Nucleon form factors in the canonically quantized
Skyrme model

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The explicit expressions for the electric, magnetic, axial and induced pseudoscalar form factors of the nucleons are derived in the \textit{ab initio} quantized Skyrme model. The canonical quantization procedure ensures the existence of stable soliton solutions with good quantum numbers. The form factors are derived for representations of arbitrary dimension of the SU(2) group. After fixing the two parameters of the model, $f_\pi$ and $e$, by the empirical mass and electric mean square radius of the proton, the calculated electric and magnetic form factors are fairly close to the empirical ones, whereas the axial and induced pseudoscalar form factors fall off too slowly with momentum transfer.

\textit{Key words:} Nucleon form factors, Skyrme model


1 Introduction

Quantum chromodynamics (QCD) describes the nucleon as a composite system with many internal degrees of freedom. In the nonperturbative region, which encompasses hadron structure and intermediate range observables, the large $N_c$ limit of the theory, which partly allows treatment in closed form, has proven to be of phenomenological utility. This limit of QCD may be realized either in terms of the constituent quark model, or, as was first suggested, in the form of effective meson field theory, in which the baryons appear as topologically stable soliton solutions [1].

The generic chiral topological soliton model with topologically stable solutions, which represent baryons is that of T.H.R. Skyrme [2]. The first comprehensive phenomenological application of the model to nucleon structure was the
semiclassical calculation of the static properties of the nucleon in ref. [3]. In terms of agreement with the static baryon observables the results fell short by typically $\sim 30\%$, which could be viewed as an expected flaw of the large $N_C$ limit. That approach to the model did however have the more principal imperfection in that its lack of stable semiclassical solutions with good quantum numbers. The quantization was therefore realised by means of rigid body rotation of the classical skyrmion solutions. A strictly canonical quantization of Skyrme model was derived later in ref.[4] and shown to yield stable quantized skyrmion solutions in refs.[5–7]. In addition the Skyrme model was generalized to representations of arbitrary dimension of the SU(2) group. It was shown that - in fact an obvious consistency check - the classical skyrmion solutions are independent of the dimension of the representation, but that in contrast the canonically quantized Skyrme model gives results for baryon observables, which are representation dependent.

An interesting consequence of the canonical \textit{ab initio} quantization of the Skyrme model is the natural appearance of a finite effective pion mass even for the chirally symmetric Lagrangian. While the finite pion mass is conventionally introduced by adding an explicitly chiral symmetry breaking pion mass term to the Lagrangian density of the model [8], the canonical quantization procedure by itself gives rise to a finite pion mass. This realizes Skyrme’s original conjecture that ”This (chiral) symmetry is, however, destroyed by the boundary condition ($U(\infty) = 1$), and we believe that the mass (of pion) may arise as a self consistent quantal effect [9].”

To derive the explicit expressions for electric, magnetic, axial and pseudoscalar form factors of the nucleon we employ the expressions for the Noether currents derived in ref.[7]. Because of the appearance of a finite ”effective” pion mass the asymptotic behavior of the chiral angle $F(r)$ has the required exponential falloff, which ensures finite radii and physical forms for the energy (mass) density. The expressions for the current operators are valid for representations of arbitrary dimension of SU(2). Numerical results are given for the representations with $j = 1/2; 1; 3/2$ and also for the reducible representation $j = 1 \oplus 1/2 \oplus 1/2 \oplus 0$. The different representations of the quantized Skyrme model may be interpreted as different phenomenological models. The best agreement with experimental data on the form factors obtain with the reducible SU(2) representation, which in fact is the SU(3) group octet (1,1) restricted to the SU(2).

This paper is divided into 5 sections. In Section 2 the canonically quantized skyrmion is reviewed. In Section 3 we derive the electroweak form factors of the nucleon. Section 4 contains the numerical results and Section 5 a concluding discussion.
2 Canonically quantized skyrmion

The chirally symmetric Lagrangian density that defines the Skyrme model may be written in the form [3]:

\[ \mathcal{L}[U(r,t)] = -\frac{f_\pi^2}{4} \text{Tr}\{R_\mu R_\mu\} + \frac{1}{32e^2} \text{Tr}\{[R_\mu, R_\nu]^2\}, \tag{1} \]

where \( R_\mu \) is the "right handed" chiral current \( R_\mu = (\partial_\mu U)U^\dagger \). The unitary field \( U(r,t) \) may, in a general reducible representation of the SU(2) group, be expressed as a direct sum of Wigner's D matrices:

\[ U(r,t) = \sum_k \oplus D^{jk}[\vec{\alpha}(r,t)]. \tag{2} \]

Here the vector \( \vec{\alpha} \) represents a triplet of Euler angles \( \alpha_1(r,t), \alpha_2(r,t), \alpha_3(r,t) \).

Quantization of the skyrmion field \( U \) is brought about by means of rotation by collective coordinates that separate the variables, which depend on time and spatial coordinates:

\[ U(r,q(t)) = A(q(t))U_0(r)A^\dagger(q(t)). \tag{3} \]

Here the matrix \( U_0 \) is the generalization of the classical hedgehog ansatz to a general reducible representation [7]. The collective coordinates \( q(t) \) (the Euler angles) are dynamical variables that satisfy the commutation relations \([q^a, q^b] \neq 0\). The energy of the canonically quantized skyrmion, which represents a baryon with spin-isospin \( \ell \), which corresponds to the Lagrangian density (1) in an arbitrary reducible representation has the form:

\[ E(j, \ell, F) = M(F) + \Delta M_j(F) + \frac{\ell(\ell + 1)}{2a(F)}, \tag{4} \]

where \( M(F) \) represents the classical skyrmion mass:

\[ M(F) = 2\pi \frac{f_\pi}{e} \int d\tilde{r} \tilde{r}^2 \left( F'^2 + \frac{\sin^2 F}{\tilde{r}^2} \left( 2 + 2F'^2 + \frac{\sin^2 F}{\tilde{r}^2} \right) \right). \tag{5} \]

The dimensionless variable \( \tilde{r} = e f_\pi r \) has been employed here. Above \( a(F) \) represents the moment of inertia of the skyrmion:

\[ a(F) = \frac{8\pi}{3} \frac{1}{e^2 f_\pi} \int d\tilde{r} \tilde{r}^2 \sin^2 F \left( 1 + F'^2 + \frac{\sin^2 F}{\tilde{r}^2} \right), \tag{6} \]
and \( \Delta M_j(F) \) is a (negative) mass term, which appears in the canonically quantized version of the model:

\[
\Delta M_j(F) = \frac{-2\pi}{15e^3 f_\pi a^2(F)} \int d\tilde{r}\tilde{r}^2 \sin^2 F \left(15 + 4d_2 \sin^2 F(1 - F')^2 + 2d_3 \frac{\sin^2 F}{\tilde{r}^2} + 2d_1 F'^2\right). \tag{7}
\]

The coefficients \( d_i \) in these expressions are given as

\[
N = \frac{2}{3} \sum_k j_k(j_k + 1)(2j_k + 1). \tag{8}
\]

\[
d_1 = \frac{1}{N} \sum_k j_k(j_k + 1)(2j_k + 1)(8j_k(j_k + 1) - 1), \tag{9}
\]

\[
d_2 = \frac{1}{N} \sum_k j_k(j_k + 1)(2j_k + 1)(2j_k - 1)(2j_k + 3), \tag{10}
\]

\[
d_3 = \frac{1}{N} \sum_k j_k(j_k + 1)(2j_k + 1)(2j_k(j_k + 1) + 1). \tag{11}
\]

Minimization of the mass expression in Eq. (5) for \( M(F) \), gives the classical solution for the chiral angle \( F(r) \), which behaves as \( 1/\tilde{r}^2 \) at large distances. In the semiclassical case, the quantum mass correction \( \Delta M_j(F) \) drops out, and variation of the expression (4) yields no stable solution \([10]\). Such a semiclassical skyrmion was considered in ref. \([3]\) as a ”rotating” rigid-body skyrmion with fixed \( F(r) \). The canonical quantization procedure in terms the collective coordinates approach leads to the expanded energy expression (4), variation of which yields a (self-consistent) integro-differential equation with the boundary conditions \( F(0) = \pi \) and \( F(\infty) = 0 \). In contrast to the semiclassical case, the asymptotic behavior of \( F(\tilde{r}) \) at large \( \tilde{r} \) falls off exponentially as:

\[
F(\tilde{r}) = k \left( \frac{\tilde{m}_\pi}{\tilde{r}} + \frac{1}{\tilde{r}^2} \right) \exp(-\tilde{m}_\pi \tilde{r}), \tag{12}
\]

with

\[
\tilde{m}_\pi^2 = -\frac{1}{3e^2 f_\pi^2 a(F)} \left( 8\Delta M_j(F) + \frac{2\ell(\ell + 1) + 3}{a(F)} \right). \tag{13}
\]

The integrals (5), (6), (7) are convergent, and therefore ensure the stability of the quantum skyrmion only if \( \tilde{m}_\pi^2 > 0 \). The positive quantity \( m_\pi = e f_\pi \tilde{m}_\pi \) admits an obvious interpretation as an effective pion mass. The appearance of this effective pion mass conforms with Skyrme’s original conjecture concerning the origin of the pion mass.
The electroweak form factors of the semiclassically quantized SU(2) skyrmion were studied systematically in ref. [11]. An extension of this work to the SU(3) was made in ref. [12]. Analogous studies of the electroweak form factors in the related SU(3) chiral Quark-Soliton Model has been made in ref. [13].

The explicit expressions for the Noether current density operators of the canonically quantized Skyrme model were derived in ref. [7]. The isoscalar part of the nucleon electromagnetic current operator is proportional to the topological baryon current operator and therefore depends on the Lagrangian density only through the chiral angle. The isovector component of the vector current of the nucleon current is proportional to vector Noether current of the Lagrangian density [7]. Linear combinations of the isoscalar and isovector charge densities yield the expressions for the proton and the neutron charge densities:

\[
\rho_p(r) = -\frac{1}{4\pi^2r^2} F'(r) \sin^2 F(r) + \frac{1}{3a(F)} \sin^2 F(r) \left( f_\pi^2 + \frac{1}{e^2} \left( F'^2(r) + \frac{\sin^2 F(r)}{r^2} \right) \right),
\]

\[
\rho_n(r) = -\frac{1}{4\pi^2r^2} F'(r) \sin^2 F(r) - \frac{1}{3a(F)} \sin^2 F(r) \left( f_\pi^2 + \frac{1}{e^2} \left( F'^2(r) + \frac{\sin^2 F(r)}{r^2} \right) \right).
\]

respectively. The Fourier transforms of these charge densities, which are spherically symmetric scalar functions, give the electric form factors of proton and the neutron in the Breit frame as:

\[
G_{E}^{p}(Q^2) = 4\pi \int \frac{drr^2}{r^2} \rho_p(r) j_0(qr),
\]

\[
G_{E}^{n}(Q^2) = 4\pi \int \frac{drr^2}{r^2} \rho_n(r) j_0(qr).
\]

Here \(j_n(qr)\) is the spherical Bessel function of n-th order and \(Q\) is the 4-momentum transfer to the nucleon \((Q^2 = -q^2)\).

The isoscalar and isovector magnetic form factors for the nucleon may be expressed as

\[
G_{M}^{S}(Q^2) = \frac{-2m}{\pi a(F)q} \int drr F'(r) \sin^2 F(r) j_1(qr),
\]

\[
G_{M}^{N}(Q^2) = 4\pi \int \frac{drr^2}{r^2} \rho_n(r) j_1(qr).
\]
\[ G_M'(Q^2) = \frac{16\pi m}{3q} \int dr \left( f_s^2 + \frac{1}{e^2} \left( F'^2(r) + \frac{\sin^2 F(r)}{r^2} \right) - \frac{2d_2 - 15}{4 \cdot 5a^2(F)} \sin^2 F(r) \right) \sin^2 F(r) j_1(qr). \] (19)

Recombination into proton and neutron form factors yields

\[ G_M^p(Q^2) = \frac{1}{2} \left( G_M^S(Q^2) + G_M'(Q^2) \right), \] (20)
\[ G_M^n(Q^2) = \frac{1}{2} \left( G_M^S(Q^2) - G_M'(Q^2) \right). \] (21)

The magnetic form factors at zero-momentum transfer give the magnetic moments of nucleons as

\[ G_M^p(0) = \mu_p, \quad G_M^n(0) = \mu_n, \] (22)

in units of nuclear magnetons.

The standard definition of the matrix element of the axial vector current of the nucleon is

\[ \langle N' | A_\mu(0) | N \rangle = \pi(p_2) \tau^i \left( \gamma_\mu \gamma_5 G_A(Q^2) \right. \]
\[ \left. + q_\mu \gamma_5 \frac{G_P(Q^2)}{2m} \right) u(p_1), \] (23)

where \( G_A(Q^2) \) and \( G_P(Q^2) \) are the axial vector and induced pseudoscalar form factors respectively and \( q = p_2 - p_1 \).

In the non-relativistic limit the axial current operator takes the form

\[ \langle N' | A_\mu(0) | N \rangle = \langle N' | \tau^a \sigma_{\mu'\nu} | N \rangle \left( (-1)^b \delta_{b,-b'} G_A(Q^2) \right) \]
\[ - q^2 \frac{\hat{q}_b \hat{q}_{b'}}{4m^2} G_P(Q^2) \] (24)

Here it is convenient to employ the circular coordinates system for spin and isospin. The momentum transfer \( q = q \hat{q} \) is then:

\[ \hat{q}_a = \frac{2\sqrt{\pi}}{\sqrt{3}} Y_{1,a}(\theta, \varphi). \] (25)

The induced pseudoscalar form factor now takes the form:
\[ G_P(Q^2) = -\frac{3\sqrt{2 \cdot 5} m^2}{\sqrt{\pi q^2}} \int d\vartheta d\varphi \sin \vartheta \left\langle p \left| A_0^1(0) \right| n \right\rangle Y_{2,0}(\vartheta, \varphi) \]
\[ = -\frac{16\pi m^2}{3q^2} \int r^2 dr \left( f^2 \left( 2F' - \frac{\sin 2F}{r} \right) - \frac{1}{e^2} \left( F'' \sin 2F \right. \right. \]
\[ \left. - \left. 4F' \sin^2 F \right) \frac{r}{r^2} + \frac{\sin^2 F \sin 2F}{r^3} \right) \frac{1}{4a^2(F)r} \left( \frac{\sin F}{r^2} + F' \sin F \right) j_2(qr). \] (26)

The axial form factor is

\[ G_A(Q^2) = \frac{1}{\sqrt{2\pi}} \int d\vartheta d\varphi \sin \vartheta \left\langle p \left| A^1_0(0) \right| n \right\rangle \left( Y_{0,0}(\vartheta, \varphi) - \frac{\sqrt{5}}{2} Y_{2,0}(\vartheta, \varphi) \right) \]
\[ = -\frac{8\pi}{9} \int r^2 dr \left( f^2 \left( F' + \frac{\sin 2F}{r} \right) + \frac{1}{e^2} \left( F'' \sin 2F \right. \right. \]
\[ \left. + \left. 2F' \sin^2 F \right) \frac{r^3}{4a^2(F)r} - \frac{5\sin^2 F \sin 2F}{4a^2(F)r} \right) j_0(qr) + \frac{q^2}{12m^2} G_P(Q^2). \] (27)

The expression (27) equals that for the axial form factor given in ref.[14], with exception for the quantum corrections \( \sim 1/a^2(F) \) which appear in the canonical quantization procedure.

The axial current operator contains terms of fourth order in the components of \( r \) [7], and consequently its Fourier transform involves terms of fourth order in \( q \). To avoid a redefinition of the axial current (23), we reduce it to \( Y_{4,a}(\vartheta, \phi) \), and terms of second and zero order in the components of \( q \).

The electromagnetic mean square radii of nucleons is determined by means of the expression:

\[ \langle r^2 \rangle = -\frac{6}{G(0)} \frac{d}{dq^2} G(-q^2) \] (28)

The effect of Lorentz boosts for these form factors may be taken into account by means of the rescalings [15]:

\[ \text{rel} G_E^{p,n}(Q^2) = G_E^{p,n} \left( \frac{Q^2}{1 + Q^2/4m^2} \right), \] (29)

\[ \text{rel} G_M^{p,n}(Q^2) = \frac{1}{1 + Q^2/4m^2} G_M^{p,n} \left( \frac{Q^2}{1 + Q^2/4m^2} \right), \] (30)
Table 1
The predicted static baryon observables for different representations with fixed empirical values for the proton radius $\langle r^2 \rangle_p^E = 0.735 \text{ fm}^2$ and nucleon mass 939 MeV. The experimental data on the nucleon mean square radii are from ref.[17].

<table>
<thead>
<tr>
<th>$j$</th>
<th>Classical [8]</th>
<th>1/2</th>
<th>$\frac{1}{2} \oplus \frac{1}{2}$</th>
<th>1</th>
<th>3/2</th>
<th>Expt.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>Input $^1$</td>
<td>Input</td>
<td>Input</td>
<td>Input</td>
<td>Input</td>
<td>939 MeV</td>
</tr>
<tr>
<td>$\langle r^2 \rangle_p^E$</td>
<td>$\infty$</td>
<td>Input</td>
<td>Input</td>
<td>Input</td>
<td>Input</td>
<td>0.735 fm$^2$</td>
</tr>
<tr>
<td>$f_\pi$</td>
<td>64.5</td>
<td>64.8</td>
<td>60.3</td>
<td>59.4</td>
<td>57.5</td>
<td>93 MeV</td>
</tr>
<tr>
<td>$e$</td>
<td>5.45</td>
<td>4.76</td>
<td>4.31</td>
<td>4.19</td>
<td>3.86</td>
<td></td>
</tr>
<tr>
<td>$\langle r^2 \rangle_n^E$</td>
<td>$\infty$</td>
<td>-0.368</td>
<td>-0.269</td>
<td>-0.249</td>
<td>-0.210</td>
<td>-0.114 fm$^2$</td>
</tr>
<tr>
<td>$\langle r^2 \rangle_p^M$</td>
<td>$\infty$</td>
<td>0.618</td>
<td>0.594</td>
<td>0.587</td>
<td>0.575</td>
<td>0.719 fm$^2$</td>
</tr>
<tr>
<td>$\langle r^2 \rangle_n^M$</td>
<td>$\infty$</td>
<td>0.687</td>
<td>0.609</td>
<td>0.594</td>
<td>0.567</td>
<td>0.637 fm$^2$</td>
</tr>
<tr>
<td>$\mu_p$</td>
<td>1.87</td>
<td>1.96</td>
<td>2.32</td>
<td>2.39</td>
<td>2.54</td>
<td>2.79</td>
</tr>
<tr>
<td>$\mu_n$</td>
<td>-1.31</td>
<td>-1.37</td>
<td>-1.73</td>
<td>-1.81</td>
<td>-1.99</td>
<td>-1.91</td>
</tr>
<tr>
<td>$g_A$</td>
<td>0.61</td>
<td>0.73</td>
<td>0.84</td>
<td>0.87</td>
<td>0.95</td>
<td>1.26</td>
</tr>
<tr>
<td>$m_\pi$</td>
<td>0</td>
<td>110</td>
<td>171</td>
<td>191</td>
<td>246</td>
<td>138 MeV</td>
</tr>
</tbody>
</table>

$$r_{rel} G_A(Q^2) = \frac{1}{\sqrt{1 + Q^2/4m^2}} G_A \left( \frac{Q^2}{1 + Q^2/4m^2} \right), \quad (31)$$

$$r_{rel} G_P(Q^2) = \frac{1}{\sqrt{(1 + Q^2/4m^2)^3}} G_P \left( \frac{Q^2}{1 + Q^2/4m^2} \right). \quad (32)$$

These boost corrections are numerically significant for large values of momentum transfer.

4 Numerical results

The nucleon form factors have been calculated numerically in the representations of the SU(2) group with $j = 1/2, 1, 3/2$ and in the reducible representation $1 \oplus 1/2 \oplus 1/2 \oplus 0$. The two parameters of the Lagrangian density, $f_\pi$ and $e$, have been determined here so that the empirical mass of the proton (938 MeV) and its electric mean square radius (0.735 fm$^2$) are reproduced for each value of $j$ (Table 1). The chiral angle $F(r)$ for each one of these representations has been determined by self consistent numerical variation of the energy expression (4). This procedure yields four pairs of model parameters $f_\pi$ and

$^1$ Ref.[8] used the $\Delta$ resonance mass 1232 MeV as an input parameter.
The value of the axial coupling constant $g_A$, which is far too small in the semiclassical version of the Skyrme model remains below 1 in all the representations considered. The reason for this systematic underestimate is the absence of a quark contribution to the helicity of the nucleon as explained by a sum rule argument in ref.[34]. The “effective” pion mass $m_\pi$ describes the behavior at infinity of the chiral angle $F(R)$ and the asymptotic falloff $e^{-2m_\pi r}$ of nucleon mass density.

The calculated electric form factor of the proton as obtained with the boost corrections (29) are plotted in Fig. 1. In this case the form factor that is calculated in the reducible representation comes closest to the dipole fit to the empirical data.

The corresponding magnetic form factors of the proton, again including the boost correction (30), are plotted in Fig. 2. In this case all the calculated form factors have a realistic falloff with momentum transfer at low values of momentum transfer, although the absolute predictions for the magnetic moment of the proton fall short by some $\sim 10$-20%. In the semiclassical case the magnetic form factor is not well defined [3].

In Fig. 3 the calculated electric form factors of neutron are shown. The results again include the boost correction (29). The experimental data in this case have too wide uncertainty margins for model discrimination. The new experimental results obtained by polarized electron scattering [20,22] indicates this form factor to much larger than what earlier data have suggested, and thus
closer to the present calculated values, even though these are still much larger than the empirical results at intermediate values of momentum transfer.

In Fig. 4 we plot the magnetic form factors of neutron as obtained with the boost correction (30). In terms of agreement with the empirical form factor values only the results for the fundamental representation in which $j = 1/2$ is found to be wanting. This form factor is also ill defined in the semiclassical case.
In Fig. 4 we plot the neutron magnetic form factor $G_M^n(Q^2)$ with relativistic corrections. The empirical values for the magnetic form factor have a dipole-like behavior. The Skyrme model form factors tend to underestimate the falloff rate with momentum considerably, although it is possible to find parameter values that bring the magnetic coupling constant close to the empirical value in the case of the quantum skyrmion.

In Fig. 5 we plot the axial form factor of nucleon with the boost correction (31). The empirical values for the axial form factor have a dipole-like behavior. The Skyrme model form factors tend to underestimate the falloff rate with momentum considerably, although it is possible to find parameter values that bring the axial coupling constant close to the empirical value in the case of the quantum skyrmion.

In Fig. 6 we plot the pseudoscalar form factor of nucleon with the boost correction. This correction represents only about a 1% correction at $Q^2 = 0.2$
The nucleon form factors are well defined in the Skyrme model if the chiral angle asymptotically falls faster than by the semiclassical rate $1/r^2$. The desired exponential fall of has to be brought about by a finite pion mass term, which implies breaking of chiral symmetry. While the pion mass term may be introduced at the classical level through an explicit chiral symmetry breaking term in the Lagrangian density, we have previously shown that such breaking of chiral symmetry also arises, without additional mass parameters, in the canonical ab initio quantization of the Skyrme model [7]. As shown here, this ensures well defined nucleon form factors, which - at least in the case of the electromagnetic form factors - do have phenomenologically adequate momentum dependence. It has also been noted elsewhere and in another context, that quantum corrections may generate a finite pion mass [33].

The present work develops the phenomenological application of the original Skyrme model to representations of arbitrary dimension of the $SU(2)$ group,
and by imposing consistent canonical quantization. This of course in no way exhaust the phenomenological freedom of the Skyrme model with only pion fields: the possibility for generalization of the Lagrangian to terms of higher order in the derivatives remains largely unexplored. Expanded versions of the topological soliton models, which besides the pion fields, also contain vector meson fields have additional mass scales and thus the parameter freedom, which makes it possible to achieve closer agreement with experiment [12].

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