Effective cosmological “constant” and quintessence

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Abstract

In this paper we present exactly solved extended quintessence models; furthermore, through a dynamical effective Q-cosmological "constant", we recover some of the $\Lambda$ decaying cases found in the literature. Finally we introduce a sort of complementarity between the Q-dominated or $\Lambda$-dominated expansions of the Universe.

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There are recent crucial informations coming from observations which seem to put new light on our nowadays knowledge of the Universe. We mention just two of them: first, the recent observations on Type Ia supernovae are strongly in favour of an accelerated Universe \cite{1} \cite{2} \cite{3}. As well, measurements of microwave background, mass power spectrum \cite{4} and lensing statistics (for example, \cite{5}) all suggest that a large amount of energy density in the Universe should have a negative pressure. That is, we have to include some missing part of the energy needed to reach the critical one. Second, as suggested by BOOMERanG \cite{6}, it seems to be very reasonable to assume that the Universe is spatially flat, that is $\Omega_k = 0$ (in agreement with inflationary scenarios) as we will consider all along the paper. In connection with the first information we will follow the assumption present in the literature assuming that the missing part of the energy density can be supposed to have the form of ”quintessence” or the form of a cosmological constant \cite{4} \cite{8} \cite{9} \cite{10}. Actually, to both these possibilities is connected a negative pressure, in order to explain the above quoted observed acceleration: then we have to require such a component to have a state equation $p = w \rho$, with $-1 < w < 0$ (very recent considerations \cite{7} require $-1 \leq w \leq -0.6$ for quintessence, see also \cite{11}, \cite{12}, \cite{13}, \cite{14}, for the $x$-matter scenario); when $w = -1$ we have the cosmological constant scenario \cite{15}.

In this letter we clarify some questions connected with these different models improving the results already published \cite{16}. In the context of non-minimally coupled quintessence theories (quintessence is, from this point of view, directly related with geometry \cite{28}, \cite{16}), we give an exact treatment of the models presenting one of the most commonly used quintessence potential, i.e. the inverse power potential $V = V_0 Q^{-\alpha}$, with $\alpha > 0$ (for example see \cite{17} \cite{18}). We, also, solve exactly the model connected, in the minimal coupling case, to another type of potential now present in the literature, i.e. $V(Q) = V_0 \sinh(\alpha Q)$, \cite{20}. In this scenario we shall discuss a definition (already given in a different context \cite{21}) of quintessence effective cosmological constant, and then, using it in the cases we are considering, we explain some of the most ad hoc $\Lambda$ decaying behaviours considered in literature \cite{26}. We will spend, also, some words on the ($\Lambda$-)fine tuning \cite{10} and the cosmic coincidence problems ($\Omega_m \sim \Omega_\Lambda$) (see for example \cite{15}), which are present in these scenarios. We believe that such an exact treatment of the models can put some further light on the two above quoted problems connected with these scenarios. Furthermore we want to mention, in these introductory remarks, that, concerning the use of $\Lambda$ to explain the supernovae results, doubts have been presented: for example it has been shown that the inhomogeneity can be used to explain them \cite{22} \cite{32}. Together with this, Riess et al. \cite{23} have opened some doubts on the interpretation of same data stressing the possibility of interpreting them entirely in the context of a standard open Friedmann-Robertson-Walker model joined with a reasonable astrophysical evolution model of the white dwarf supernovae progenitors. Finally we will exhibit a sort of complementarity relation showing the usual mutually exclusive presence, in these models, of the quintessence evolving dynamics and of the given (effective quintessence) cosmological constant evolving dynamics. This result, together with finding a very well studied $\Lambda$-decaying
from our quintessence defined effective cosmological constant, we believe is the bridge which makes evident the difference in using a quintessence dominated Universe or a quintessence defined effective Λ dominated Universe. This means that the last one can be "deduced" from the former one and, in a dynamical way, it could be connected to an asymptotic really constant Λ at the end (this last aspect is interesting also because we think it can reintroduce a new attention in the cosmological NoHair theorem [27]).

We start considering the general (field) action that describes the model we are going to use:

$$A = \int \sqrt{-g} (F(Q) R + \frac{1}{2} g^{\mu \nu} Q,_{\mu} Q,_{\nu} - V(Q) + \mathcal{L}_m) d^4x$$

(1)

being \((F(Q), V(Q))\) the (generic) functions describing respectively the coupling and the potential, \(R\) is the curvature scalar, \(\frac{1}{2} g^{\mu \nu} Q,_{\mu} Q,_{\nu}\) is the kinetic energy of the quintessence field and \(\mathcal{L}_m\) describes the standard matter content. In units \(8\pi G = \hbar = c = 1\) we recover the standard gravity when \(F\) is equal to \(-\frac{1}{2}\).

In the flat Friedman–Robertson–Walker cosmologies, (1) gives rise to the "point-like" Lagrangian \(\mathcal{L}\) (we use this expression intending that the field density Lagrangian connected to (1), because of the cosmological principle, can be considered as defined in the minisuperspace where the remnant two field variables \((a, Q)\) have to be considered functions only of the cosmological time and then considered as describing a two degrees of freedom model). In this sense \(\mathcal{L}\) is defined on a two dimensional configuration space (actually we are treating the quintessence field which is not homogeneous, as a function of the time only)

$$\mathcal{L} = 6Fa\dot{a}^2 + 6F'\dot{Q}a^2\dot{a} + a^3 p_Q - Da^{-3(\gamma - 1)}$$

(2)

where \(a\) is the scale factor, \(p_Q \equiv \frac{1}{2} \dot{Q}^2 - V(Q)\) and \(\gamma\) is given by using the standard matter \(p_m = (\gamma - 1)\rho_m\) state equation (completely independent of the second order differential Eqs. connected to (2)). Prime denotes the (total) derivative with respect to \(Q\), dot the same with respect to time.

From Lagrangian (2) we get the same equations which come from the field Eqs. derived from (1) in the FRW metric. Variation of \(Q\) gives the Klein–Gordon equation, whereas the Bianchi identity gives rise to (standard) \(\rho_m = Da^{-3\gamma}\), being the constant \(D\) given by \(D = \rho_{m0} a_0^{3\gamma}\) (the expressions \((,)_0\) denote the value of the quantity \((,\) now). We find from (2)

$$H^2 + \frac{\dot{F}}{F} H + \frac{\rho_Q}{6F} + \frac{\rho_m}{6F} = 0,$$

(3)

$$2\frac{\ddot{a}}{a} + H^2 + \frac{\dot{F}}{F} + 2H\frac{\dot{F}}{F} - \frac{1}{2F} P_Q - \frac{1}{2F} P_m = 0,$$

(4)

where \(\rho_Q \equiv \frac{1}{2} \dot{Q}^2 + V(Q)\). In this presentation the Klein–Gordon equation is derived from (3), (4); it is interesting that Eq.(3) is precisely \(E_\mathcal{L} = 0\), being \(E_\mathcal{L}\)
the (constant) energy associated with $\mathcal{L}$. We will consider here only the dust case, i.e. $p_m = 0$. According to Noether theorem (the Noether symmetry is studied in the quintessence minisuperspace), we could get a further information (the possible existence of that symmetry) very useful to exactly integrate the system (3), (4) as well as to find a form for the two unknowns ($F(Q), V(Q)$). The study of the existence of this symmetry actually leads to an infinite set of Lagrangians (2) if the following relation between the two functions $F(Q), V(Q)$ is satisfied:

$$V = V_0(F(Q))^{2p(s)},$$

where $p = \frac{3(s + 1)}{2s + 3}$, and if $F(Q)$ is of the form

$$F = F_0(s)Q^2,$$

with $F_0(s) = \frac{3s+2}{48(s+1)(s+2)}$. The parameter $s$ labels each Lagrangian belonging to the class of infinite Lagrangians (of type (2)) admitting a Noether symmetry. The value $s = 0$ is a permitted value and gives rise to a model admitting a Noether symmetry, but has to be treated in a different way (for a complete treatment of this approach see [19]). Requiring $F(Q) < 0$ (attractive gravity) and $V(Q)$ of inverse power-law type, we get $s \in (-3/2, -1)$. In Fig. (1) we plot the two functions $F_0(s), p(s)$: (pratically all the interesting expressions of

![Figure 1](image-url)

Figure 1: In (a) we plot the coefficient of the function $F(Q)$. In (b) we show all the possible exponents for the inverse power-law potential: pratically, all exponent values are available.

the inverse power law potential are available: for example, $V = \frac{V_0}{Q^2}$ is relative to the model we pick up fixing the value $s = -1.257571$). The existence of the Noether symmetry gives a further first integral of the second order Euler-Lagrangian Eqs. related to (2). Using it we can exactly solve those equations;
their solutions \( a(t), Q(t) \) are

\[
a(t) = \delta_2(s) [k_1 t + k_2]^{s+2} \left\{ k_3 [k_1 t + k_2]^{s+6} + b_1 t + b_0 \right\}^{\frac{s+1}{s+3}} \tag{7}
\]

\[
Q(t) = \delta_1(s) \left\{ [k_1 t + k_2]^{\frac{s+2}{s+3}} \left\{ k_3 [k_1 t + k_2]^{s+6} + b_1 t + b_0 \right\} \right\}^{\frac{1}{s+3}} \tag{8}
\]

being

\[
\delta_1(s) = \left[ \left( \frac{\chi(s)}{3} \right) \right]^{\frac{s+2}{s+3}}, \quad \chi(s) = -\frac{6s}{2s + 3}, \quad \delta_2(s) = \delta_1 \left( \frac{s+2}{s+3} \right),
\]

\[
b_1 = -\frac{sD}{3\Sigma_0}, \quad k_1 = \frac{s + 3}{s} \Sigma_0 \gamma(s), \quad k_2 = \frac{s+6}{s} \omega_0, \quad k_3 = -\frac{V_0(s + 3)^2}{3k_1^2 \gamma(s)(s + 6)}
\]

\[
\gamma(s) = \frac{2s + 3}{12(s + 1)(s + 2)^2},
\]

where \( \Sigma_0, \omega_0, b_0 \) and \( b_1 \) are the four initial data (the constant \( \Sigma_0 \) comes from the existence of the Noether symmetry); considering the condition \( E_L = 0 \), we get \( 3\Sigma_0 b_1 + sD = 0 \), representing the only constraint on the four initial data for the system of the two second order differential Eqs. (7), (8) which, then, become three initial data as usual. From this constraint we see that neither \( \Sigma_0 \) nor \( b_1 \) can be zero because we are studying models with nonzero standard matter. The constant \( D \) comes from the Bianchi identity for the standard matter as we have already stressed, and it cannot be considered like an initial datum for the system (7) and (8) because the state equation is used. These informations, together with \( V_0 \neq 0 \), which is quite obvious, tell us that \( k_1 \) and \( k_3 \) have to be different from zero. It is important to stress that \( \frac{s+6}{s+3} > 1 \), for \( s \in (-3/2, -1) \); then, for large \( t \) the two functions (7), (8) become \( a(t) = A_0 r(s), \quad Q = Q_0 t N(s) \), where

\[
r(s) = \frac{6 + 9s + 2s^2}{s(3 + s)} \quad (> 0, \quad \text{for} \quad s \in (-3/2, -1)), \tag{9}
\]

\[
N(s) = \frac{-(3 + 2s)}{s} \quad (> 0, \quad \text{for} \quad s \in (-3/2, -1)) \tag{10}
\]

which we plot in Fig. 2. We have indicated with \( (A_0, Q_0) \) the two coefficients which come from (7), (8) for \( t \gg 0 \); these two constants are parametrized by \( s \), that is they are dependent on the model; they also depend on \( \Sigma_0 \) and \( k_1 \). It is important to stress here that this does not give rise to any real limitation. Actually, we are just facing a typical situation found in standard cosmology. If we consider, for example, the de Sitter solution which is given by \( a(t) = a_0 e^{\sqrt{2\Lambda/3} t} \), of course we have to impose \( a_0 > 0 \) from the very beginning, and this does not imply that such a behaviour depends on initial data. Actually this is required by considering an expanding Universe; then the initial data have to
Figure 2: The plots of the two exponents we find in the large $t$ behaviour of the two functions $Q$ and $a$ respectively. It is interesting to see the these two functions give rise to a monotone $t$-dependence for both the functions we are considering: then the inverse functions (say $t = t(a)$) are always well defined.

belong to the semiplane $a_0 > 0$. The same concerns $(A_0, Q_0)$, that is for any of those data the asymptotic behaviour of $a(t)$ and $Q(t)$ is the same, they both cannot be zero for all the physical initial data one can choose. We believe that, in this way, we do not have any (initial data) fine tuning in our models, this being strictly connected to the control we have on those data; that’s why is so important to have the exact solutions of the evolution of the cosmological variables. It is also relevant that we find most of the (large $t$) solutions spread out in the literature. We see that $Q(t)$ diverges for any $s \in (-3/2, -1)$, the same is true for $a(t)$: in the same $s$-interval $\dot{Q}$ goes to zero for large $t$.

Of course, we can fix a special value of $s$ (which identifies the model) in order to get special time dependences for the scale factor: for example, we can have

$$a \sim t^{2/3}$$

(11)

for $s = -1.0788$. For this value of $s$, the potential $V(Q)$ is given by

$$V_0 = F_0(s)^2p(s)\bar{V_0},$$

$Q(t)$ diverges like $t^{0.790}$ (for $s = -1.0788$). In Table 1 are given particular values of the $s$-functions which play a relevant role (actually, we recover almost all of the studied behaviours for $a(t)$). Furthermore we see that, being $\tilde{\rho}_Q = \frac{\dot{F}}{F}H + \frac{\rho Q}{6F}$, and substituting the solutions we found, we have $\tilde{\rho}_Q \sim \frac{1}{t^2} = \frac{1}{a^{2/r}}$, in the case $s = -1.0778$-model, we get $r = 3/2$, and $\tilde{\rho}_Q \sim \frac{1}{a^3}$. Then, among the infinite models under considerations, there is an exactly integrated one which shows a scaling-type behaviour [24], $\tilde{\rho}_Q \sim \frac{1}{a^3}$ scales like $\rho_m$ (we also see that there are
Table 1: For different choices of the parameter $s$ we give the values of the functions describing the most important behaviours appearing in each models we consider. It is interesting to stress that for $s = -1.499$ we get the radiation behaviour for the scale factor.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$r(s)$</th>
<th>$1/r(s)$</th>
<th>$N(s)$</th>
<th>$1/N(s)$</th>
<th>$2p(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0788</td>
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<td>1.5001</td>
<td>-0.780</td>
<td>1.280</td>
<td>-0.280</td>
</tr>
<tr>
<td>-1.1</td>
<td>0.708</td>
<td>1.412</td>
<td>-0.727</td>
<td>-1.375</td>
<td>-0.75</td>
</tr>
<tr>
<td>-1.2</td>
<td>0.888</td>
<td>1.125</td>
<td>-0.5</td>
<td>2</td>
<td>-2.</td>
</tr>
<tr>
<td>-1.3</td>
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<td>0.952</td>
<td>-0.307</td>
<td>3.25</td>
<td>-4.5</td>
</tr>
<tr>
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<td>0.8358</td>
<td>-0.142</td>
<td>7</td>
<td>-0.280</td>
</tr>
<tr>
<td>-1.499</td>
<td>1.332</td>
<td>0.750</td>
<td>0.001</td>
<td>749</td>
<td>-4.5</td>
</tr>
</tbody>
</table>

a large number of models exhibiting $2/r \leq 3$ (tracker behaviour, see [9], [10]). In Table 1 we find also the relative values for the power in the expression of the potential.

The treatment reported before requires $s \neq 0$ [19]; as we said the case connected to the value $s = 0$ has to be discussed separately. It can be shown that two subcases are found: there is a Noether symmetry for the Lagrangian (2) if:

(i) $F = K_0 Q^2$, $(K_0 < 0)$ and $V = V_0 Q^2$ $(V_0 > 0)$; (ii) $F = -\frac{1}{2}$ (minimal coupling), and $V = V_0 (A e^{\mu Q} - B e^{-\mu Q})^2$, with $\mu = \sqrt{3/2}$ [20].

It is easy to see that in this last case the starting Lagrangian we have to consider is $\mathcal{L} = 3a \dot{a}^2 - a^3 (\frac{1}{2} Q^2 - V(Q)) - D$ for the standar dust case. Everything goes like the $s \neq 0$-cases and we will not discuss here the problems related to the initial data as well as the constraint (Einstein (0, 0) Eq.) imposes on the four initial data. We will discuss only the minimal coupling case which splits in two subcases (for the complete discussion of both cases, see [19]). Depending on the relative signs of the two free parameters $(A, B)$, the general solutions of FRW Eqs. are:

$AB > 0$

\[
a(t) = \frac{\omega_0 (\sqrt{AB}t + z_0)^2 - \omega_0^2 \sin^2(\sqrt{AB}t + \omega_1)}{4AB}^{1/3} \quad (12)
\]

\[
Q(t) = \sqrt{\frac{2}{3}} \ln \sqrt{\frac{B}{A}} \left[ \frac{(\omega_0 \sqrt{AB}t + z_0) + \omega_0 \sin(\sqrt{AB}t + \omega_1)}{(\omega_0 \sqrt{AB}t + z_0) - \omega_0 \sin(\sqrt{AB}t + \omega_1)} \right] \quad (13)
\]
\( AB < 0 \)

\[
a(t) = \left[ \frac{-(\omega_0 \sqrt{-AB}t + z_0)^2 + \omega_0^2 \sin h^2(\sqrt{-AB}t + \omega_1)}{4|AB|} \right]^{1/3} \quad (14)
\]

\[
Q(t) = \sqrt{\frac{2}{3}} \ln \left\{ \left[ \frac{B}{A} \right] \frac{-(\omega_0 \sqrt{-AB}t + z_0) + \omega_0 \sin h(\sqrt{-AB}t + \omega_1)}{-(\omega_0 \sqrt{-AB}t + z_0) + \omega_0 \sin h(\sqrt{-AB}t + \omega_1)} \right\} \quad (15)
\]

where \( \omega_0, \omega_1, z_0 \) are the three integration constants (of course \( A, B \) and \( D \) are different from zero). In both cases, \( Q \to \text{const.} \) for large \( t \); in the first case \( a(t) \) behaves as \( t^{4/3} \) for small \( t \) and as \( t^{2/3} \) for large \( t \) (self tuning solution see [24]), in the second \( a(t) \) has a de Sitter asymptotic behaviour. Anyway, the asymptotic behaviours are independent of initial data in the same sense that we have clarified above.

After having exhibited exact solutions for inverse power law potential, and for some kind of exponential potential, let us go now to discuss a way of introducing an effective, time dependent, cosmological "constant" [21] (we hope in this way to solve the connected fine tuning problem [30]). Before presenting our definition of an effective, cosmological "constant" it is noteworthy to recall that: i) standard \( \Lambda \) is introduced by hands; ii) it determines the (in general asymptotic) time behaviour of \( a(t) \) through the (0, 0)-FRW Eqs., which can be rewritten as:

\[
(H - \sqrt{\frac{3\Lambda}{2}})(H + \sqrt{\frac{3\Lambda}{2}}) = \rho_m. \quad \text{An expanding universe requires, asymptotically, } H = \sqrt{\frac{3\Lambda}{2}}. \quad \text{If we look at our (3) we see that it is still possible (in the dust case) to put it in the similar form } (H - \Lambda_{\text{eff,1}})(H + |\Lambda_{\text{eff,2}}|) = -\frac{\rho_m}{6F} = -\frac{-\dot{\rho}_m}{3},
\]

being

\[
\Lambda_{\text{eff,1}} = -\frac{\dot{F}}{2F} + \sqrt{\left( \frac{\dot{F}}{2F} \right)^2 - \frac{\rho_Q}{6F}} \quad (> 0, \quad \text{because } F < 0). \quad (16)
\]

The second root, i.e. \( \Lambda_{\text{eff,2}} \), is less than zero and does not affect the asymptotic time behaviour of the cosmological quantities. Two comments on definition (17) are in order: first, it is completely defined on the quintessence side of the quintessence-tensor theories we are considering, i.e. it is defined only using \( Q \) once we have \( F(Q) \) and \( V(Q) \); from this point of view, \( \Lambda_{\text{eff,1}} \) is not introduced by hands but using the same procedure we have mentioned above, that is using the roots of the (0, 0) Einstein Eqs. We, also, want to stress that it could be of some interest to study, in the effective cosmological constant scenario, under which conditions (17) becomes, for \( t \gg 0 \), a constant giving back the condition for having a de Sitter asymptotic behaviour (the complete discussion of how the asymptotic cosmological NoHair Theorem can be generalized to this case is in [21]). From definition (17) we have that, using the solutions of the system (3) and (4), \( \Lambda_{\text{eff,1}} \) is a function of \( (Q(t), \dot{Q}(t)) \), and of the parameters connected with \( F(Q) \) and \( V(Q) \). It is noteworthy that for \( s \in (-3/2, -1) \) it is found that \( \Lambda_{\text{eff,1}} \) decays in a way very well studied in literature, even if in all these
discussions the decaying of $\Lambda_{\text{eff}}$ is given ad hoc (see the exhaustive paper by Overduin and Cooperstock [26]).

Using the solutions (7), (8) for $t \gg 0$, we find:

$$\Lambda_{\text{eff}} = \frac{B}{a^m}, \quad m = m(s) = \frac{1}{r(s)}.$$ (17)

Even if the constant $B$ is depending on $s$ and on the initial data of the problem as well as on the parameter $V_0$, as we have stressed till now, the way $\Lambda_{\text{eff}}$ behaves in time does not depend on the initial data. Then Eq. (18) shows that the form of the decaying is general, that is $\Lambda_{\text{eff}}$ goes to zero independently of any initial data. In [26] the form (18) is given a-priori and then an explanation of the constant $B$ is not found (in terms of $H$ we have also that $\Lambda_{\text{eff}} \sim H$).

When $a(t) \sim t^{2/3}$ we have $m = 1.50016$, that is $\Lambda_{\text{eff}} \sim \frac{1}{a^{1.5}}$ compatible with lens statistics, power spectrum of matter density perturbations [26]. Also the value $m = 2$ is permitted for $s = -1.0002$. In the case $s = 0$, with $AB > 0$ we have that $\Lambda_{\text{eff}} \sim \frac{B}{a^{3/2}}$. ($\bar{B}$ is the equivalent of $B$ introduced above, in the case $s = 0$, $Q$ goes to const.). For $AB < 0$ we get the interesting result $\Lambda_{\text{eff}} \rightarrow \Lambda(A, B) = \text{const.}$, independently of any initial data and then we recover a de Sitter asymptotic behaviour for the scale factor (also in this case $Q$ goes to const.).

In summary, in the context of Noether symmetry approach to non-minimally and minimally coupled quintessence tensor theories of gravity we have obtained a class of models exhibiting two important types of potentials, respectively, inverse power law and exponential, and we have exactly solved it. If we look at the solutions, we can state a property which shows a sort of complementarity between quintessence and the dynamically defined $\Lambda_{\text{eff}}$. Before writing down this property, in Table 2 we give the behaviours we have found for $(a(t), Q(t))$, $\dot{Q}(t))$ and for the effective $\Lambda$-term we have introduced. The relation we propose is the following:

$$\Lambda_{\text{eff}}, f(Q) = \text{cost.}, \quad t \gg 0,$$ (18)

where $f$ depends not only on the $Q$-field but is determined also by $s$ (for example is given by $Q^{\omega(t)}$ in the cases $s \in (-3/2, -1)$, i.e. it is determined by the model we choose. Relation (19) is true, in general, if we assume the existence of Noether symmetries and that $\frac{\dot{F}}{F} = \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \overset{t \gg 0}{\rightarrow} 0$ (see [19] for a better understanding of the hypothesis on $\frac{\dot{F}}{F} = \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \overset{t \gg 0}{\rightarrow} 0$). We see that (19) gives rise to the already mentioned complementarity between the $\Lambda$-term and the (divergent or convergent to a constant) quintessence content in the universe. From this point of view we do not have that dark energy, or quintessence, and cosmological constant are completely different forms of energy: first, because they are different forms of the same (quintessence) energy; in fact, we derive the $\Lambda$-term from the presence of quintessence. Furthermore, they are complementary in the sense given by (19) (actually, the quintessence kinetic energy has no
\( s \in (-3/2, -1) \) & \( s = 0, \ AB > 0 \) & \( s = 0, \ AB < 0 \) \\
\( a(t) \sim t^{1(s)} \) & \( a(t) \sim t^{2/3} \) & \( a(t) \sim \exp[\Lambda_{\text{eff}, t}] \) \\
\( Q(t) \sim \infty \) & \( Q(t) \sim \text{const.}(\neq 0) \) & \( Q(t) \sim \text{const.}(\neq 0) \) \\
\( \dot{Q}(t) \sim 0 \) & \( \dot{Q}(t) \sim 0 \) & \( \dot{Q}(t) \sim 0 \) \\
\( F(Q(t)) \sim \infty \) & \( F(Q(t)) = -1/2 \) & \( F(Q(t)) = -1/2 \) \\
\( \dot{F}/F \sim 0 \) & \( \dot{F}/F = 0 \) & \( \dot{F}/F = 0 \) \\
\( V(Q(t)) \sim 0 \) & \( V(Q(t)) \sim 0 \) & \( V(Q(t)) \sim \text{const.}(\neq 0) \) \\
\( \Lambda_{\text{eff}, t} \sim 1/a^m \) & \( \Lambda_{\text{eff}, t} \sim 1/a^3 \) & \( \Lambda_{\text{eff}, t} \sim \text{const.}(\neq 0) \) \\

Table 2: The \( t \gg 0 \) behaviour of all the physical quantities introduced. It is noteworthy that in all the cases the potential rolls down to its minimum. In the \( AB > 0 \) case its minimum is zero, whereas in the \( AB > 0 \) case the minimum is different from zero, and then we recover a real cosmological constant. We see also that the kinetical-Q energy has no asymptotical role

Asymptotic role in all the cases we have presented, from which we deduce that the dominant ingredient is connected with the large \( t \) behaviour of the Q-field and with the form of the potential minimum. When this minimum is zero we have that the Q-field dominates, when this minimum is non zero we have the asymptotic dominance of the effective cosmological "constant". In this paper we do not discuss the way \( \Lambda \), as well the solutions we have found, depends on the potential parameter; to this purpose see our [16]. From Table2 it is clear that the two possible evolutions of the scale factor are quite different, then the expansion history of the Universe is different: its acceleration and its age will strongly depend on the dominance of the Q-matter or of the effective \( \Lambda \)-Q. In all cases the potential rolls down to its minimum and then, as we have stressed, the value of this minimum plays a very important role: in the cases we have presented, when this minimum is different from zero, we have an asymptotic cosmological constant (de Sitter behaviour for the scale factor); in the cases this minimum is zero, we do not recover any cosmological constant (power expansion of the scale factor). More precisely, we have that in the (Noether) nonminimal coupling case the minimum is always zero, whereas the coupling is divergent; then we get that \( \dot{G}_{\text{eff}}/G_{\text{eff}} \stackrel{t \gg 0}{\rightarrow} 0 \) and that \( \Lambda \) decays. In the (Noether) minimal coupling case, when \( AB > 0 \), the potential minimum is zero, whereas in the \( AB < 0 \) case the minimum is different from zero, and then we recover an asymptotical true cosmological constant. In [16] we discussed in more details the possible values of the parameter \( w \), finding a range of values, at least for the case \( s \in (-3/2, -1) \), coherent with the currently used range of values as given
in the introductory remarks. We have to say that, concerning $w$, compared to
what is found in the literature, we have followed here a different approach: we
have not introduced this new parameter but we have reconstructed it using the
knowledge of the exact solutions: more precisely the state equation $p_Q = p_Q(\varrho)$
has been found knowing $\rho_Q = (t(\rho_Q))$, that is using $a(t)$ and $Q(t)$ for $t \gg 0$.
Concerning the large scale structure connected with this approach, see our pa-
per [31]. We believe that, through the knowledge of the exact solutions we have
presented, it is possible to have a better control of the role of initial data; actu-
ally, all along the paper we have shown that using our approach we do not
have that problem. Furthermore, using our approach, we hope to have clarified,
among the reported other features, some aspects of quintessence models: more
precisely we have given relation (5), we believe interesting, between the coupling
and the potential as well as the complementarity relation (19) which holds for
the types of couplings and potentials we have studied.

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