(F1, D1, D3) Bound State, Its Scaling Limits and $SL(2,\mathbb{Z})$ Duality

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Abstract

We investigate the properties of the bound state (F1, D1, D3) in IIB supergravity in three different scaling limits and the $SL(2,\mathbb{Z})$ transformation of the resulting theories. In the simple decoupling limit with finite electric and magnetic components of NS $B$ field, the worldvolume theory is the $\mathcal{N}=4$ SYM and the supergravity dual is still the $AdS_5 \times S^5$. In the large magnetic field limit with finite electric field, the theory is the noncommutative super Yang-Mills (NCSYM), and the supergravity dual is the same as that without the electric background. We show how to take the decoupling limit of the closed string for the critical electric background and finite magnetic field, and that the resulting theory is the noncommutative open string (NCOS) with both space-time and space-space noncommutativity. It is shown that under the $SL(2,\mathbb{Z})$ transformation, the SYM becomes itself with different coupling constant, the NCSYM is mapped into the NCOS, but the NCOS in general transforms into another NCOS and reduces to a NCSYM in a special case.

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1 Introduction

Over the past years one of the important progresses in string and field theories is the observation due to Maldacena [1]: open string excitations on D-branes decouple from gravity under appropriate low energy limit, the so-called decoupling limit: string/M theories on the anti-de Sitter space (AdS) is dual to a certain large $N$ conformal field theory (CFT) which lives on the boundary of the AdS. One important example of this AdS/CFT correspondence is that IIB string theory on the $AdS_5 \times S^5$ is believed to be dual to the $\mathcal{N}=4$ super Yang-Mills theory (SYM). The geometry $AdS_5 \times S^5$ comes from the decoupling limit of D3-branes (hereafter referred to as SYM limit). Through this “duality” one can learn something about the large $N$ SYM in the strong ’t Hooft coupling from the low energy limit of superstring/M theory: supergravity. Indeed a lot of knowledge has been acquired from the correspondence, which is consistent with our expectation.

Recently it has been noticed that when a constant NS $B$ field is present, the worldvolume coordinates of D-branes become noncommutative [2, 3, 4]. In an appropriate limit the worldvolume theory of D-branes is a noncommutative SYM (NCSYM) [5]-[12] (hereafter called NCSYM limit). Furthermore, it has been found that the resulting theory is unitary if space-space coordinates are noncommutative. More recently when electric background is introduced, the resulting theory has been found to be a noncommutative open string (NCOS) theory with space-time noncommutativity in an appropriate limit (hereafter NCOS limit) [13, 14]. For related discussions, see also [15]-[22].

In this paper we consider the above three limits for the bound state (F1, D1, D3) with both electric and magnetic $B$ field. In the simple $\alpha' \rightarrow 0$ limit, we show the resulting theory is still the $\mathcal{N}=4$ SYM in ordinary spacetime; in the NCSYM limit the theory is the NCSYM with space-space noncommutativity again, namely without space-time noncommutativity; and in the NCOS limit the resulting theory is most nontrivial and is the NCOS with not only space-time noncommutativity, but also space-space noncommutativity. Another subject we will discuss is the S-duality or more generally $SL(2, \mathbb{Z})$ duality of these solutions. The S-duality has been discussed for theories with either electric or magnetic background [14], but full understanding of the behavior of these theories with both backgrounds under the S-duality does not seem to be obtained. In the NCOS theory,
there remains nonvanishing axion field and the simple S-duality is not general enough to get clear understanding of the relations of these theoirs. Instead we will show that these theories are nicely related with each other by the $SL(2, \mathbb{Z})$ transformation of IIB supergravity.

As a preparation of our following discussions, in sect. 2 we review the Seiberg-Witten relation between the open and closed string moduli when both the electric and magnetic components of the NS $B$ field are present. In sect. 3 we discuss supergravity duals for three different limits. In particular we show how we are uniquely lead to the nontrivial decoupling limit for space-time and space-space noncommutativities. In sect. 4, we discuss general $SL(2, \mathbb{Z})$ transformations of these solutions and show that under the $SL(2, \mathbb{Z})$ transformation, the NCSYM is always mapped into the NCOS, but the NCOS transforms in general into another NCOS with different parameters and gives rise to a NCSYM in a special case. Concluding remarks are given in sect. 5.

## 2 Seiberg-Witten relation

When a constant NS $B$ field is present, the open string ending on a D-brane has the following boundary condition:

$$g_{ij} \partial_s X^j + 2\pi \alpha' B_{ij} \partial_\tau X^j = 0, \quad \delta X^a = 0.$$  \hspace{1cm} (2.1)

The open string moduli occur in the disk correlators on the open string worldsheet boundaries

$$< X^i(\tau) X^j(0) > = -\alpha' G^{ij} \ln(\tau)^2 + \frac{i}{2} \Theta^{ij} \epsilon(\tau).$$  \hspace{1cm} (2.2)

The open and closed string moduli are connected by the Seiberg-Witten relation [4]

$$G_{ij} = g_{ij} - (2\pi \alpha')^2 (Bg^{-1}B)_{ij},$$

$$\Theta^{ij} = 2\pi \alpha' \left( \frac{1}{g + 2\pi \alpha' B} \right)^{ij}_A,$$

$$G^{ij} = \left( \frac{1}{g + 2\pi \alpha' B} \right)^{ij}_S,$$

$$G_s = g_s \left( \frac{\det G_{ij}}{\det(g_{ij} + 2\pi \alpha' B_{ij})} \right)^{1/2},$$  \hspace{1cm} (2.3)
where \( (\cdot)_A \) and \( (\cdot)_S \) denote the antisymmetric and symmetric parts, respectively.

The constant NS \( B \) field is equivalent to a gauge field on the worldvolume of the D-brane because only \( F_{ij} = B_{ij} + F_{ij} \) is gauge invariant. Therefore the electric and magnetic components of \( B \) field can always be rotated so that they are parallel to each other. On the other hand, we are mainly concerned with the D3-brane case in this paper, restricting ourselves to the worldvolume of D3-branes. Suppose we have the closed string metric

\[
g_{ij} = g_1(\delta^1_i \delta^1_j - \delta^0_i \delta^0_j) + g_2(\delta^3_i \delta^3_j + \delta^3_i \delta^3_j),
\]

(2.4)

and the constant \( B \) field has components

\[
B_{ij} = E(-\delta^1_i \delta^0_j + \delta^0_i \delta^1_j) + B(\delta^3_i \delta^3_j - \delta^3_i \delta^3_j).
\]

(2.5)

Defining

\[
e = \frac{E}{E_{\text{crit}}}, \quad b = \frac{B}{B_0},
\]

(2.6)

where

\[
E_{\text{crit}} = \frac{g_1}{2\pi \alpha'}, \quad B_0 = \frac{g_2}{2\pi \alpha'},
\]

(2.7)

and using the Seiberg-Witten relation, one has the open string metric

\[
G^{ij} = \frac{1}{g_1(1 - e^2)}(-\delta^0_i \delta^0_j + \delta^1_i \delta^1_j) + \frac{1}{g_2(1 + b^2)}(\delta^3_i \delta^3_j + \delta^3_i \delta^3_j),
\]

(2.8)

the noncommutativity matrix

\[
\Theta^{ij} = \frac{2\pi \alpha' e}{g_1(1 - e^2)}(\delta^0_i \delta^0_j - \delta^1_i \delta^1_j) + \frac{2\pi \alpha' b}{g_2(1 + b^2)}(-\delta^3_i \delta^3_j + \delta^3_i \delta^3_j),
\]

(2.9)

and the open string coupling constant

\[
G_s = g_s \sqrt{(1 - e^2)(1 + b^2)}.
\]

(2.10)

Let us now consider three different limits.

(1) SYM limit. From (2.8), (2.9) and (2.10), we see that when \( \alpha' \to 0 \) with finite electric \( E \) and magnetic components \( B \), the open and closed string moduli are equal; the noncommutativity matrix \( \Theta^{ij} \) vanishes such that the whole spacetime is an ordinary commutative one. In this case, the oscillation modes of open string and gravity are decoupled, and the worldvolume theory is the \( \mathcal{N}=4 \) SYM in the low energy limit. Note
that the constant $B$ field converts into a constant part of gauge field on the worldvolume of D3-brane and remains in that limit. If the D3-brane worldvolume is noncompact, the constant part of the gauge field is physically unmeasurable in the flat infinite space [14].

On the other hand, if the worldvolume is compact, the constant part is quantized and the resulting low energy theory is the $\mathcal{N}=4$ SYM with both quantized electric and magnetic fluxes.

(2) NCSYM limit. Taking the limit $\alpha' \to 0$ with $g_1 = 1$, $g_2 = (2\pi \alpha' B)^2$, and $g_s = 2\pi \alpha' B G_s$ while keeping $E$ and $B$ finite, one can obtain

$$G^{ij} = \eta^{ij}, \quad \Theta^{ij} = \frac{1}{B} (-\delta_2^i \delta_3^j + \delta_3^i \delta_2^j),$$

and the open string coupling $G_s$ is finite. In this case, the resulting theory is the noncommutative SYM (because of $\alpha' G^{ij} = 0$) with space-space noncommutativity $\Theta^{23} \neq 0$; the Yang-Mills coupling constant is $g_{YM}^2 = 2\pi G_s$. The magnetic background gives rise to the space-space noncommutativity. The constant electric component of NS $B$ field is converted into a constant electric part of the gauge field, which is physically unmeasurable if the worldvolume is noncompact again. If the worldvolume is compact, the resulting theory is the NCSYM with quantized electric flux.

(3) NCOS limit. In contrast to the magnetic component $B$, on which there is no restriction, the electric component cannot be beyond its critical value $E_{\text{crit}}$. When the electric field approaches its critical value in a certain manner, one may obtain a noncritical NCOS theory, from which the closed string sector is decoupled [13, 14], in spacetime with the space-time noncommutativity. If a finite magnetic component is also present, the space-space coordinates are also noncommutative. This has been noticed as well in [19]. For example, taking the scaling limit

$$e = 1 - \alpha^m e_0 / 2, \quad (n > 0),$$

$$g_1 = \frac{1}{\alpha_{\text{eff}} e_0 \alpha^{m-1}}, \quad g_2 = \frac{2\pi \alpha' b}{(1 + b^2) \theta_1},$$

$$g_s = \frac{G_s}{\alpha^{m/2} \sqrt{e_0 (1 + b^2)}},$$

(2.12)
with $e_0$ a constant, while keeping $B$ a constant, one may get

$$\frac{\alpha'}{\alpha'_{\text{eff}}} G^{ij} = -(\delta^i_0 \delta^j_0 - \delta^i_1 \delta^j_1) + \frac{\theta_1}{\theta_0 b}(\delta^i_2 \delta^j_2 + \delta^i_3 \delta^j_3),$$

(2.13)

$$\Theta^{ij} = \theta_0 (\delta^i_0 \delta^j_1 - \delta^i_1 \delta^j_0) + \theta_1 (-\delta^i_2 \delta^j_3 + \delta^i_3 \delta^j_2),$$

(2.14)

where $\alpha'_{\text{eff}} = \theta_0/(2\pi)$ and the open string coupling constant $G_s$ is finite. In this case, the critical electric field $E_{\text{crit}} = \frac{1}{e_0 \alpha'_{\text{eff}} \alpha_n}$. Although the closed string coupling constant is divergent in this limit, the open string metric and coupling constant are well defined.

It is rather nontrivial how to derive the dual gravity description of this NCOS theory. One of our purposes in this paper is to elucidate this problem. This will be discussed in the next section.

### 3 Supergravity duals

When both the electric and magnetic components of the NS $B$ field are present on a D3-brane, the D3-brane becomes a (F1, D1, D3) bound state [23, 24, 25]. The supergravity configuration can be constructed as follows [5]. Starting from a D3-brane without NS $B$ field with worldvolume coordinates $(x_0, x_1, x_2$ and $x_3$), and making a T-duality along $x_3$, one gets a D2-brane with a smeared coordinate $x_3$. Uplifting the D2-brane yields an M2-brane in the 11-dimensional supergravity. Performing a coordinate rotation with parameter angles $\varphi$ and $\theta$

$$x_4 = x_4' \cos \varphi + (x_2' \cos \theta + x_3' \sin \theta) \sin \varphi,$$

$$x_2 = -x_4' \sin \varphi + (x_2' \cos \theta + x_3' \sin \theta) \cos \varphi,$$

$$x_3 = -x_2' \sin \theta + x_3' \cos \theta,$$

(3.1)

and then reducing along $x_4'$, one obtains a new D2-brane. Acting a T-duality along $x_3'$, one reaches a (F1, D1, D3) bound state. For the case of black configurations, the procedure is applicable as well.

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1Here $n$ is an arbitrary positive parameter. It was chosen to be 2 in the S-duality consideration in ref. [14], and 1 in ref. [21]. We will show in the next section that this freedom is allowed in this limit.
Using the above approach, we obtain
\[ ds^2 = F^{-1/2}[h'(-f dx_0^2 + dx_1^2) + h(dx_2^2 + dx_3^2)] + F^{1/2}[f^{-1}dr^2 + r^2d\Omega_5^2], \]
\[ e^{2\phi} = g_s^2 h h', \quad \chi = \frac{1}{g_s F} \tan \varphi \sin \theta, \]
\[ B_{01} = H^{-1} \coth \alpha \sin \varphi, \quad A_{01} = H^{-1} \coth \alpha \sin \theta \cos \varphi / g_s, \]
\[ B_{23} = \frac{\tan \theta}{G}, \quad A_{23} = \frac{\tan \varphi}{g_s G \cos \theta}, \]
\[ D_{0123} = \frac{\coth \alpha}{g_s G \cos \theta \cos \varphi}, \quad (3.2) \]

where
\[ f = 1 - \frac{r_0^4}{r^4}, \quad H = 1 + \frac{r_0^4 \sinh^2 \alpha}{r^4}, \]
\[ h = F/G, \quad h' = F/H, \]
\[ F = 1 + \cos^2 \varphi \frac{r_0^4 \sinh^2 \alpha}{r^4}, \quad G = 1 + \cos^2 \varphi \cos^2 \theta \frac{r_0^4 \sinh^2 \alpha}{r^4}. \quad (3.3) \]

The bound state solution includes several special cases. When \( \varphi = \theta = 0 \), it reduces to the D3-brane solution; when \( \varphi = \pi/2 \) and \( \theta \) is arbitrary, it goes to the F-string solution with two smeared coordinates; when \( \varphi = 0 \) and \( \theta = \pi/2 \), it becomes the D-string solution with two smeared coordinates; when \( \varphi = 0 \) and \( \theta \) is arbitrary, the solution reduces to the (D1, D3) bound state; when \( \theta = 0 \) and \( \varphi \) is arbitrary, it becomes the (F1, D3) bound state; and finally as \( \theta = \pi/2 \) and \( \varphi \) is arbitrary, it goes back to the (F1, D1) bound state with two smeared coordinates.

Some thermodynamic quantities, the ADM mass \( M \), the Hawking temperature \( T \), and the entropy \( S \), associated with the solution (3.2) are
\[ M = \frac{5\pi^3 r_0^4 V_3}{16\pi G} (1 + \frac{4}{5} \sinh^2 \alpha), \]
\[ T = \frac{1}{\pi r_0 \cosh \alpha}, \]
\[ S = \frac{\pi^3 r_0^5 V_3}{4G} \cosh \alpha, \quad (3.4) \]

where \( V_3 \) is the spatial volume of worldvolume of the bound state. A remarkable feature of these thermodynamic quantities is their independence of the parameter angles \( \varphi \) and \( \theta \).

\[ ^2 \text{In this section a factor } 2\pi \alpha' \text{ is absorbed into the } B \text{ field.} \]
θ. This means that the thermodynamics is the same for all special cases discussed above. It also guarantees the thermodynamic equivalence among three theories coming from different scaling limits which will be discussed shortly.

For the (F1, D1, D3) bound state, the charges of three kinds of branes are

\[
Q_{D3} = \frac{4\pi^3 V_3 r_0^4 \sinh \alpha \cosh \alpha}{16 \pi G} \cos \varphi \cos \theta, \\
Q_{D1} = \frac{4\pi^3 V_3 r_0^4 \sinh \alpha \cosh \alpha}{16 \pi G} \cos \varphi \sin \theta, \\
Q_F = \frac{4\pi^3 V_3 r_0^4 \sinh \alpha \cosh \alpha}{16 \pi G} \sin \varphi.
\] (3.5)

In the extremal limit: \( \alpha \rightarrow \infty, \ r_0 \rightarrow 0 \) with finite charges, we have

\[
M_{ext}^2 = Q_{D3}^2 + Q_{D1}^2 + Q_F^2,
\] (3.6)

which indicates the bound state is a non-threshold one. The charges satisfy the following relations

\[
\frac{Q_{D1}}{Q_{D3}} = \tan \theta, \quad \frac{Q_F}{Q_{D3}} = \frac{\tan \varphi}{\cos \theta}.
\] (3.7)

Furthermore, in this bound state the number of D3-branes is

\[
N_3 = \frac{R^4 \cos \varphi \cos \theta}{4\pi g_s \alpha'^2},
\] (3.8)

the number of D-strings

\[
N_1 = \frac{R^4 \cos \varphi \sin \theta}{4\pi g_s \alpha'^2} \frac{V_2}{(2\pi)^2 \alpha'},
\] (3.9)

and the number of F-strings

\[
N_F = \frac{R^4 \sin \varphi}{4\pi g_s^2 \alpha'^2} \frac{V_2}{(2\pi)^2 \alpha'},
\] (3.10)

where \( R^4 = r_0^4 \sinh \alpha \cosh \alpha \) and \( V_2 \) is the area of worldvolume coordinates \( x_2 \) and \( x_3 \).

We are now going to discuss the various decoupling limits for the supergravity duals.

### 3.1 SYM limit

In this subsection, we first discuss the SYM limit. Taking the usual decoupling limit:

\[
\alpha' \rightarrow 0: \quad r = \alpha' u, \quad r_0 = \alpha' u_0,
\] (3.11)
and keeping \( \cos \theta \) and \( \cos \varphi \) finite, we have

\[
ds^2 = \alpha' \left[ \frac{u^2}{R^2} (\cos^2 \varphi (-\tilde{f} dx_0^2 + dx_1^2) + \cos^{-2} \theta (dx_2^2 + dx_3^2)) + \frac{\tilde{R}^2}{u^2} (\tilde{f}^{-1} du^2 + u^2 d\Omega^2) \right],
\]

(3.12)

where \( \tilde{R}^4 = 4\pi g_s N_3 \cos \varphi / \cos \theta \) and \( \tilde{f} = 1 - u_0^4 / u^4 \). The dilaton, axion and \( B \) fields reduce to

\[
e^{2\phi} = g_s^2 \cos^2 \varphi / \cos^2 \theta, \quad \chi = 0, \quad B_{01} = \alpha'^2 \sin \varphi \cos^2 \varphi u^4 / \tilde{R}^4, \quad B_{23} = \alpha'^2 \tan \theta u^4 / (\tilde{R}^4 \cos^2 \theta).
\]

(3.13)

Obviously, rescaling the closed string coupling constant and worldvolume coordinates

\[
g_s = \frac{\cos \theta}{\cos \varphi} \tilde{g}, \quad x_{0,1} = \frac{1}{\cos \varphi} \tilde{x}_{0,1}, \quad x_{2,3} = \cos \theta \tilde{x}_{2,3},
\]

(3.14)

we can convert the metric (3.12) into a standard form of \( AdS_5 \times S^5 \):

\[
ds^2 = \alpha' \left[ \frac{u^2}{R^2} (-\tilde{f} dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{\tilde{R}^2}{u^2} (\tilde{f}^{-1} du^2 + u^2 d\Omega_5^2) \right].
\]

(3.15)

The dilaton, axion and \( B \) fields become

\[
e^{2\phi} = \tilde{g}^2, \quad \chi = 0, \quad B_{01} = \alpha'^2 \sin \varphi u^4 / \tilde{R}^4, \quad B_{23} = \alpha'^2 \tan \theta u^4 / \tilde{R}^4.
\]

(3.16)

From the above solution, we see that although both the electric and magnetic components are present in the D3-brane bound state, the resulting theory in the SYM limit is still the \( \mathcal{N}=4 \) SYM with gauge group \( U(N_3) \) without noncommutativity, just as was noticed in [25]. This can be also understood from the boundary condition (2.1) of open string. In the SYM limit, the mixed boundary condition reduces to the ordinary Neumann one. The constant \( B \) field has no effect on the open string ending on the D3-branes in that limit.

It is worthwhile to stress here that there is a remarkable difference between the D3-brane with finite \( B \) field and the case without \( B \) field depending on whether the D3-brane is compact or not.

The bound state solution (3.2) implies that there are a constant electric component \( B_{01} = \sin \varphi \) and a constant magnetic component \( B_{23} = \tan \theta \) of NS \( B \) field on the worldvolume of D3-brane, which gives a constant part of the worldvolume field strength
Although the constant part has no effect on the open string ending on the D3-branes in the SYM limit, it remains in that limit. If the D3-brane is compact on a torus, this part should be quantized. We find that the constant part gives $N_1$ units of magnetic flux and $N_F$ units of electric flux, with $N_1$ and $N_F$ being the numbers of D-strings and F-strings in the bound state, respectively. Thus the resulting low energy theory is the $\mathcal{N}=4$ SYM with both quantized electric and magnetic fluxes. On the other hand, if the worldvolume of D3-brane is not compact, the constant part is physically unmeasurable in the flat infinite space as mentioned before [14]. The resulting theory is then just the $\mathcal{N} = 4$ SYM as for the case of D3-branes without $B$ field.

Thus we conclude that in the SYM limit the worldvolume theory on the (F1, D1, D3) bound state is the $\mathcal{N}=4$ SYM without noncommutativity. We will see in the next section that the theory is self-dual in the sense that under the $SL(2,\mathbb{Z})$ transformation the SYM converts again into the SYM with different coupling constant. If flux is present, the electric flux converts into the magnetic one, and vice versa under S-duality.

### 3.2 NCSYM limit

Taking the decoupling limit

$$\alpha' \to 0 : \quad \tan \theta = \frac{\tilde{b}}{\alpha'}, \quad x_{0,1} = \tilde{x}_{0,1}, \quad x_{2,3} = \frac{\alpha'}{\tilde{b}} \tilde{x}_{2,3},$$

$$r = \alpha' u, \quad r_0 = \alpha' u_0, \quad g_s = \alpha' \tilde{g},$$

while keeping $\cos \varphi$ finite, one has

$$ds^2 = \alpha' \left[ \frac{u^2}{R_y^4} \cos^2 \varphi (-\tilde{f} d\tilde{x}_0^2 + d\tilde{x}_1^2) + \tilde{h}(d\tilde{x}_2^2 + d\tilde{x}_3^2) \right] + \frac{R_y^2}{u^2} [\tilde{f}^{-1} du^2 + u^2 d\Omega_5^2],$$

(3.18)

where $R_y^4 = 4\pi \tilde{g} \tilde{b} N_3 \cos \varphi$, and

$$\tilde{h}^{-1} = 1 + (au)^4, \quad a^4 = \tilde{b}^2 / R_y^4.$$  

(3.19)

and

$$e^{2\phi} = \tilde{g}^2 \tilde{b}^2 \tilde{h} \cos^2 \varphi, \quad \chi = 0,$$

$$B_{01} = \alpha' \sin \varphi \cos^2 \varphi u^4 / R_y^4, \quad B_{23} = \frac{\alpha'}{\tilde{b}} \frac{(au)^4}{1 + (au)^4}. \quad (3.20)$$

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After further rescaling the string coupling constant $\tilde{g}$ and the coordinates $\tilde{x}_{0,1}$ as

$$\tilde{g} \rightarrow \frac{\tilde{g}}{\cos \varphi}, \quad \tilde{x}_{0,1} \rightarrow \frac{1}{\cos \varphi} \tilde{x}_{0,1},$$

we reach

$$ds^2 = \alpha' \left[ \frac{u^2}{R^4} \left( -\tilde{f} \tilde{x}_0^2 + d\tilde{x}_1^2 \right) + \tilde{h} (d\tilde{x}_2^2 + d\tilde{x}_3^2) \right] + \frac{R^2}{u^2} \left[ f^{-1} du^2 + u^2 d\Omega_5^2 \right],$$

and

$$e^{2\phi} = \tilde{g}^2 \tilde{b}^2 \tilde{h}, \quad \chi = 0,$$

$$B_{01} = \alpha' \sin \varphi u^4 / R^4, \quad B_{23} = \frac{\alpha'}{\tilde{b}} \frac{(au)^4}{1 + (au)^4},$$

where $a^4 = \tilde{b}^2 / R^4$ and $R^4 = 4 \pi \tilde{g} \tilde{b} N_3$. We notice that the geometry (3.22) is completely the same as the case for black D3-branes with only spatial component of $B$ field; for the latter see [5, 6]. It has been claimed that the geometry (3.22) is the gravity dual configuration of the $\mathcal{N}=4$ NCSYM with gauge group $U(N_3)$ with space-space noncommutativity $[\tilde{x}_2, \tilde{x}_3] = i \tilde{b}$.

In the NCSYM limit, the infinitely large magnetic field gives rise to the noncommutativity of space-space while the electric field is kept finite $F_{01} = \sin \varphi$. The electric field has no effect on the field theory limit of open string (it gives rise to a quantized electric flux if the worldvolume is compact). This is in accordance with the belief that the field theory with space-time noncommutativity may not be unitary [18]. As a result, the low energy field theory of the bound state $(F_1, D_1, D_3)$ in the NCSYM limit is the $\mathcal{N}=4$ NCSYM without space-time noncommutativity, and only the spatial coordinates $(\tilde{x}_{2,3})$ are noncommutative. In addition, we note that the NCSYM limit implies that $\theta \rightarrow \pi/2$ and $\varphi$ is arbitrary in this case. So the decoupling geometry goes to that of the bound state $(F_1, D_1)$ with two smeared coordinates as $au \gg 1$.

For low energies, $au \ll 1$, the geometry is that of ordinary SYM, $AdS_5 \times S_5$. It significantly deviates from that for high energies, and the deviation appears at the scale of $u \sim 1/a = R/\sqrt{\tilde{b}}$.

### 3.3 NCOS limit

The NCOS limit in the dual supergravity is drastically different from the NCSYM limit, and we study how such a limit can be uniquely determined in our setting.
In order to get NCOS limit, we should keep $\Theta^{01}$ finite. This means that the electric field $e$ should tend to its critical value and other quantities should scale as given in eq. (2.12). Note that $\alpha' G^{ij} \neq 0$ in this limit and the oscillating modes of open string do not decouple. This critical behavior is translated in the supergravity solution (3.2) into

$$\cos \varphi = \left(\frac{\alpha'}{b}\right)^{n/2},$$

with $\theta$ kept finite.

Next suppose the scaling behavior of $r$ is given as

$$r = \alpha'^m u. \quad (3.25)$$

We would like to make all our metric in (3.2) scale as $\alpha'$. First consider the $dr^2$ term. This is transformed into

$$F^{1/2} dr^2 = \left(1 + \frac{R^4}{\alpha'^{4m-2} u^4}\right)^{1/2} \alpha'^{2m} du^2, \quad (3.26)$$

where $R^4 = 4\pi \tilde{g} N_3/(\tilde{b}^{n/2} \cos \theta)$ and $g_s = \tilde{g} \alpha'^{-n/2}$. We thus see that as long as $m \geq \frac{1}{2}$, this scales as

$$\alpha' \left(1 + \frac{R^4}{u^4}\right)^{1/2} du^2, \quad \text{for } m = \frac{1}{2},$$

$$\alpha' \frac{R^2}{u^2} du^2, \quad \text{for } m > \frac{1}{2}. \quad (3.27)$$

Thus all values $m \geq \frac{1}{2}$ are allowed at this point.

We next examine the behavior of other components of the metric.

$$H \sim \frac{\tilde{b}^n}{\alpha'^{4m-2+n}} \frac{R^4}{u^4},$$

$$G = 1 + \frac{\cos^2 \theta}{\alpha'^{4m-2}} \frac{R^4}{u^4}. \quad (3.28)$$

From eqs. (3.27) and (3.28), we see that all nontrivial functions of $u$ disappear from the solution for $m > \frac{1}{2}$, and the resulting metric is $AdS_5 \times S_5$ up to rescaling of the coordinates. This is supposed to correspond to the SYM limit and not the limit we are looking for. Thus we are uniquely lead to the special scaling $m = \frac{1}{2}$:

$$r = \sqrt{\alpha'} u. \quad (3.29)$$
Note that the parameter $n$ remains arbitrary as long as it is positive.

Having understood how everything scales, we find that the dual gravity solution corresponding to the NCOS is given by

$$ds^2 = \alpha' \tilde{F}^{1/2} \left[ \frac{u^4}{R^4} (-\tilde{f} d\tilde{x}_0^2 + d\tilde{x}_1^2) + \frac{1}{\tilde{G}} (d\tilde{x}_2^2 + d\tilde{x}_3^2) + \tilde{f}^{-1} du^2 + u^2 d\Omega_5^2 \right], \quad (3.30)$$

where

$$\tilde{F} = 1 + \frac{R^4}{u^4}, \quad \tilde{G} = 1 + \frac{R^4 \cos^2 \theta}{u^4}.$$  \quad (3.31)

and the coordinates are rescaled as

$$x_{0,1} = \frac{\tilde{b}^{n/2}}{\alpha'(n-1)/2} \tilde{x}_{0,1}, \quad x_{2,3} = \sqrt{\alpha'} \tilde{x}_{2,3}.$$  \quad (3.32)

The dilaton, axion and $B$ fields are

$$e^{2\phi} = \tilde{g}^2 \frac{\tilde{F}^2}{\tilde{b}^n R^4}, \quad \chi = \frac{\tilde{b}^{n/2} \sin \theta}{\tilde{g} \tilde{F}},$$

$$B_{01} = \alpha' u^4/R^4, \quad B_{23} = \alpha' \tan \theta / \tilde{G}.$$  \quad (3.33)

We note that the axion field is nonvanishing here, which is quite different from other solutions.

The geometry (3.30) is supposed to be the supergravity dual of the NCOS theory with both space-time and space-space noncommutativities. It can be regarded as an extension of [14], in which the supergravity dual of NCOS has been given with only space-time noncommutativity by applying the S-duality to the supergravity dual of NCSYM. In the NCOS limit, the electric field approaches its critical value while the magnetic field remains finite. The critical electric field leads to the space-time noncommutativity; the magnetic field also gives rise to space-space noncommutativity although it is finite.

Again the geometry is $AdS_5 \times S_5$, that of ordinary SYM for small $u$, which indicates the low energy limit of NCOS is also the ordinary SYM. For large $u$, it deviates in two ways from that, one (due to $\tilde{F}$) involving space-time coordinates, the other (due to $\tilde{G}$) space-space coordinates. These are the reflections of space-time and space-space noncommutativities, and they arise at the scales of $R$ and $R \sqrt{\cos \theta}$, respectively.
4 \( SL(2, \mathbb{Z}) \) duality

It is well known that IIB superstring has the \( SL(2, \mathbb{Z}) \) symmetry and its low energy approximation, IIB supergravity, has also this symmetry. It is interesting to consider relations of the above theories in different decoupling limits under the \( SL(2, \mathbb{Z}) \) transformation.

As it is already pointed out in ref. [14], the S-dual of the NCSYM gives a NCOS theory without space-space noncommutativity. This case is simple because the axion field \( \chi \) vanishes for the supergravity dual of NCSYM and the S-duality is achieved just by taking a simple inverse of the dilaton field. This relation is also natural because the NCSYM theory is affected only by the magnetic component of the \( B \) field, while the NCOS with \( \theta = 0 \) has only the electric background. The question we are asking here is what is the general situation. It might appear that this is again simple, but the above consideration suggests that these two theories cannot be just S-dual of each other since the axion field does not vanish for the NCOS theory. Indeed we find that the relation is rather nontrivial.

In the course of our work, two papers [26, 27] have appeared where similar topics were discussed. In ref. [26], it has been claimed that the S-dual of the NCOS theory is always a NCSYM, but the S-dual of the NCSYM theory is singular in some cases and does not always give a NCOS even though S-duality is an invertible transformation. When the axion field is nonvanishing, it is more appropriate to consider the \( SL(2, \mathbb{Z}) \) transformation. We show here that under the \( SL(2, \mathbb{Z}) \) transformation, the NCSYM is always mapped into the NCOS since the axion field in the NCSYM vanishes. The converse is rather nontrivial. Under the \( SL(2, \mathbb{Z}) \) transformation, we find that the NCOS theory transforms in general into another NCOS, and for some special case this reduces to a NCSYM theory. In ref. [27], the authors have discussed this problem from the open string point of view. Our discussion is in terms of dual supergravity description.\(^3\)

Our IIB supergravity has an \( SL(2, \mathbb{Z}) \) invariance. The metric in the Einstein frame is inert under this transformation, so that two solutions after a general \( SL(2, \mathbb{Z}) \) transfor-

\(^3\)More recently, related discussion on S-duality has been given in refs. [28, 29].
Information are related as
\[
ds_E^2 = e^{-\phi/2}ds_{st,1}^2 = e^{-\phi'/2}ds_{st,2}^2,
\] (4.1)
where the latter two expressions are two string-frame metrics related by \(SL(2, \mathbb{Z})\) transformations, and \(\phi, \phi'\) are the dilatons for each solution. Using the notation \(\tau = \chi + ie^{-\phi}\), they are related by
\[
\tau' = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1,
\] (4.2)
for integers \(a, b, c, d\) (the usual S-duality corresponds to the case \(\tau' = -1/\tau\)). This gives us
\[
e^{-\phi'} = \frac{e^{-\phi}}{|c\tau + d|^2},
\] (4.3)
resulting in
\[
ds_{st,2}^2 = |c\tau + d|ds_{st,1}^2.
\] (4.4)

Let us first discuss the \(SL(2, \mathbb{Z})\) transformation of the SYM theory (3.12). Since the axion field vanishes and the dilaton is a constant, the supergravity dual \(AdS_5 \times S^5\) is still of the form \(AdS_5 \times S^5\) after the \(SL(2, \mathbb{Z})\) transformation. This means that the ordinary SYM theory is itself again, but with different coupling constant
\[
\tilde{g}' = \tilde{g}(d^2 + c^2\tilde{g}^2).
\] (4.5)

Next, for the NCSYM (3.22), the axion is again zero and
\[
\tau = i/\tilde{h}^{1/2}\tilde{g}\tilde{b}.
\] (4.6)

The \(SL(2, \mathbb{Z})\) transformation then leads to
\[
ds^2 = \alpha'\tilde{h}'^{-1/2}\left[\frac{u^2}{\tilde{R}^2}(-\tilde{f}\tilde{dx}_0^2 + \tilde{dx}_1^2) + \tilde{h}(\tilde{dx}_2^2 + \tilde{dx}_3^2) + \frac{\tilde{R}^2}{u^2}f^{-1}du^2 + u^2d\Omega_5^2\right],
\]
\[
\tilde{h}'^{-1} = d^2 + \frac{c^2}{\tilde{g}^2\tilde{b}^2}\tilde{h}^{-1}, \quad e^{2\phi} = \tilde{g}^2\tilde{b}^2\tilde{h}\tilde{h}'^{-2}.
\] (4.7)

This metric can be cast into the form (3.30) with
\[
\cos^2 \theta' = \frac{c^2}{c^2 + (db\tilde{g})^2},
\] (4.8)
up to coordinate and parameter rescaling. This means that the NCSYM transforms into a NCOS with both space-time and space-space noncommutativities after the transformation. We can choose \( d = 0 \) in this case, and then space-space noncommutativity disappears. This is the S-duality transformation considered in ref. [14].

Finally let us consider the NCOS theory (3.30) with both space-time and space-space noncommutativities where the axion field does not vanish. In the above two cases, the simple S-duality \( \tau \to -\frac{1}{\tau} \) was useful to get information on their relations, but here it is important to consider the \( SL(2, \mathbb{Z}) \) transformation with

\[
\tau = \frac{\tilde{b}^{n/2} \sin \theta}{\tilde{g} \tilde{F}} + i \frac{\tilde{b}^{n/2} \tilde{G}^{1/2} R^2}{\tilde{g} \tilde{F} u^2}.
\]

The factor in eq. (4.4) then becomes

\[
|c\tau + d| = \left( (c\tilde{b}^{n/2} \sin \theta + d\tilde{g} \tilde{F})^2 + c^2 \tilde{b}^n \tilde{G} R^4 u^4 \right)^{1/2} \frac{1}{\tilde{g} \tilde{F}},
\]

which, with the help of eq. (3.31), is transformed into

\[
\frac{\tilde{F}^{1/2}}{\tilde{F}^{1/2}},
\]

where

\[
\tilde{F} \equiv \left( d + \frac{\tilde{c} \tilde{b}^{n/2}}{\tilde{g}} \sin \theta \right)^2 + \left( d^2 + \frac{c^2 \tilde{b}^n}{\tilde{g}^2} \cos^2 \theta \right) \frac{R^4}{u^4}.
\]

When eq. (4.11) is used in eqs. (4.4) and (3.30), we find that the \( SL(2, \mathbb{Z}) \)-transformed solution is the same as the original one (3.30) with \( \tilde{F} \) replaced by \( \tilde{F} \). New \( \theta' \) is given by

\[
\cos^2 \theta' = \frac{(d\tilde{g} + c\tilde{b}^{n/2} \sin \theta)^2}{d^2 \tilde{g}^2 + c^2 \tilde{b}^n \cos^2 \theta} \cos^2 \theta.
\]

We thus find that a NCOS theory transforms into another NCOS with different noncommutativity parameter under the \( SL(2, \mathbb{Z}) \) transformation. In general both the NCOS’s have space-time and space-space noncommutativities. However, in a special case when the first term in eq. (4.12) vanishes,

\[
d + \frac{\tilde{c} \tilde{b}^{n/2}}{\tilde{g}} \sin \theta = 0,
\]

we find that the \( SL(2, \mathbb{Z}) \)-transformed solution reduces to the form (3.22), which describes a NCSYM. This is possible only for the case in which the asymptotic value of the axion
\[ \frac{\beta}{g} \] is a rational number, in agreement with the conclusion in ref. [27] derived from the open string point of view. This also includes the case \( \theta = 0 \) discussed in ref. [14]. In this case one can also reach the same conclusion by using a simple S-duality.

5 Conclusions and Discussions

In this paper, we have considered in the framework of IIB supergravity three different scaling limits for the bound state (F1, D1, D3) and \( SL(2, \mathbb{Z}) \) transformations of the resulting theories. These three theories are, respectively, the ordinary \( \mathcal{N}=4 \) SYM with or without \( N_F \) units of electric flux and \( N_1 \) units of magnetic flux depending on whether the worldvolume of D3-brane is compact or not (here \( N_1 \) and \( N_F \) are, respectively, the numbers of D-string and F-string in the bound state); the NCSYM with or without \( N_F \) units of electric flux; and NCOS with both space-time and space-space noncommutativities. Under a general \( SL(2, \mathbb{Z}) \) transformation, the gravity dual \( AdS_5 \times S^5 \) of the SYM still have the same form up to the rescaling of parameters and coordinates. This implies that the SYM becomes another SYM with different coupling constants after the transformation. The gravity dual of NCSYM takes the form of the NCOS in general with both the space-time and space-space noncommutativities. This means that a NCSYM transforms into a NCOS after the \( SL(2, \mathbb{Z}) \) transformation. Finally, the gravity dual of NCOS with both space-time and space-space noncommutativities remains in the same form after the \( SL(2, \mathbb{Z}) \) transformation. Our result implies that in general a NCOS transforms into another NCOS with different noncommutativity parameter. In a special case when the asymptotic value of the axion is a rational number, it is possible to transform a NCOS into a NCSYM, and when the asymptotic value vanishes, one can reach this conclusion by using a simple S-duality alone.

As mentioned before, the thermodynamic quantities (3.4) are independent of the parameter angles \( \varphi \) and \( \theta \). This implies that thermodynamics are the same for three different theories, SYM, NCSYM, and NCOS, in this supergravity approximation. Also the thermodynamics remains unchanged under the \( SL(2, \mathbb{Z}) \) transformation, since the latter does not change the form of Einstein metric.

From the solution (3.2) we see that when \( \theta = \pi/2 \) and \( \varphi \) is arbitrary, the bound
state solution goes to the one for the (F1, D1) bound state. Taking a critical electric field limit, one has a 2-dimensional NCOS from the bound state (F1, D1) [14, 22]. From our previous discussions on the $SL(2, \mathbb{Z})$ transformation, one can see that in general the 2-dimensional NCOS becomes another 2-dimensional NCOS with different noncommutativity parameters, as in the case of 4-dimensions. When the asymptotic value of the axion is a rational number, it is possible to transform the 2-dimensional NCOS into an ordinary 2-dimensional SYM with a quantized electric flux [22]. Of course, when the asymptotic value of the axion vanishes, one can transform the 2-dimensional NCOS into a 2-dimensional SYM only by using S-duality, which is precisely the case in [22].

An interesting extension of our work is to consider more general solutions like (F1, D1, NS5, D5) bound states and their scaling limits. We suspect that in the bound state, SYM, NCSYM, NCOS and the so-called little string theory are connected with each other through the $SL(2, \mathbb{Z})$ transformation, which is currently under investigation.

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