The chiral WZNW phase space
as a quasi-Poisson space

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Abstract

It is explained that the chiral WZNW phase space is a quasi-Poisson space with respect to the ‘canonical’ Lie quasi-bialgebra which is the classical limit of Drinfeld’s quasi-Hopf deformation of the universal enveloping algebra. This exemplifies the notion of quasi-Poisson-Lie symmetry introduced recently by Alekseev and Kosmann-Schwarzbach.

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1 Introduction

The WZNW model [1] is perhaps the most important model of 2-dimensional conformal field theory due to its applications in string theory and to the fact that many other interesting models can be understood as its reductions. The reduction procedures, such as Hamiltonian reduction and the Goddard-Kent-Olive coset construction, rely on the affine Kac-Moody symmetry of the model. Besides this linear symmetry, the WZNW model also provides an arena in which non-linear, quantum group symmetries appear [2].

One of the quantum group aspects of the model is that the fusion rules of the WZNW primary fields reflect [3] certain truncated Clebsch-Gordan series for tensor products of representations of a Drinfeld-Jimbo quantum group $U_q(G)$. Another very interesting result [4] is that the monodromy representations of the braid group that arise on the solutions of the Knizhnik-Zamolodchikov (KZ) equation, which governs the chiral WZNW conformal blocks, are equivalent to corresponding representations generated by the universal R-matrix of $U_q(G)$. This aspect of the connection between quantum groups and WZNW models is most elegantly summarized by Drinfeld’s construction of a canonical quasi-Hopf deformation $A_h(G)\otimes G$ of the universal enveloping algebra $U(G)$ that encodes the monodromy properties of the KZ equation [5, 6]. As a coalgebra $A_h(G) = U(G)[[h]]$, it has a quasi-triangular structure given by $R_h = \exp(h\Omega)$, where $\Omega$ is the appropriately normalized tensor Casimir of $G$, and a coassociator $\Phi \in A_h(G) \otimes A_h(G) \otimes A_h(G)$ defined by means of the KZ equation. An advantage of the category of quasi-Hopf algebras is that it admits a notion of twisting, under which many different looking objects could be equivalent. In particular, for generic $q U_q(G)$ is twist equivalent to $A_h(G)$, which in many ways is a simpler object to consider. For a review, see e.g. [7].

The classical limits of the Hopf-algebraic formal deformations of $U(G)$ are the Lie bialgebra structures on $G$, which integrate to Poisson-Lie groups, while in the quasi-Hopf case the analogous objects are Lie quasi-bialgebras [6] and quasi-Poisson-Lie groups [8]. There have been many studies devoted to the description of the possible Poisson-Lie symmetries of the so called chiral WZNW phase space given by

$$\mathcal{M}_{\text{chir}} := \{ g \in C^\infty(\mathbb{R}, G) \mid g(x + 2\pi) = g(x)M, \quad M \in G \}, \quad (1)$$

where $G$ is a connected Lie group corresponding to $G$. The upshot (see e.g. [9, 10]) is that such a symmetry can be defined globally on $\mathcal{M}_{\text{chir}}$ whenever an appropriately normalized solution $\hat{r} \in G \wedge G$ of the modified classical Yang-Baxter equation is available, which is used to define the Poisson-Lie structure on $G$ and appears also in the definition of the appropriate Poisson structure on $\mathcal{M}_{\text{chir}}$. In fact, the Poisson structure on $\mathcal{M}_{\text{chir}}$ can be encoded by a local formula of the following form:

$$\left\{ g(x) \otimes g(y) \right\} = \frac{1}{\kappa} \left( g(x) \otimes g(y) \right) \left( \hat{r} + \Omega \text{sign} (y - x) \right), \quad 0 < x, y < 2\pi. \quad (2)$$

Here $\Omega = \frac{1}{2} \sum_i T_i \otimes T^i$, where $T_i$ and $T^i$ denote dual bases of $G$ with respect to a symmetric, invariant, non-degenerate bilinear form $(\cdot, \cdot)$ on $G$, $(T_i|T^j) = \delta^j_i$, and $\kappa$ is a constant. In terms of the usual tensorial notation, the Jacobi identity of (2) requires that

$$[\hat{r}_{12}, \hat{r}_{23}] + [\hat{r}_{12}, \hat{r}_{13}] + [\hat{r}_{13}, \hat{r}_{23}] = -[\Omega_{12}, \Omega_{13}]. \quad (3)$$
It is worth noting that for a compact simple Lie algebra such an ‘exchange r-matrix’ does not exist, because of the negative sign on the right hand side of (3).

On the basis of the above mentioned results (see also [11]), it is natural to expect that the classical analogue of the quasi-Hopf algebra \( \mathcal{A}_h(\mathcal{G}) \) can also be made to act as a symmetry on \( \mathcal{M}_{\text{chir}} \), but as far as we know this question has not been investigated so far. Apparently, the notion of a classical symmetry action of a quasi-Poisson-Lie group itself has only been defined very recently [12, 13]. The purpose of this letter is to point out that the expectation just alluded to is indeed correct, since the group \( \mathcal{G} \) equipped with the so called canonical quasi-Poisson-Lie structure, which is the classical limit of \( \mathcal{A}_h(\mathcal{G}) \), acts as a symmetry in the sense of [12] on the chiral WZNW quasi-Poisson space. We shall see that the simplest quasi-Poisson structure on \( \mathcal{M}_{\text{chir}} \) for which this holds can still be defined by (2) but with \( \hat{r} = 0 \). This symmetry is thus available for any Lie algebra (compact or not) with a invariant scalar product, since it relies only on the Casimir \( \Omega \).

In the next section we briefly recall the necessary notions and then deal with the quasi-Poisson structure of \( \mathcal{M}_{\text{chir}} \) in sections 3 and 4.

### 2 The notion of a quasi-Poisson space

We first recall from [6, 12] that a Manin quasi-triple is a Lie algebra \( D \) with a non-degenerate, invariant scalar product \( \langle \cdot | \cdot \rangle \) and decomposition \( D = \mathcal{G} \oplus \mathcal{H} \), where \( \mathcal{G} \) and \( \mathcal{H} \) are maximal isotropic subspaces and \( \mathcal{G} \) is also a Lie subalgebra in \( D \). \( (D, \mathcal{G}, \mathcal{H}) \) becomes a Manin triple if \( \mathcal{H} \) is a Lie subalgebra, too. Choosing a basis \( \{ e_i \} \ i = 1, 2, \ldots, n \) in \( \mathcal{G} \) and corresponding dual basis \( \{ \varepsilon^j \} \ j = 1, 2, \ldots, n \) in \( \mathcal{H} \) (satisfying \( \langle e_i | \varepsilon^j \rangle = \delta^j_i \)) the commutation relations can be characterized in terms of the structure constants \( f_{ij}^k, F_{ij}^k \) and \( \varphi^{ijk} \) as follows:

\[
\begin{align*}
[e_i, e_j] &= f_{ij}^k e_k, \quad (4) \\
[e_i, \varepsilon^j] &= f_{ki}^j \varepsilon^k + F_{ij}^k \varepsilon_k, \quad (5) \\
[\varepsilon^i, \varepsilon^j] &= F_{ij}^{\quad k} e_k + \varphi^{ijk} e_k. \quad (6)
\end{align*}
\]

Here \( f_{ij}^k \) are the structure constants of the Lie algebra \( \mathcal{G} \), \( F_{ij}^k \) is antisymmetric in \( j, k \), \( \varphi^{ijk} \) is totally antisymmetric; and the usual summation convention is in force. It is easy to write down the quadratic equations in terms of the structure constants that express the Jacobi identities of the commutation relations (4-6). One of the equations (consisting of terms of the form \( fF \)) can be interpreted as requiring the \( \mathcal{G} \to \mathcal{G} \wedge \mathcal{G} \) mapping \( \hat{F} \) defined by \( \hat{F}(e_i) = F_{ij}^k e_j \otimes e_k \) to be a 1-cocycle. Further there are equations of \( F\varphi \) and also of \( FF + f\varphi \) type.

The triple \( (\mathcal{G}, \hat{F}, \hat{\varphi}) \) is called a Lie quasi-bialgebra, where \( \hat{\varphi} = \varphi^{ijk} e_i \otimes e_j \otimes e_k \) is here interpreted as a special element in \( \wedge^3(\mathcal{G}) \). It follows from (6) that \( (\mathcal{G}, \hat{F}, \hat{\varphi}) \) becomes a Lie bialgebra if \( \hat{\varphi} = 0 \). Note that \( \hat{F} \) and \( \hat{\varphi} \) correspond to the classical limits of the coproduct and the coassociator of a quasi-Hopf deformation of \( U(\mathcal{G}) \).

A simple and important solution of the Jacobi constraints is given by \( \hat{F} = 0 \) and \( \hat{\varphi} \) invariant with respect to the adjoint action of \( \mathcal{G} \). Below we shall use the following ‘canonical’ realization
of this special case. We consider a (real or complex) simple Lie algebra \( G \) and use the invariant metric \( \gamma_{ij} = (e_i | e_j) \) (where \( ( | ) \) denotes a fixed invariant scalar product on \( G \)) to raise and lower Lie algebra indices. \( D \) is given by \( G \oplus G \) and the scalar product is defined by

\[
\langle (a,b) | (c,d) \rangle = k \left[ (a|c) - (b|d) \right],
\]

where \( k \) is an arbitrary constant, which is real in the case of a real Lie algebra. Now in terms of a basis \( \{ T_i \} \ i = 1,2,\ldots,n \) in \( G \) the elements of the dual \( D \) basis are written as

\[
e_i = (T_i, T_i), \quad \varepsilon^j = \frac{1}{2k} (T^j, -T^j).
\]

In this case \( \hat{\varphi} \) is given by

\[
\varphi^{ijk} = C f^{ijk},
\]

where \( C = \frac{1}{4k^2} \). Note that while the Jacobi identities for the commutation relations (4-6) are algebraically satisfied with \( \hat{F} = 0 \) and \( \hat{\varphi} \) given by (9) with any \( C \), for a real Lie algebra \( G \) the canonical realization defined above always gives \( C > 0 \).

Manin quasi-triples and Lie quasi-bialgebras can be deformed by the following transformation called ‘twist’. In terms of the dual basis the twisted quasi-triple is defined by \( \tilde{e}_i = e_i \) and

\[
\tilde{\varepsilon}^i = \varepsilon^i + t^j e_j,
\]

where \( t^{ij} \) is an arbitrary antisymmetric matrix. The twisted structure constants \( f^{ijk} \), \( \tilde{F}^{ijk} \) and \( \tilde{\varphi}^{ijk} \) are easily obtained by substituting (10) into (5,6). In particular, the 1-cocycle transforms as

\[
\hat{\tilde{F}}(X) = \hat{\tilde{F}}(X) + [X \otimes 1 + 1 \otimes X, \hat{t}], \quad \forall X \in G,
\]

where \( \hat{t} = t^j e_i \otimes e_j \).

The importance of twisting is due to the fact that the canonical Lie quasi-bialgebra defined by the above example can be twist transformed into a Lie bialgebra. In this case the 1-cocycle is a 1-coboundary,

\[
\hat{\tilde{F}}(X) = [X \otimes 1 + 1 \otimes X, \hat{t}],
\]

and the requirement \( \hat{\tilde{\varphi}} = 0 \) is equivalent to the modified classical Yang-Baxter equation

\[
[[\hat{t}, \hat{t}]] + C \hat{f} = 0,
\]

where \( [[\hat{t}, \hat{t}]] = [\hat{t}_{12}, \hat{t}_{23}] + [\hat{t}_{13}, \hat{t}_{23}] + [\hat{t}_{12}, \hat{t}_{13}] \) and \( \hat{f} = f^{ijk} e_i \otimes e_j \otimes e_k \). It is well-known that for a real compact Lie algebra (13) has solutions for \( C < 0 \) only. Thus the twisting from the canonical Lie quasi-bialgebra to a Lie bialgebra is possible if \( G \) is a real simple algebra for non-compact real forms only. In the complex case it is always possible.

The global objects corresponding to Lie quasi-bialgebras \((G, \hat{F}, \hat{\varphi})\) are the quasi-Poisson-Lie groups \([8, 12]\). Such a Lie group with Lie algebra \( G \) is equipped with a multiplicative bivector that corresponds to \( \hat{F} \) and satisfies certain conditions involving \( \hat{\varphi} \). The precise definition is not needed in this letter.
The final notion to be recalled is that of a quasi-Poisson space [12]. This is a manifold $\mathcal{M}$ which is equipped with a bivector $P_M$ and on which a (left) action of a connected quasi-Poisson-Lie group $G$ is given subject to some conditions. These conditions are specified below in terms of the corresponding Lie quasi-bialgebra. To describe them, note that on any manifold a bivector can be encoded by the associated ‘quasi-Poisson bracket’ $\{ , \}$. The quasi-Poisson bracket is associated with the bivector in the same way as the Poisson bracket on a Poisson manifold is associated with the Poisson bivector [14]. Thus $\{ , \}$ is a bilinear, antisymmetric derivation on the functions on $\mathcal{M}$, whose properties differ from the usual Poisson bracket only in that it does not necessarily satisfy the Jacobi identity. Using this language a quasi-Poisson space is required to satisfy

$$X_i \{ f, g \} - \{ X_i f, g \} - \{ f, X_i g \} = -F_{ij}^k \left( X_j f \right) \left( X_k g \right)$$

and

$$\{ \{ f, g \}, h \} + \{ \{ h, f \}, g \} + \{ \{ g, h \}, f \} = -\varphi^{ijk} \left( X_i f \right) \left( X_j g \right) \left( X_k h \right).$$

Here $X_i$ are the infinitesimal generators of the left action of the (quasi-Poisson-Lie) group $G$ on functions on $\mathcal{M}$ satisfying $[X_i, X_j] = t_{ij}^k X_k$.

It is easy to prove that if $\mathcal{M}$ equipped with the quasi-Poisson bracket $\{ , \}$ is a quasi-Poisson space in the above sense with respect to some Lie quasi-bialgebra then it is also a quasi-Poisson space with respect to the twisted Lie quasi-bialgebra using the twisted bracket

$$\{ \tilde{f}, g \} = \{ f, g \} - t_{ij}^k \left( X_i f \right) \left( X_j g \right).$$

Note that the requirements (14) and (15) ensure that the quasi-Poisson bracket can be restricted to the space of invariant functions where it becomes a genuine Poisson bracket, which is also invariant under twisting [12].

The above conditions become the usual conditions of a Poisson-Lie action on a Poisson manifold if $\varphi = 0$. Another simplification occurs in the case of $F = 0$, that is e.g. for the canonical structure, since in this case (14) simply means that the bivector $P_M$ must be invariant under the action of $G$.

3 $\mathcal{M}_{\text{chir}}$ as a quasi-Poisson space

After this short review we turn to the study of our example, the chiral WZNW phase space $\mathcal{M}_{\text{chir}}$. We here show that this phase space can be equipped with a very simple and natural quasi-Poisson bracket which makes it a quasi-Poisson space with respect to the canonical Lie quasi-bialgebra. In this example $G$ has to be identified with the WZNW group and the corresponding (left) action on functions of $g(x)$ is infinitesimally generated by

$$X_i g(x) = g(x)T_i.$$  \hspace{1cm} (17)

It is natural to equip $\mathcal{M}_{\text{chir}}$ with such a bivector which guarantees that the components of the current $J_i(x) = \kappa(g'(x)g^{-1}(x)|T_i)$ generate the affine Kac-Moody algebra and $g(y)$ is a primary field of this algebra:

$$\{ J_i(x), g(y) \} = -T_i g(x) \delta(x - y).$$  \hspace{1cm} (18)
The motivation for this second property is that it ensures the analogous property of the full $G$-valued WZNW field of which we are here considering the chiral part. Now the simplest Poisson bracket on $\mathcal{M}_{\text{chir}}$ consistent with the requirements that the classical KM algebra relations together with the primary field nature of the chiral WZNW field are reproduced is given by (2). Analogously, the simplest quasi-Poisson bracket still consistent with the above two requirements is

$$\{g(x) \otimes g(y)\} = \frac{1}{\kappa} \left( g(x) \otimes g(y) \right) \Omega \text{sign} (y - x), \quad 0 < x, y < 2\pi,$$

(19)

which is simply the exchange relation (2) with $\hat{r} = 0$. Similarly to the Poisson brackets [10], the quasi-Poisson brackets are only defined for a class of ‘admissible’ functions, which are typically smeared out functions of the chiral WZNW field $g(x)$ and the relation (19) has to be interpreted in a distributional sense.

Now it is very easy to see that the equivariance requirement (14) (with $\hat{F} = 0$) is satisfied by the bracket (19) so we are left with (15). Since (19) is understood in a distributional sense, we keep the arguments $x, y$ and $z$ strictly different from each other and then find

$$\{\{g(x) \otimes g(y)\} \otimes g(z)\} + \text{cycl. perm.} = -\frac{1}{4\kappa^2} g(x) \otimes g(y) \otimes g(z) \hat{f}.$$

(20)

By taking into account (9) and (17), we conclude by comparing (15) with (20) that $\mathcal{M}_{\text{chir}}$ with the bracket (19) is a quasi-Poisson space with respect to the canonical Lie quasi-bialgebra if the constant $C$ in (9) is chosen as $C = \frac{1}{4\kappa^2}$. We make this choice by identifying the Lie quasi-bialgebra parameter $k$ in (7) with the classical KM level parameter $\kappa$.

As we have mentioned, a canonical quasi-Poisson structure may be twisted to a genuine Poisson structure. In our example the twist that transforms (19) into (2) is given by

$$\hat{t} = -\frac{1}{\kappa} \hat{r}.$$

(21)

Recalling (13) and using $C = 1/4\kappa^2$ we notice that this transformation is possible if $\hat{r}$ satisfies the modified classical YB equation $[[\hat{r}, \hat{r}]] = -(1/4) \hat{f}$. Thus we see once more that $\mathcal{M}_{\text{chir}}$ can be equipped with a genuine Poisson structure of the form of (2) for complex or non-compact real groups only. On the other hand, the quasi-Poisson structure (19) is perfectly well-defined for compact groups as well.

It is interesting to note that (19) is not the only possible choice for a quasi-Poisson structure on $\mathcal{M}_{\text{chir}}$. Another solution is defined as follows. The quasi-Poisson brackets are still generated by the formula (2) but the constant exchange $r$-matrix $\hat{r}$ is replaced by a ‘dynamical’ $r$-matrix $\hat{r}_{\text{dyn}}(M)$ depending on the dynamical variables through the monodromy matrix $M$. The $M$-dependence is given by the formula

$$r_{\text{dyn}}(M) = -\frac{1}{2} \tanh \left( \frac{\mathcal{Y}}{2} \right),$$

(22)

where $\mathcal{Y} = 2\pi \text{ad} \Gamma$ using an exponential parametrization of the monodromy matrix, $M = e^{2\pi \Gamma}$. In (22) $r_{\text{dyn}}(M)$ has to be interpreted as a linear operator on $\mathcal{G}$ corresponding to $\hat{r}_{\text{dyn}}(M)$ by means of the natural identification.
$\mathcal{M}_{\text{chir}}$ equipped with this ‘dynamical’ quasi-Poisson structure is also a quasi-Poisson space with respect to the canonical Lie quasi-bialgebra, with $C = 1/\kappa^2$, at least on that subspace of the chiral WZNW phase space on which the exponential parametrization of $M$ is valid. There it can also be twisted to a genuine Poisson structure by adding a constant r-matrix, but now the constant r-matrix defining the twist (21) has to be normalized according to $[\hat{r}, \hat{r}] = -\hat{f}$. The resulting twisted dynamical r-matrix,

$$r + r_{\text{dy}}(M) = r - \frac{1}{2} \tanh \left( \frac{Y}{2} \right) = r + \frac{1}{2} \coth \left( \frac{Y}{2} \right) - \coth (Y),$$

is the $\nu = 1$ member of the family of dynamical r-matrices constructed in (section 5 of) [10], which were shown to satisfy a generalization of the modified classical YB equation and were interpreted in terms of certain Poisson-Lie groupoids.

4 The monodromy matrix as momentum map

Finally we would like to point out that the quasi-Poisson structure (19) on $\mathcal{M}_{\text{chir}}$ admits a group valued momentum map, which is provided by the monodromy matrix of $g(x) \in \mathcal{M}_{\text{chir}}$, as might be expected. We first recall the definition of the generalized momentum map introduced in [12]. For this we need to consider the connected Lie groups $D$ and $G \subset D$ integrating $\mathcal{D}$ and $\mathcal{G}$ respectively as well as the coset space $S = D/G$. The action of $D$ on itself by left multiplications induces an action of $D$ on the coset space $S$. We denote the infinitesimal generators of this left action on functions on $S$ by $X_i$ and $Y^j$ corresponding to $e_i$ and $\varepsilon^j$ respectively. In the ‘canonical example’ of our interest $\mathcal{D} = \mathcal{G} \oplus \mathcal{G}$, and thus we can represent $D = G \times G$ as $D = \{(g_1, g_2)\}$ with $g_1, g_2 \in G$ and the elements of the form $(g, g)$ give the subgroup $G$ embedded diagonally. Then the coset space can be represented by $\{\sigma = g_1 g_2^{-1}\}$, and hence $S$ in this case is identified with the group $G$ itself. As a consequence of (8), the infinitesimal generators of the $D$-action on $S \simeq G$ are explicitly given in the canonical case by

$$X_i \sigma = -[T_i, \sigma], \quad Y^j \sigma = -\frac{1}{2k} [T^j, \sigma]_+ .$$

Here $T_i$ and $\sigma$ are matrices in some linear representation corresponding to the Lie algebra generators and the group element respectively and $[ , , ]_+$ means matrix anticommutator.

The momentum map $\mu : \mathcal{M} \to S$ introduced in [12] is required to satisfy the following two conditions. First, it must be equivariant with respect to the corresponding $G$-actions, which means that

$$(X_i \Psi) \circ \mu = X_i (\Psi \circ \mu)$$

for any function $\Psi$ on $S$. The second condition may be formulated as the equality

$$\{ \Psi \circ \mu, f \} = (X_i f) (Y^i \Psi) \circ \mu,$$

where $\Psi$ is an arbitrary function on $S$ and $f$ an arbitrary function on $\mathcal{M}$. More precisely, one also requires a non-degeneracy condition, which ensures that (26) can be solved for $X_i f$ in
terms of the quasi-Poisson brackets of $f$ with the coordinate functions of $\mu$; so that $\mu$ generates the vector fields $\overline{X}_i$ on $M$ through $\{ \cdot, \cdot \}$. For the precise details the reader may consult [12]. There it is also shown that the momentum map $\mu$ is always a bivector map, i.e. it satisfies

$$\{ \Psi_1 \circ \mu, \Psi_2 \circ \mu \} = \{ \Psi_1, \Psi_2 \}_S \circ \mu,$$

where $\{ \cdot, \cdot \}_S$ is the quasi-Poisson bracket on the coset space $S$, which is a quasi-Poisson space itself [12]:

$$\{ \Psi_1, \Psi_2 \}_S = (Y^i \Psi_1)(X_i \Psi_2).$$

In the canonical example, we can rewrite (25) and (26) more explicitly as

$$-[T_i, \mu] = \overline{X}_i \mu$$

and

$$\{ \mu, f \} = -\frac{1}{2k} (\overline{X}_i f) [T^i, \mu]_+,$$

where we use some matrix representation of the elements of $S \simeq G$ like in (24).

Now our point is that for the quasi-Poisson structure (19) on $\mathcal{M}_{\text{chir}}$ the momentum map is given by

$$\mu : \mathcal{M}_{\text{chir}} \ni g(x) \mapsto M = g^{-1}(x) g(x + 2\pi) \in G \simeq S.$$

In fact, (17) implies immediately that $\mu = M$ satisfies (29). Furthermore, one easily obtains from (19) the relation

$$\{ M \otimes g(x) \} = -\frac{1}{2k} [T^i, M]_+ \otimes g(x) T_i,$$

which is nothing but (30) in our case after the identification $k = \kappa$. We conclude that the monodromy matrix plays the role of the group valued momentum map with respect to the quasi-Poisson structure (19), similarly to its well-known [9] role in the Poisson-Lie context for the Poisson bracket given by (2). Incidentally, the non-degeneracy condition mentioned above is satisfied upon restriction to a domain of $\mathcal{M}_{\text{chir}}$, where $M$ is near enough to the unit element.

For definiteness, so far we have taken $G$ to be a simple Lie algebra since the WZNW model is usually studied in this case thanks to the relationship with the affine Kac-Moody algebras. However, as a classical field theory the model is equally well-defined for any finite dimensional Lie group $G$ whose Lie algebra carries an invariant scalar product. It is clear that $\mathcal{M}_{\text{chir}}$ with (19) is a quasi-Poisson space in the same manner in all these examples as well.

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