Path Integral Quantisation of Finite Noncommutative Geometries

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Abstract

We present a path integral formalism for quantising the spectral action of noncommutative geometry, based on the principle of summing over all Dirac operators. The approach is demonstrated on two simple finite noncommutative geometries. On the first, the graviton is described by a Higgs field, and on the second, it is described by a gauge field. We start with the partition function, and calculate the propagator and Greens functions for the gravitons. The expectation values for the distances are evaluated, and we discover that distances can shrink with increasing graviton excitations. We find that adding fermions reduces the effects of the gravitational field. We also make a comparison with Rovelli’s canonical quantisation approach described in [?], and briefly discuss the quantisation of a Riemannian manifold.

1 Introduction

One of the greatest successes of noncommutative geometry has been the unification of the forces of nature into a single gravitational action - the spectral action [?, ?]. This has been achieved at the classical level, for an Euclidean signature. It does this using the Kaluza-Klein idea of rewriting all the gauge fields as components of a metric on a more structured spacetime. Noncommutative geometry succeeds where Kaluza-Klein fails as it is not limited to pure Riemannian manifolds. For introductions to noncommutative geometry see [? , ?, ?, ?]. The question of how to quantise a theory on a general noncommutative geometry remains largely unresolved. Conventional techniques work on Riemannian-like manifolds, and have been used on noncommutative extensions, such as, almost commutative geometries1 and the noncommutative torus [?]. Beyond this, most efforts have focused on quantising a particular noncommutative geometry [?, ?, ?]. In this paper, we present a path integral approach that is applicable to any noncommutative geometry. It has been developed with the spectral action in mind, which is the natural geometric action for a noncommutative geometry. Our approach builds on and complements the work done by Rovelli in [?].

The outline of this paper is as follows. We begin in section 2 with a detailed description of our path integral formalism. Then, in sections 3 and 4, it is demonstrated on two simple finite noncommutative geometries; the two-point space and a matrix geometry. For these, the path integral is a standard integral, so the technical difficulties associated with functional integration are avoided. To keep the examples clear and concise we restrict ourselves to (\(A, H, D\)) spectral triples. That is, we ignore real structure, orientability and Poincaré duality, which do not play an essential role in the discussion. As such, the examples can be considered as

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1 The tensor product of a Riemannian manifold with a finite noncommutative geometry.
fragments of a larger noncommutative geometry that does conform to all the axioms set out in [?]. In section 5, we make a comparison with the canonical quantisation approach taken in [?]. Section 6 contains a brief discussion on the quantisation of a Riemannian manifold, and we end with the conclusion in section 7.

Note: we work with an Euclidean signature, i.e. Riemannian means Riemannian not pseudo-Riemannian.

2 Path Integral Quantisation

We decided to work on a path integral approach rather than a canonical approach, because it requires knowledge of only the fields and not their dynamics. To be able to do canonical quantisation on a noncommutative geometry, we would need a general procedure for finding the phase space and constructing a symplectic structure on it. Conventionally, this amounts to finding the canonical momenta and using the Poisson bracket. In contrast, path integrals need a (symmetry invariant) measure on the space of histories. Deciding how to parameterise this space is thus an important consideration. The advantage lies in that this does not depend on the details of the action, unlike finding the phase space. The only things that really matter are the fields, because they determine the measure. One of the other benefits of using path integrals is that they are explicitly covariant.

A good starting point for developing a path integral formalism for noncommutative geometry is the conventional formalism. It has lead to standard model predictions that agree spectacularly with experiment, so it should be incorporated as a special case. Since the standard model action can be expressed as a spectral action on a noncommutative geometry, a dictionary can be set up between noncommutative geometry and field theory. This made it apparent that the fields parameterise the Dirac operator, so the space of histories of the fields is equivalent to the space of histories of the Dirac operator. From the noncommutative geometry point of view then, the degrees of freedom of the Dirac operator correspond to the fields in the spectral action, and hence give the path integration measure. Thus, in principle, we can path integral quantise a general spectral action. Schematically, the general partition function can be written as

\[ Z = \int \mathcal{D}D \ e^{-\text{Tr} f(D^2/\Lambda^2)}, \]

where \( D \) is the Dirac operator. The function \( f \) and parameter \( \Lambda \) are the cutoffs for the spectral action.

3 Two-Point Space and Higgs Gravity

The two-point space is the simplest example of a noncommutative geometry. It consists of just two points which we label \( L \) and \( R \). The spectral triple is given by

\[ A = \mathbb{C} \oplus \mathbb{C} = \left\{ f = \left( \begin{array}{cc} f_L & 0 \\ 0 & f_R \end{array} \right) \right\} \]
\[ \mathcal{H} = \mathbb{C} \oplus \mathbb{C} \]
\[ D = \frac{1}{\hbar} \left( \begin{array}{cc} 0 & m \\ \overline{m} & 0 \end{array} \right), \]

where \( m \) is a complex constant which fixes the distance between the two points.

Some may be unsettled by the appearance of \( \hbar \) in the Dirac operator before quantisation. It is used only to follow the convention that \( m \) has units of mass rather than inverse length, and so can be omitted. Alternatively, one could view \( \hbar \) as the noncommutative geometry version of \( c \). In the same way that \( c \) relates space and time on a Lorentzian manifold, \( \hbar \) relates space and (inverse) mass on a noncommutative geometry (“spacemass”). The spectral action is naturally dimensionless, so no \( \hbar \) is needed for quantisation. We, however, will take our actions to have the usual dimensions of \( \hbar \).

To move from a static (flat) space to a dynamic (curved) space, we promote the constant \( m \) to a variable \( \phi \), which will play the role of the gravitational field. This is the analogue of moving from \( \eta_{\mu\nu} \) to \( g_{\mu\nu}(x) \) on a
Lorentzian manifold. In fact, \( \phi \) is really a connection, so it plays the role of a vierbein/spin connection rather than a metric. In the context of the standard model, \( \phi \) is interpreted as the Higgs field, hence we refer to this as Higgs gravity.

The spectral action is taken to be
\[
S = \frac{1}{G} \text{Tr} \ D^2 = \frac{2 l_p^2}{\hbar} |\phi|^2,
\]
where \( G \) is the gravitational coupling and \( l_p = \frac{1}{\sqrt{8\pi}} \) is the Planck length. Varying the action, the equations of motion are
\[
\phi = 0, \quad \bar{\phi} = 0.
\]

Using Connes’ distance formula,
\[
d(x, y) = \sup_{f \in A} \left\{ |\langle x| f |x\rangle - \langle y| f |y\rangle| : \|D, f\| \leq 1 \right\},
\]
the distance between the two points is
\[
d(L, R) = \frac{\hbar}{|\phi|} = \frac{m_p}{|\phi|} l_p,
\]
where \( m_p \) is the Planck mass. So on-shell, the metric structure \( D \) vanishes and the distance is infinite.

Now we quantise by doing path integrals over \( \phi \) and \( \bar{\phi} \), the degrees of freedom of \( D \). The partition function is
\[
Z = \int dD \ e^{-S/\hbar} = \int d\phi \ d\bar{\phi} \ \exp \left( -\frac{2|\phi|^2}{m_p^2} \right).
\]

However, the action has a \( U(1) \) symmetry so we need to employ some gauge-fixing. The symmetries of the spectral action are related to the automorphisms of \( A \). We can use the gauge freedom to remove the complex phase from \( \phi \) and reduce it down to a real, positive field. The partition function then reads,
\[
Z = \int_0^\infty d\phi \ \exp \left( -\frac{2\phi^2}{m_p^2} \right) = \frac{\sqrt{2\pi}}{4} m_p.
\]

Expectation values are calculated in the usual fashion. For example,
\[
\langle \phi \rangle = \frac{1}{Z} \int_0^\infty d\phi \ \phi \ \exp \left( -\frac{2\phi^2}{m_p^2} \right) = \frac{m_p}{\sqrt{2\pi}}
\]
\[
\langle d(L, R) \rangle = \frac{1}{Z} \int_0^\infty d\phi \ \frac{m_p}{\phi} \ l_p \ \exp \left( -\frac{2\phi^2}{m_p^2} \right) = \infty.
\]

Here we see that in the vacuum state, the distance still maintains its classical value, but \( \phi \) has now acquired a v.e.v. Clearly, the classical distance relation (6) no longer holds.

In general,
\[
\int_0^\infty d\phi \ \phi^n \ \exp \left( -\frac{2\phi^2}{m_p^2} \right) = \frac{1}{2} \Gamma \left( \frac{n+1}{2} \right) \left( \frac{m_p}{\sqrt{2}} \right)^{n+1}.
\]
Thus, the Greens functions are
\[
\langle \phi^n \rangle = \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} \left( \frac{m_p}{\sqrt{2}} \right)^n.
\]
In particular, the propagator functions can be expressed as
\[
\langle (\phi \phi)^n \rangle = \frac{(2n)!}{n!} \left( \frac{m_p^2}{8} \right)^n
\]
for \( n \in \mathbb{Z} \). These reproduce the usual propagator combinatorics for a real scalar field.

We should expect \( \phi \) to be a real field, since there is no concept of a time dimension for a space consisting of just two points - a real field does not feel the arrow of time. In contrast, a complex field uses the arrow of time to distinguish between propagating particles and antiparticles (antiparticles being particles that travel backwards through time).

In an excited state, the distance \( d(L, R) \) is given by its expectation value in a background of propagators. So for the \( N \)th particle state,
\[
d_N \equiv \langle d(L, R) \rangle_N = \frac{1}{Z_N} \langle \phi^N d(L, R) \phi^N \rangle,
\]
where \( Z_N = \langle (\phi \phi)^N \rangle \). This evaluates to
\[
d_N = \frac{\sqrt{2} \Gamma(N)}{\Gamma(N + \frac{1}{2})} l_p,
\]
which can be rewritten as
\[
d_N = \lim_{\varepsilon \to 0} \prod_{n=1}^{N} \left( \frac{2n - 2 + \varepsilon}{2n - 1} \right) d_0(\varepsilon),
\]
where
\[
d_0(\varepsilon) = \sqrt{\frac{2}{\pi}} \frac{2l_p}{\varepsilon}
\]
is the v.e.v. (10) with the infinity regularised by \( \varepsilon \). The distance thus gets successively smaller as the number of gravitons (Higgs particles) is increased. Using the Stirling series, we find the distance shrinks to zero in the \( N \to \infty \) limit, and so the two points merge into one. The metric, \( D \), correspondingly becomes infinite since the description of the geometry as two points is no longer valid. This resembles the behaviour of a high curvature limit, i.e. gravitational collapse to a black hole.

The spectral action can be supplemented with the fermionic term
\[
S_F = \langle \bar{\psi}, D \psi \rangle = \bar{\psi}_L \phi \psi_R + \bar{\psi}_R \phi \psi_L.
\]
Note that this is purely an interaction term - the fermions are fixed at the points and do not propagate. Quantising as before, we write down the partition function,
\[
Z = \int d\phi \bar{d}\psi d\psi \exp \left( -\frac{2 \phi^2}{m_p^2} - \langle \bar{\psi}, D \psi \rangle \right).
\]
Remember that the Hilbert space is complex and not Grassmann, so
\[
Z = \int d\phi \det D \exp \left( -\frac{2 \phi^2}{m_p^2} \right) = -\int \frac{d\phi}{\phi^2} \exp \left( -\frac{2 \phi^2}{m_p^2} \right) = \infty.
\]
This makes the v.e.v.s \( \langle d(L, R) \rangle \) and \( \langle \phi \rangle \) ill-defined, and the propagator \( \langle \phi \phi \rangle \) will be zero. For the excited states \( (N \geq 1) \), the expectation values continue to be well-behaved. The effect of the fermions is to shield out the gravitational field by lowering the states by one. If we tensor product the two-point space with a Riemannian manifold, then a more conventional spinor Hilbert space is obtained. In this case, the fermions would enhance the gravitational field by raising the states by one.

Note, for a generic finite noncommutative geometry, the fermion contribution will be \( (\det D)^{-k} \) where \( k \) is the number of fermion generations fixed by the Hilbert space.
Next we look at the quantisation of the simplest matrix geometry, $M_2(\mathbb{C})$. Its spectral triple is

$$A = M_2(\mathbb{C}) = \left\{ f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \right\}$$

$$H = M_2(\mathbb{C})$$

$$D = \frac{1}{\hbar}\begin{pmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{pmatrix},$$

where $D$ is an SU(2) gauge field with $A_1$ real and $A_2$ complex. The spectral action evaluates to

$$S = \frac{1}{G} \text{Tr} D^2 = \frac{2l_p^2}{\hbar} (A_1^2 + |A_2|^2),$$

which is invariant under SU(2) gauge transformations. The gauge transformations are the analogue of the diffeomorphisms of conventional general relativity.

As before, we will need to gauge-fix the action in order to perform the path integral quantisation. The SU(2) gauge freedom allows us to reduce $D$ down to the form

$$D = \frac{1}{\hbar}\begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix},$$

where $\phi = \sqrt{A_1^2 + |A_2|^2}$. This will give us the same gauge-fixed action as for the two-point space, therefore the quantisation will also be the same. Note that $\phi$ is a gauge invariant quantity, as it must be, because gauge invariant observables calculated with a gauge-fixed $D$ should be gauge invariant.

Although the path integrals will be essentially identical to those for the two-point space, the interpretation of the observables may not be. For instance, the distance between the points (pure states) will be different from (6). Evaluating the condition $\| [D, f] \| \leq 1$ in the distance formula (5) we find

$$\frac{\hbar}{\phi} \geq \begin{cases} \frac{|(f_1 - f_4) - (f_2 - f_3)|}{|(f_1 - f_4) - (f_3 - f_2)|} & \text{depending on which is larger} \end{cases}$$

Thus, the distance is

$$d(1, 4) = \sup_{f \in A} \{|f_1 - f_4| : \| [D, f] \| \leq 1 \} = \infty.$$  

Unlike the two-point space, the observable $\hbar/\phi$ does not correspond to a distance.

The fermion action is

$$S_F = \text{Tr} \Psi^\dagger D \Psi = \bar{\psi}_1 \phi \psi_3 + \bar{\psi}_3 \phi \psi_1 + \bar{\psi}_2 \phi \psi_4 + \bar{\psi}_4 \phi \psi_2.$$  

It contains twice as many fermions as (18) due to the larger Hilbert space. So, its contribution to the partition function will be $(\det D)^{-2} = \phi^{-4}$. This will have the effect of lowering the states by two.

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For the record, the transformation $D \rightarrow U D U^{-1}$ is given by

$$U = \frac{1}{2 \sqrt{\phi (\phi - A_1)}} \begin{pmatrix} \phi - A_1 + A_2 & \phi - A_1 - A_2 \\ -(\phi - A_1 - A_2) & \phi - A_1 + A_2 \end{pmatrix}.$$
5 Comparison with Rovelli’s Canonical Quantisation

We tried to compare our path integral approach with Rovelli’s canonical approach (see [? for details), but found problems with his model example. Rovelli modified the spectral action in an effort to obtain non-trivial equations of motion. After careful examination, we found this actually had\(^3\) the opposite effect. The action in question is

\[ S = \frac{1}{2} \text{Tr} \ D \tilde{M} D \]

But this can be factorised as

\[ = \frac{1}{2G} (\overline{m}_1 + e^{-i\phi} m_2) (m_1 + e^{i\phi} m_2) \]

\[ = \frac{|m|^2}{2G}, \tag{28} \]

where \( m = m_1 + e^{-i\phi} m_2 \). Thus, we end up with a much simpler action and set of equations of motion. Which incidentally, are the same as the ones we have considered. Canonical quantisation in these variables is a very different problem from the one considered by Rovelli.

Physically, the interaction terms in (27) allow the particles \( m_1 \) and \( m_2 \) to spontaneously change into one another. This is like a mixing term, so \( m_1 \) and \( m_2 \) will not make good eigenstates. As we have seen in (28), the linear combination given by \( m \) will make a good eigenstate.

Although the action (27) is not spectral \( \textit{per se} \), we can in fact still quantise it with our path integral approach. We begin by rewriting the action in terms of an effective Dirac operator, \( D' \), so it is spectral:

\[ S = \frac{1}{2} \text{Tr} \ D \tilde{M} D = \frac{1}{2} \text{Tr} \ D' P \dagger P D = \frac{1}{2G} \text{Tr} \ D'^\dagger D'. \tag{29} \]

Solving \( P \dagger P = \tilde{M} \) gives

\[ P = \frac{1}{\sqrt{2G}} \begin{pmatrix} 1 & e^{-i\phi} & 0 \\ 1 & e^{i\phi} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{30} \]

thus

\[ D' = \sqrt{G} P D = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & m \\ 0 & 0 & m \\ 0 & 0 & 0 \end{pmatrix}. \tag{31} \]

Further, a self-adjoint \( D'' \) can be constructed by \( D'^\dagger D' = \left( \frac{D'^\dagger + D'}{\sqrt{2}} \right)^2 = D''^2 \), as \( D' \) is nilpotent. The degrees of freedom of \( D'' \) are \( m \) and \( \overline{m} \), just as we have proposed. After gauge-fixing, we end up with path integrals equivalent to those for the two-point space.

The problem with trying to canonically quantise finite noncommutative geometry spectral actions is that they have no phase space as such. This could be taken to mean that they simply cannot be quantised, but we have shown otherwise using path integrals. Perhaps some noncommutative generalisation of phase space is needed (like tangent groupoids, see [?, Sec. 6]), or maybe the path integral approach is just more fundamental. We could try to reverse-engineer our path integral approach, and find the canonical equivalent. For example, the quantised Fourier mode expansion of \( \phi \) could be taken to be \( \phi = \frac{1}{\sqrt{2}} (\hat{a}^\dagger + \hat{a}) \), where \( \hat{a} \) and \( \hat{a}^\dagger \) are the usual creation and annihilation operators. This would mean the canonical momentum operator would be \( \hat{\pi} = \frac{i}{\sqrt{2}} (\hat{a}^\dagger - \hat{a}) \). So, what does the classical quantity \( \pi \) correspond to? We leave the further exploration of these ideas for another time.

\(^3\)Somewhat ironically.
Also in [?], Rovelli put forward a proposal for a path integral approach. It was suggested that the integration measure should be given by the eigenvalues of the Dirac operator. This leads to path integrals which we term **spectral integrals**. It turns out that our gauge-fixed path integrals correspond exactly to spectral integrals. The eigenvalues give of course a natural gauge invariant measure. However, whether the two approaches remain equivalent for general noncommutative geometries (e.g. Riemannian manifolds) is not clear, at least not to the author.

### 6 Riemannian Manifolds

We now move on to outline how our approach might work for less trivial noncommutative geometries, particularly Riemannian manifolds. The Dirac operator for a Riemannian manifold is

\[
D = \gamma^\alpha e^{\alpha}_{a}(x) \left( \frac{\partial}{\partial x^\mu} + \frac{1}{4} \omega_{bc\mu}(x) \gamma^b \gamma^c \right),
\]

where \(e^\alpha_{a}\) is the vierbein and \(\omega^{ab}_{\mu}\) is the spin connection. Computing the spectral action for it yields the Einstein-Hilbert action (ignoring higher order terms) [?].

Usually, the metric, \(g_{\mu\nu}\), is considered as the dynamical field and hence gives the measure for path integrals. In our approach, the vierbein and spin connection would be used instead, these being the degrees of freedom of the Dirac operator. This resembles the conventional connection-based way of quantising Yang-Mills theories. So, one might hope that this would make things more tractable.

We can go further. Let us now use a Dirac operator with a self-dual spin connection \(A_{ab}^\mu\). Since we work in an Euclidean signature, \(A_{ab}^\mu\) is real as \(A_{ab}^\mu = \frac{1}{2} \epsilon_{abcd} A_{cd}^\mu\). It is reasonable to assume that the spectral action will be the Einstein-Hilbert action, but with a self-dual curvature. This is the Ashtekar formulation of general relativity. Thus, canonical quantisation will take us down the path that leads to loop quantum gravity [?]. While, path integral quantisation makes contact with the spin foam approach [?] (the sum over spin foams is a discretised version of the path integral).

### 7 Conclusion

We have developed a path integral approach for quantising a general spectral action. It has been used successfully on two finite noncommutative geometries. We have found that the graviton on these geometries behaves like a real scalar field. On the two-point space, the effect of graviton excitations is to shrink the distance between the two points. In the extreme, the two-point space collapses to a single point ("a black point"). The matrix geometry did not exhibit this behaviour, as the distances were naturally infinite. Introducing fermions on to the geometries had the effect of shielding out the gravitational field. All the graviton states were lowered by an amount equal to the number of fermion generations.

Comparing our approach with [?] led us to question the validity of their results. We found that their equations of motion could be expressed in much simpler terms, which result in a smaller phase space. This will modify the canonical quantisation. Despite this, both approaches seem to support the qualitative result that distances shrink with increasing graviton excitations. The idea of spectral integrals is very appealing, since it is consistent with the philosophy of spectral invariance. We know that our path integral approach, by construction, coincides with the conventional one, so it would be interesting to see in what ways (if any) spectral integration differs from this.

The next step, to obtaining a better understanding of the generic features of quantisation on a noncommutative geometry, would be to investigate some more substantial examples than the ones we have considered here. For example, the spectral triple associated with the finite part of the standard model algebra, i.e. \(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})\). It would also be worth examining whether the connection we suggested between the spectral action and the Ashtekar action can be made more precise. More speculatively, we have an idea for another way of quantising
noncommutative geometries based on topological quantum field theory. A topological quantum field theory is a functor from the category of \((n - 1)\)-dimensional manifolds, \(\mathcal{Cob}\), to the category of Hilbert spaces, \(\mathbb{Hilb}\). Using the Gelfand-Naimark cofunctor, it might be possible to construct a functor from the category of \(C^*\)-algebras to \(\mathbb{Hilb}\). This would give us a quantisation for spectral triples: a classical triple \((A, \mathcal{H}, D)\) would become the quantum triple \((A, \mathcal{H} \otimes K, D)\), where \(K\) is the quantum Hilbert space. The main drawback of topological quantum field theories is that they have no local degrees of freedom. We could think about introducing local degrees of freedom by extending our functor to the Dirac operator. So, quantising \((A, \mathcal{H}, D)\) yields \((A, \mathcal{H} \otimes K, \hat{D})\), this looks remarkably similar to the quantised triples consider in \([?]\)! We hope to pursue these ideas in the future.

Noncommutative geometry has introduced a new twist in the search for a theory of quantum gravity. The biggest problem we face may not be one of quantisation, but one of finding the right geometry to quantise.

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