phase error (recall that under CNOT, $I \otimes Z \rightarrow Z \otimes Z$).
It also maps $Z \mapsto X \mapsto Y$. When this operation is applied immediately before the controlled-$X$ to the ancilla and immediately after it for every $Y$ in the stabilizer, the $Z$-type construction applies again. However, for the purpose of concatenation we need to check that the procedure is still fault-tolerant of stabilizer errors. The normalizer generators now have the form of Eqs. (5.1),(5.2). Every time a Hadamard or $Q$ operation is applied, $Z \mapsto X$ in a single position in $\mathbf{Z}$. Similarly, $Z \mapsto X$, or $X \mapsto Z$ (if Hadamard) or $Y$ (if $Q$) in a single position in $\mathbf{X}$.

The case of the transformed $\mathbf{Z}$ is trivial: if $Z \mapsto X$ anywhere then the transformed $\mathbf{Z}$ anticommutes with the original $\mathbf{Z}$. Consider the transformed $\mathbf{X}$. The possibilities are: (i) $X_1 \mapsto Z_1$ or $Y_1$, (ii) $Z \mapsto X$ in the $N_2$ part, (iii) $X \mapsto Z$ or $Y$ in the $M_X$ part. In all these cases it is easily verified that the transformed $\mathbf{X}$ anticommutes with the original $\mathbf{X}$. Therefore the measurement procedure is fault-tolerant also in the non-CSS case.

VIII. OUTLOOK: IMPLICATIONS FOR THE INDEPENDENT-ERRORS MODEL

The methods we have introduced in this paper need not be restricted to stabilizer-errors. In this section we briefly touch upon the implications of our construction for universal quantum computation in the independent errors model, when stabilizer-errors are taken into account as well. We thus generalize the standard treatment of stabilizer codes [34], where stabilizer errors that may occur during the course of gate implementation are ignored. However, we are here only able to consider independent single-qubit errors, so that the inclusion of the special type of correlated many-body errors represented by the stabilizer-errors is a rather unrealistic error model. The main importance of the result presented here is that it suggests an alternative route to universal quantum computation that is fault tolerant with respect to error detection, and is highly parallelizable. We believe that this may lead to an improved threshold for fault tolerant computation in the setting of concatenated codes [51].

Let us recall the error detection and correction criteria for a stabilizer code $Q = \{q_k\}$ to be able to deal with all single qubit errors:

$$\forall i, j, \alpha, \beta \ \exists k \text{ s.t. } \{q_k, \sigma_i^\alpha \otimes \sigma_j^\beta\} = 0 \quad (8.1)$$

Can we implement encoded $SU(2)$ operations in the independent errors model similarly to what we did above for stabilizer-errors? To do so we need to make sure that the errors do not become part of the sequence of transformed normalizers. The important difference compared to the stabilizer-errors case is that now the errors are “small” (single-body), which means that we must avoid using a single-qubit Hamiltonian as a central gate (for it is a normalizer element which will not be distinguishable from an error). If we restrict ourselves to using two-body Hamiltonians as central gates (which we can always do – recall the comment at the end of Section IV D), then we run into a similar problem regarding the two-body form of Eq. (8.1), i.e., if the central gate uses the Hamiltonian $\sigma_i^\alpha \otimes \sigma_j^\beta$ then we will not be able to correct the two errors $\sigma_i^\alpha$ and $\sigma_j^\beta$. However, as we now show, as long as we use a two-body central gate it is nearly always possible to satisfy the error detection criterion, $\forall i, \alpha \ \exists k \text{ s.t. } \{q_k, \sigma_i^\alpha\} = 0$.

Let us demonstrate this explicitly for Steane’s 7-qubit code [12]. This is a CSS code encoding one qubit into seven, and in standard form has the normalizer:

$$\mathbf{X} = X_1X_5X_6$$
$$\mathbf{Z} = Z_1Z_3Z_4. \quad (8.2)$$

Consider the gate construction [derived from Eq. (5.5)]

$$\exp(i\theta \mathbf{Z}) = T_{X_1Z_3} \circ \exp(i\theta Y_1Z_4). \quad (8.3)$$

The normalizer transforms as:

$$\mathbf{X} \xrightarrow{X_1Z_3} -\mathbf{X} Y_1Z_4 \cos(2\theta)\mathbf{X} + i\sin(2\theta)Z_1Z_4X_5X_6 \xrightarrow{X_1Z_3} \cos(2\theta)\mathbf{X} + \sin(2\theta)\mathbf{Y} = \mathbf{X} \exp(2i\theta \mathbf{Z})$$
$$\mathbf{Z} \xrightarrow{X_1Z_3} Y_1Z_4 Y_1Z_4 \xrightarrow{X_1Z_3} \mathbf{Z}. \quad (8.4)$$

We see that at no point does a single-qubit error become part of the transformed normalizer, so that all single qubit errors are detectable. On the other hand, while we can always detect the occurrence of both the $Y_1$ and $Z_1$ errors, we cannot distinguish between them after the first gate has been applied (since our normalizer is $Y_1Z_4$ at that point). Since we might accidentally try to reverse the error $Y_1$ when in fact the error $Z_4$ has taken place, this means that our construction is fault tolerant only for error detection. Similarly, the gate construction

$$\exp(i\theta \mathbf{X}) = T_{Z_1X_5} \circ \exp(i\theta Y_1X_6) \quad (8.5)$$
which also satisfies the error detection (but not correction) condition for single-qubit errors, in that no single-qubit error becomes part of the transformed stabilizer.

Let us now consider the general stabilizer case. Recall once more the standard form of the normalizer, Eqs. (5.1),(5.2). Our gate construction acts by transforming one of the normalizer elements to two-body form, where it is applied as the central \( \theta \)-gate, and then is transformed back to its standard form. All other normalizer elements are left unchanged until the application of the central gate, with which they anticommute. At this point each \( Z \) [\( X \)] is multiplied by \( \exp(-2i\theta X) \) [\( \exp(2i\theta Z) \)]. The final sequence of gates flips these normalizer elements back and forth between \( \exp(-2i\theta X) \) and \( \exp(-2i\theta Y) \) [\( \exp(2i\theta Z) \) and \( \exp(2i\theta Y) \)] (recall the analysis in Section V B). All these operations have the effect of expanding, rather than shrinking the normalizer elements, as seen in the example of the 7-qubit code above.

The ability to error-detect at each point thus translates to the question of whether any normalizer element ever becomes a single-body Hamiltonian under this sequence of transformations. It is not hard to see from the above description of the orbit of the normalizer that this can only be the case if in the standard form the normalizer contains a single-body element to begin with. This is certainly possible, as indeed shown in our \( Q_{2X} \) example (Section V C), where \( X = X_{131} \). However, it is not the case for most interesting stabilizer codes, i.e., those offering protection against arbitrary single-qubit errors. Such codes must have “large” normalizer elements since they may not contain any single-qubit errors to begin with. We conclude that our \( SU(2) \) construction using just two-qubit Hamiltonians works for all stabilizer codes of interest, in the sense that it is fault-tolerant with respect to error detection.

To complete the repertoire of universal operations the \( \text{CNOT} \) gate is still needed. The discussion given in Section VI applies here as well, with the modification that for non-CSS stabilizer codes it is once again necessary to apply two-body central gates. Fault tolerant measurement of the error syndrome can be done using the standard techniques available for stabilizer codes [34].

### IX. SUMMARY AND CONCLUSIONS

In a previous paper [29] we derived conditions for the existence of class of decoherence-free subspaces (DFSs) defined by having Abelian stabilizers over the Pauli group. In this sequel paper we addressed the problem of universal, fault-tolerant quantum computation on this class of DFSs. The errors in this model are the elements of the stabilizer, and thus are necessarily correlated. This model is complementary to the standard model of quantum computation using stabilizer quantum error correcting codes (QECCs), where the errors that are correctable by the code anticommute with the stabilizer (rather than being part of it). The correlation between errors in the present model implies no spatial symmetry in the system-bath interaction, unlike in most previous studies of computation on DFSs (which considered the “collective decoherence” model, and where the stabilizer is non-Abelian). Therefore our present results significantly enlarge the scope of the theory of DFSs.

It turns out that even though the class of DFSs we considered are Pauli-group stabilizer codes, the usual universality constructions do not apply, because of the different error-model we assume. Our alternative construction of a set of universal quantum gates resorts to the early ideas about universal quantum computation, except that our operations all act on \( \text{encoded} \) (DFS) qubits: we showed how to implement arbitrary single-encoded-qubit operations [the \( SU(2) \) group] and \( \text{CNOT} \) gates between pairs of encoded qubits. The challenge here was to show how to accomplish this implementation using only physically reasonable Hamiltonians, i.e., those involving no more than two-body interactions. To do so, we switched from the usual point of view treating the normalizer elements (i.e., the operations that preserve the DFS) as gates, to one where these elements are considered as many-body Hamiltonians. We then introduced a procedure whereby these Hamiltonians could be simulated using at most two-body interactions. The gate sequence implementing this simulation does not preserve the DFS except at the beginning and end. Throughout the execution of the gates the DFS states are exposed to the stabilizer-errors. However, we showed that in fact the DFS is transformed into a sequence of stabilizer codes, each of which is capable of detecting and correcting the original stabilizer-errors. Moreover, we showed that these errors can be diagnosed in a fault-tolerant manner, i.e., without introducing new errors as a result of the associated measurements. In all, we showed how by using this type of hybrid DFS-QECC approach, universal, fault-tolerant quantum computation can be implemented.

Our results have implications beyond computation on DFSs. We briefly considered here also the question of whether our techniques can be used to compute fault-tolerantly in the standard stabilizer error-model. We found the answer
to be affirmative for the purpose of single-qubit error detection, but not correction. While this is interesting in its
own right because of the new universality construction we introduced, it may also have important implications for the
question of quantum computing using concatenated codes. The reason is that our construction is highly paralleliz-
able, meaning that it requires a very small number of operations during which the encoded information is exposed to
events. We speculate that this can significantly reduce the threshold for fault-tolerant quantum computation.

Finally, an interesting open question is whether the methods developed here are applicable to the problem of
universal quantum computation on other classes of DFSs.

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APPENDIX A: WHY THE 4-BLOCK IMPLEMENTATION OF CNOT IS NOT FAULT-TOLERANT FOR
NON-CSS STABILIZERS

The construction of the CNOT in Ref. [34] uses a series of bitwise CNOT's (along with some other operations)
acting between pairs of qubits in 4 different blocks. Let us calculate the result of applying bitwise CNOT's on
I⊗K ⊗X
(i.e., on two out of the four blocks). Recall that for a non-CSS code X = X ⊗NZ ⊗MX [Eq. (5.2)]. Therefore it
follows from Eq. (6.3) that

\[ I⊗K ⊗X \rightarrow [I ⊗ NZ ⊗ I⊗K−1−r] ⊗ X, \]  \hspace{1cm} (A1)

i.e., the Z's are copied backwards into the first block. Therefore the normalizer on the first block now contains
I ⊗ NZ ⊗ I⊗K−1−r. This element obviously commutes with both the original X and Z [Eq. (5.1)], but does not equal
either. Therefore it must be in the original stabilizer Q. Turning this around, we see that an error e ∈ Q has become
part of the new normalizer N_j(Q_j)/Q_j which is catastrophic since this error is now undetectable.

Verlag, Berlin, 1987).
FIG. 1. Fault-tolerant circuit implementing exp(iθZ) for the Q_{2X} subgroup. The transformed Z is shown at each gate, and directly below the original normalizer element that it anticommutes with.

FIG. 2. Parallel implementation of θZ for the Q_{2X} subgroup.

FIG. 3. Fault-tolerant implementation of exp(iθZX) needed to generate CNOT.

FIG. 4. Measurement of the stabilizer element XZYX.