T-duality of axial and vector dyonic integrable models

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ABSTRACT

A general construction of affine Non Abelian Toda(NA) models in terms of axial and vector gauged two loop WZNW model is discussed. We study the off-critical T-duality between certain families of axial and vector type of integrable models for the case of affine NA-Toda theories with one global U(1) symmetry. In particular we find the Lie algebraic condition defining a subclass of T-selfdual torsionless NA Toda models and their zero curvature representation.

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1 Introduction

Two dimensional integrable models represent an important laboratory for testing new ideas and developing new methods for constructing exact solutions as well as for the nonperturbative quantization of 4-D non-abelian gauge theories, gravity and string theory. Among the numerous technics for constructing 2-D integrable models(IM’s) and their solutions [1], [2], the two loop $\hat{G}$ -WZNW and gauged $\hat{G}/\hat{H}$-WZNW models [3] have the advantage in providing a simple and universal method for derivation of the zero curvature representation and a consistent path integral formalism for their description as well. The power of such method was demonstrated in constructing (multi) soliton solution of the abelian affine Toda models [3] and certain nonsingular nonabelian (NA) affine Toda models [4].

The present paper is devoted to the systematic construction of the simplest class of singular torsionless affine NA Toda models characterized by the fact that the space of physical fields $g_0^F$ lies in the coset $\mathcal{G}_0/\mathcal{G}_{0} = \frac{SL(2)}{U(1)} \otimes U(1)^{rank G-1}$. Our main result is that such models exists only for the following three affine Kac-Moody algebras, $B_n^{(1)}$, $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ under certain specific restrictions on the choice of the subgroup $\mathcal{G}_0$ (i.e. equivalently the choice of gradation $Q$ and constant grade $\pm 1$ elements $\epsilon_\pm$). It turns out that these models are T-selfdual (i.e. the axial and the vector gauging of the $U(1)$ factor in the coset $\frac{SL(2)}{U(1)} \otimes U(1)^{rank G-1}$ leads to identical actions ). They appear to be natural generalization of the Lund-Regge (“complex Sine-Gordon”) model [5] and exactly reproduce the family of models proposed by Fateev [6]. Our construction provide a simple proof of their classical integrability. The generalization of the conformal abelian T-duality to the family of axial and vector models IM’s is discussed.

It is important to mention that relaxing the structure of the coset $\mathcal{G}_0/\mathcal{G}_{0} = \frac{SL(2)}{U(1)} \otimes U(1)^{rank G-1}$, i.e. gauging specific combinations of the Cartan subalgebra of $\mathcal{G}_0$ we find two new families of integrable models, axionic (for axial gauging) and torsionless (for vector gauging) for all affine (twisted and untwisted ) Kac-Moody algebras which are T-dual (but not self dual)[8].

An important motivation for the construction of the above singular NA Toda models is the fact that their soliton solutions (for imaginary coupling )carries both electric and magnetic (topological ) charges and have properties quite similar to the 4-D dyons of the Yang-Mills-Higgs model [8],[16].

This paper is organized as folows.Sect.2 contains the functional integral derivation of the effective actions for generic conformal, affine and conformal affine NA -Toda theories.In the particular case when these models manifest $\mathcal{G}_0^0=U(1)$ gauge symmetry ,the actions for the corresponding singular affine NA Toda models of axial and vector type are obtained.Two particular examples based on the affine algebras $A_r^{(1)}$ and $B_n^{(1)}$ are presented.Sect.3 is devoted to the analysis of the abelian off-critical T-duality relating the axial and vector type of IM’s. We derive the Lie algebraic condition deffining the family of T-selfdual torsionless singular affine NA Toda theories in Sect.4.These are the IM’s based on the following affine Kac-Moody algebras $B_n^{(1)}$, $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$. In Sect.5 we present the zero curvature representations of all IM’s in consideration.
2 Gauged WZNW Construction of NA Toda Models

The generic NA Toda models are classified according to a $G_0 \subset G$ embedding induced by the grading operator $Q$ decomposing an finite or infinite dimensional Lie algebra $G = \oplus_i \mathcal{G}_i$ where $[Q, \mathcal{G}_i] = i \mathcal{G}_i$ and $[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}$. A group element $g$ can then be written in terms of the Gauss decomposition as

$$g = NBM$$

where $N = \exp \mathcal{G}_< \in H_-$, $B = \exp \mathcal{G}_0$ and $M = \exp \mathcal{G}_> \in H_+$. The physical fields $B$ lie in the zero grade subgroup $\mathcal{G}_0$ and the models we seek correspond to the coset $H_- \backslash G/H_+$. For consistency with the hamiltonian reduction formalism, the phase space of the $G$-invariant WZNW model is reduced by specifying the constant generators $\epsilon_\pm$ of grade $\pm 1$. In order to derive an action for $B \in \mathcal{G}_0$, invariant under

$$g \rightarrow g' = \alpha_- g \alpha_+, \quad (2.2)$$

where $\alpha_+(z, \bar{z}) \in H_+$ we have to introduce a set of auxiliary gauge fields $A \in \mathcal{G}_<$ and $\bar{A} \in \mathcal{G}_>$ transforming as

$$A \rightarrow A' = \alpha_- A \alpha_-^{-1} + \alpha_- \partial A^{-1}, \quad \bar{A} \rightarrow \bar{A}' = \alpha_+^{-1} \bar{A} \alpha_+ + \bar{\alpha}_+ \bar{\alpha}_-^{-1} \alpha_+. \quad (2.3)$$

The result is given by the gauged WZNW action,

$$S_{G/H}(g, A, \bar{A}) = S_{WZNW}(g) - \frac{k}{2\pi} \int dz^2 Tr \left( A(\partial gg^{-1} - \epsilon_+) + \bar{A}(g^{-1} \partial g - \epsilon_-) + Ag\bar{A}g^{-1} \right). \quad (2.5)$$

Since the action $S_{G/H}$ is $H$-invariant, we may choose $\alpha_- = N_-^{-1}$ and $\alpha_+ = M_+^{-1}$. From the orthogonality of the graded subspaces, i.e. $Tr(\mathcal{G}_i \mathcal{G}_j) = 0, i + j \neq 0$, we find

$$S_{G/H}(g, A, \bar{A}) = S_{G/H}(B, A', \bar{A}') = S_{WZNW}(B) - \frac{k}{2\pi} \int dz^2 Tr[A' \epsilon_+ + \bar{A}' \epsilon_- + A'B\bar{A}'B^{-1}], \quad (2.4)$$

where

$$S_{WZNW} = -\frac{k}{4\pi} \int \partial_z Tr(g^{-1} \partial gg^{-1} \partial g) - \frac{k}{24\pi} \int D \epsilon_{ijk} Tr(g^{-1} \partial_i gg^{-1} \partial_j gg^{-1} \partial_k g), \quad (2.5)$$

and the topological term denotes a surface integral over a ball $D$ identified as space-time.

Action (2.4) describe the non singular Toda models among which we find the Conformal and the Affine abelian Toda models where $Q = \sum_{i=1}^r \frac{2\lambda_i H}{\alpha_i^2}$, $\epsilon_\pm = \sum_{i=1}^r c_{\pm i} E_{\pm \alpha_i}$, and $Q = \sum_{i=1}^r \frac{2\lambda_i H}{\alpha_i^2}$, $\epsilon_\pm = \sum_{i=1}^r c_{\pm i} E_{\pm \alpha_i} + E_{\pm 00}^{(0)}$ respectively, where $-\alpha_0$ denotes the highest root, $\lambda_i$, the fundamental weights, $h$ the coxeter number of $\mathcal{G}$ and $H_i$ are the Cartan subalgebra generators in the Cartan Weyl basis satisfying $Tr(H_i H_j) = \delta_{ij}$.

Performing the integration over the auxiliary fields $A$ and $\bar{A}$, the functional integral

$$Z_\pm = \int DA_- D\bar{A}_+ \exp(-F_\pm), \quad (2.6)$$

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where
\[ F_\pm = -\frac{k}{2\pi} \int \left( Tr(A - B \epsilon_+B^{-1})B(\bar{A} - B^{-1}\epsilon_+B)B^{-1} \right) d^2x \] (2.7)
yields the effective action
\[ S = S_{WZNW}(B) - \frac{k}{2\pi} \int Tr(\epsilon_+B\epsilon_-B^{-1}) \] (2.8)
The action (2.8) describe integrable perturbations of the $G_0$-WZNW model. Those perturbations are classified in terms of the possible constant grade $\pm 1$ operators $\epsilon_\pm$.

More interesting cases arise in connection with non abelian embeddings $G_0 \subset G$. In particular, if we suppress one of the fundamental weights from $Q$, the zero grade subspace acquires a nonabelian structure $sl(2) \otimes u(1)^{rank\mathcal{G} - 1}$. Let us consider for instance $Q = h'd + \sum_{i \neq a} \frac{2\alpha_iH}{\alpha_i}$. where $h' = 0$ or $h' \neq 0$ corresponding to the Conformal or Affine nonabelian (NA) Toda respectively. The absence of $\lambda_a$ in $Q$ prevents the contribution of the simple root step operator $E^{(0)}_{\alpha_0}$ in constructing $\epsilon_+$. It in fact, allows for reducing the phase space even further. This fact can be understood by enforcing the nonlocal constraint $J_{Y,H} = J_{Y,H} = 0$ where $Y$ is such that $[Y \cdot H, \epsilon_\pm] = 0$ and $J = g^{-1}\partial gg$ and $\bar{J} = -\bar{\partial}gg^{-1}$. Those generators of $G_0$ commuting with $\epsilon_\pm$ define a subalgebra $G_0^0 \subset G_0$. Since the potential in (2.8) is invariant under transformations generated by $G_0^0$, it allows the construction of an action invariant under
\[ g \longrightarrow g' = \alpha_0g\alpha_0'. \] (2.9)
Two inequivalent cases arise, namely, the axial gauging where $\alpha_0' = \alpha_0(z, \bar{z}) \in G_0^0$ and the vector gauging, where $\alpha_0' = \alpha_0^{-1}(z, \bar{z}) \in G_0^0$. Auxiliary gauge fields $A_0 = a_0Y \cdot H$ and $\bar{A}_0 = \bar{a}_0Y \cdot H \in G_0^0$ are introduced to construct an invariant action under transformations (2.9)
\[ S(B, A_0, \bar{A}_0) = S(g_0^{f}, A_0, \bar{A}_0) = S_{WZNW}(B) - \frac{k}{2\pi} \int Tr \left( \pm A_0 \bar{\partial}BB^{-1} + \bar{A}_0B^{-1}\partial B \pm A_0B\bar{A}_0B^{-1} + A_0\bar{A}_0 \right) - \frac{k}{2\pi} \int Tr\epsilon_+B\epsilon_-B^{-1} \] (2.10)
where the auxiliary fields transform as
\[ A_0 \longrightarrow A'_0 = A_0 - \alpha_0^{-1}\partial\alpha_0, \quad \bar{A}_0 \longrightarrow \bar{A}'_0 = \bar{A}_0 - \bar{\partial}\alpha'_0(\alpha'_0)^{-1}. \]
and the $\pm$ signs correspond to axial/vector gaugings respectively. Both, axial and vector theories are in fact related by dual transformation in gauge fields (apart from the quadratic counter term). Such residual gauge symmetry allows us to eliminate an extra field. Notice that the physical fields $g_0^{f}$ lie in the coset $G_0/G_0^0 = (sl(2) \otimes u(1)^{rank\mathcal{G} - 1})/u(1)$ of dimension $rank\mathcal{G} + 1$ and are classified according to the gradation $Q$. It therefore follows that $S(B, A_0, \bar{A}_0) = S(g_0^{f}, A'_0, \bar{A}'_0)$.

In [9] a detailed study of the gauged WZNW construction for finite dimensional Lie algebras leading to Conformal NA Toda models was presented. The study of its symmetries
was given in refs. [10] and in [11]. Here we generalize the construction of ref. [9] to infinite dimensional Lie algebras leading to NA Affine Toda models characterized by the broken conformal symmetry and by the presence of solitons.

Consider the Kac-Moody algebra $\hat{G}$

$$[T^a_m, T^b_n] = f^{abc} T^c_{m+n} + \hat{c} m \delta_{m+n} \delta^{ab}$$

$$[\hat{d}, T^a_n] = n T^a_n; \quad [\hat{c}, T^a_n] = [\hat{c}, \hat{d}] = 0$$

(2.11)

The NA Toda models we shall be constructing are associated to gradations of the type $Q_a(h') = h'd + \sum_{i \neq a} \frac{2}{\alpha^2} \hat{h}_i H_i$, where $h'$ is chosen such that the zero grade subalgebra $\hat{G}_0$ defined by $Q_a(h')$ acting on $\hat{G}$, coincide with the zero grade subalgebra $G_0$ defined by $Q_a(h' = 0)$ when acting on $G$ (apart from two commuting generators $\hat{c}$ and $\hat{d}$). Since they commute with $G_0$, their kinetic term decouples such that the conformal and the affine singular NA-Toda models differ only by the potential term.

The integration over the auxiliary gauge fields $A_0$ and $\bar{A}_0$ require explicit parametrization of $B$,

$$B = \exp(\bar{\chi} E_{-\alpha_a}) \exp(RY^j H_j + \Phi(H) + \nu \hat{c} + \eta \hat{d}) \exp(\bar{\psi} E_{\alpha_a})$$

(2.12)

where $\Phi(H) = \sum_{r=1}^r \sum_{i=1}^{r-1} \phi_i X^j_i \hat{h}_j$, $Y \cdot X_i = \sum_{j=1}^r Y^j X^j_i = 0$, $i = 1, \ldots, r - 1$ and $h_j = \frac{2\alpha_j H_j}{\alpha_j^2}$, $j = 1, \ldots, r$.

### 2.1 Axial Gauging

After gauging away the nonlocal field $R$ according to eqn. (2.9) with $\alpha'_a = \alpha_0 = e^{-\frac{1}{\alpha^2} Y \cdot H R}$, the factor group element becomes

$$g_0^f = \exp(\chi E_{-\alpha_a}) \exp(\Phi(H) + \nu \hat{c} + \eta \hat{d}) \exp(\bar{\psi} E_{\alpha_a})$$

(2.13)

where $\chi = \bar{\chi} e^{\frac{1}{\alpha^2} Y \cdot a R}$, $\psi = \bar{\psi} e^{\frac{1}{\alpha^2} Y \cdot a R}$. We therefore get for the zero grade component of the action (2.10)

$$F_0 = -\frac{k}{2\pi} \int Tr \left( A_0 \bar{\partial} g_0^f (g_0^f)^{-1} + \bar{A}_0 (g_0^f)^{-1} \partial g_0^f + A_0 g_0^f \bar{A}_0 (g_0^f)^{-1} + A_0 A_0 \right)$$

$$= -\frac{k}{2\pi} \int \left( a_0 \bar{a}_0 2Y^2 \Delta - \left( \frac{2\alpha_a Y}{\alpha_a^2} \right) (\bar{a}_0 \bar{\psi} \partial \chi + a_0 \chi \bar{\partial} \psi) e^{\bar{\Phi}(\alpha_a)} \right)$$

(2.14)

where $\Delta_a = 1 + \frac{(Y \cdot a a)^2}{\alpha_a^2} \bar{\psi} \chi e^{\bar{\Phi}(\alpha_a)}$ and $[\Phi(H), E_{\alpha_a}] = \Phi(\alpha_a) E_{\alpha_a}$.

The effective action is obtained by integrating over the auxiliary fields $A_0$ and $\bar{A}_0$

$$Z_0 = \int D A_0 D \bar{A}_0 \exp(F_0) \sim e^{-S_0}$$

(2.15)
where \( S_0 = -\frac{k}{2\pi}(2\gamma_a^2)\int \frac{\psi\bar{\psi}\partial\psi\partial\bar{\psi}}{2\gamma^2} e^{2\Phi(a_0)} \). The total effective action (2.10) is therefore given as

\[
S_{\text{eff}} = -\frac{k}{4\pi} \int \left( \text{Tr} (\partial\Phi(H)\bar{\partial}\Phi(H)) + \frac{4}{\alpha^2} \bar{\partial}\psi\partial\chi + \varphi_0 \alpha \varphi_0 \partial\psi\partial\chi - 2T(\varepsilon \bar{g}_0 \varepsilon (g_0)^{-1}) \right)
\]

(2.16)

Note that the second term in (2.16) contains both symmetric and antisymmetric parts:

\[
\bar{\partial}\psi\partial\chi = \frac{\alpha}{\Delta} (g^{\mu\nu} \partial\mu \psi \partial\nu \chi + \epsilon_{\mu\nu} \partial\mu \psi \partial\nu \chi),
\]

(2.17)

where \( g_{\mu\nu} \) is the 2-D metric of signature \( g_{\mu\nu} = \text{diag}(1,-1) \), \( \partial = \partial_0 + \partial_1 \), \( \bar{\partial} = \partial_0 - \partial_1 \). For \( r = 1 \) (\( G \equiv A_1 \), \( \Phi(a_0) \) is zero) the antisymmetric term is a total derivative:

\[
\epsilon_{\mu\nu} \partial\mu \psi \partial\nu \chi = \frac{1}{2} \epsilon_{\mu\nu} \partial\mu \left( \ln \{1 + \psi\chi\} \partial\nu \ln \frac{\psi}{\chi} \right),
\]

(2.18)

and it can be neglected. This \( A_1 \)-NA-Toda model (in the conformal case), is known to describe the 2-D black hole solution for (2-D) string theory [12]. The G-NA conformal Toda model can be used in the description of specific \((r+1)\)-dimensional black string theories [14], with \(r\)-1-flat and 2-non flat directions \((g^{\mu\nu} G_{ab}(X) \partial_\mu X^a \partial_\nu X^b, X^a = (\psi, \chi, \varphi_i))\), containing axions \((\epsilon_{\mu\nu} B_{ab}(X) \partial_\mu X^a \partial_\nu X^b)\) and tachions \((\exp \{-\epsilon_{ij}\varphi_j\})\), as well.

It is clear that the presence of the \( \Phi(a_0) \) in (2.16) is responsible for the antisymmetric tensor generating the axionic terms. On the other hand, notice that \( \Phi(a_0) \) depend upon the subsidiary nonlocal constraint \( J_{Y: H} = \bar{J}_{Y: H} = 0 \) and hence on the choice of the vector \( Y \). It is defined to be orthonal to all roots contained in \( \epsilon_\pm \).

### 2.2 Vector Gauging

The vector gauging is implemented by \( \alpha'_0 = \alpha_0^{-1} = e^{\frac{1}{2}Y \cdot RR'} \), where \(-e^{\frac{1}{2}Y \cdot RR'} \bar{\chi} = e^{-\frac{1}{2}Y \cdot a RR'} \bar{\psi} = \bar{\chi} \). The factor group element may be then parametrized as

\[
g_0' = \exp(-\bar{\chi} E_{-a}) \exp(\Phi(H) + \nu \bar{c} + \eta \bar{d}) \exp(\bar{\chi} E_{a})
\]

(2.19)

where \( \Phi(H) = Y \cdot HR + \sum_{j=1}^{r-1} \varphi_j X_j^i h_i \). The zero grade component of the action (2.10) then becomes

\[
F_0 = -\frac{k}{2\pi} \int \left( -\frac{2}{\alpha^2} (Y \cdot \alpha a_0 + \varphi_0 \bar{\chi} e^{\frac{1}{2}Y \cdot RR'} \bar{\chi} - \bar{\chi} \bar{\varphi}_0 \alpha \varphi_0 \partial\psi\partial\chi - \bar{\alpha}_0 (Y a \varphi_0 \partial\psi\partial\chi) \partial\psi\partial\chi e^{\Phi(a_0)} \right)
\]

\[
+ a_0 (Y \partial\psi\partial\chi + \frac{2}{\alpha^2} \bar{\chi} \partial\psi\partial\chi e^{\Phi(a_0)})
\]

(2.20)

Integrating over \( a_0 \) and \( \bar{a}_0 \), we find for the total effective action (2.10),

\[
S_{\text{eff}} = -\frac{k}{4\pi} \int \sum_{i,j=1}^{r-1} \text{Tr} (X_i \cdot h X_j \cdot h) \partial\varphi_i \partial\varphi_j + Y \partial\psi\partial\chi + \partial\eta \partial\nu \partial\varphi_0 \partial\psi\partial\chi + \partial\eta \partial\varphi_0 \partial\psi\partial\chi - 2T(\varepsilon \bar{g}_0 \varepsilon (g_0)^{-1})
\]

(2.21)
where \( \rho(\alpha) = \frac{Y^2 \alpha^2}{2(Y - \alpha)} \). Defining the new variables

\[
E = e^{dR}, \quad F = E^{-1}(1 - c\chi^2 \exp \Phi(\alpha))
\]

(2.22)

the effective action (2.21) becomes

\[
S_{\text{eff}} = -\frac{k}{2\pi} \int \left( \frac{1}{2} \sum_{i,j=1}^{r-1} Tr(X_i \cdot hX_j \cdot h) \partial \varphi_i \partial \varphi_j + \frac{1}{2} \partial R \partial R (Y^2 + 2d^2 \rho(\alpha)) \right)
\]

\[
- d\Gamma(\partial R \partial \Phi(\alpha_a) + \bar{\partial} R \partial \Phi(\alpha_a)) + \partial \eta \bar{\partial} \nu + \partial \nu \bar{\partial} \eta - \Gamma \frac{(\partial E \bar{\partial} F + \bar{\partial} E \partial F)}{1 - EF}
\]

\[
- Tr(\hat{\epsilon} + g_0^f \hat{\epsilon} - (g_0^f)^{-1})
\]

(2.23)

where \( 2d\Gamma = \frac{2}{\alpha^2} \rho(\alpha) \) and \( 2c\Gamma = \frac{2}{\alpha^2} \) are chosen in order to eliminate the variable \( \bar{\chi} \). Notice that the \( E, F \)- term in the action (2.23) is symmetric. The vector gauging, therefore provides a construction of torsionless actions dual to its axionic counterpart. This fact raises the question whether exists self-dual torsionless actions, i.e. when the axial and vector gauging leads to same action.

2.2.1 Example1. Torsionless \( B_r \) model

Let \( Y = \frac{2\lambda_r}{\alpha_r} - \frac{2\lambda_{r-1}}{\alpha_{r-1}} = e_r, \ a = r, \ \rho(\alpha_r) = \frac{1}{2} \) where

\[
\lambda_r = \frac{1}{2}(e_1 + e_2 + \cdots + e_r), \ \lambda_{r-1} = (e_1 + e_2 + \cdots + e_{r-1})
\]

(2.24)

Parametrizing \( \Phi(H) = \sum_{i=1}^{r-1} \varphi_i e_i \cdot H + R e_r \cdot H \), we find

\[
S_{\text{eff}} = -\frac{k}{2\pi} \int \left( \frac{1}{2} \sum_{i,j=1}^{r-1} \partial \varphi_i \bar{\partial} \varphi_i + \partial \eta \bar{\partial} \nu + \partial \nu \bar{\partial} \eta - \frac{(\partial E \bar{\partial} F + \bar{\partial} E \partial F)}{1 - EF} \right)
\]

\[
- Tr(\hat{\epsilon} + g_0^f \hat{\epsilon} - (g_0^f)^{-1})
\]

(2.25)

where we have chosen \( \Gamma = 1, \ d = -\frac{1}{2}, \ c = -1 \) in order to eliminate terms \( \partial R \bar{\partial} R \).

2.2.2 Example2. \( A_r \) vector model

Let \( Y = \lambda_1 \). We will get a simpler and cleaner result if we parametrize \( \Phi(H) = \sum_{i=1}^r \varphi_i h_i \).

\[
S_{\text{eff}} = -\frac{k}{4\pi} \int \left( \sum_{i,j=1}^r \eta_{ij} \partial \varphi_i \bar{\partial} \varphi_j + 2\frac{\partial \varphi_i \bar{\partial} \varphi_j}{\chi e^{-\varphi^2}} + \partial \eta \bar{\partial} \nu + \partial \nu \bar{\partial} \eta \right)
\]

\[
+ 2(\partial \varphi_1 \bar{\partial} \ln \bar{\chi} + \bar{\partial} \varphi_1 \partial \ln \chi) - 2Tr(\hat{\epsilon} + g_0^f \hat{\epsilon} - (g_0^f)^{-1})
\]

(2.26)

Defining

\[
E = e^{\varphi_1}, \quad F = E^{-1}(1 - \chi^2 e^{-\varphi^2})
\]

(2.27)
In the case of abelian T-duality called isometric target-space coordinates. Lagrangeans are related by the generating function $F$ Noether charges $P$ critical transformation \cite{17}, in the string phase space T-duality\cite{17} is known to be an important property of the string theory. It acts as canon-

3 Off-Critical T-Duality

T-duality\cite{17} is known to be an important property of the string theory. It acts as canonical transformation \cite{18},\cite{17] in the string phase space $P = \{ X^M(\sigma), \Pi_M(\sigma) = g_{MN}X^N + b^N_MX^N; b,c \}$ mapping the original conformal $\sigma$-model \cite{20},\cite{17]:

$$S^{conf}_\sigma = \frac{1}{4\pi}\int d^2z \left( (g_{MN}(X)\eta^{\mu \nu} + b_{MN}(X)e^{\mu \nu})\partial_\mu X^M \partial_\nu X^N + \frac{\alpha'}{2}R^{(2)}(\varphi(X)) \right)$$

(\mu, \nu = 0, 1; M, N = 1, 2, \cdots D and $R^{(2)}$ is the worldsheet curvature ) to its T-dual model $S^{conf}_\sigma(G_{MN}(\tilde{X}), E_{MN}(\tilde{X}), \phi(\tilde{X}))$. Curved string backgrounds with d-isometric directions provide an example of an abelian T-duality transformation\cite{20},\cite{17]:

$$E_{\alpha \beta} = (e^{-1})_{\alpha \beta}, \quad E_{mn} = e_{mn} - e_{m\alpha}(e^{-1})^{\alpha \beta}e_{\beta n}$$

$$E_{am} = (e^{-1})^\beta_\alpha e_{m \beta}, \quad E_{ma} = -e_{m\beta}(e^{-1})^\beta_\alpha$$

$$\phi = \varphi - \ln det(E_{\alpha \beta}), \quad \alpha, \beta = 1, 2, \cdots d, \quad m, n = d + 1, \cdots D$$

where $E_{MN} = G_{MN} + B_{MN}$. The canonical transformation $(\Pi_X, X) \rightarrow (\Pi_{\tilde{X}}, \tilde{X})$ that generates the background maps (3.30) has the following simple form \cite{18}:

$$\Pi_{\tilde{X}_a} = -X'_a, \quad \Pi_X = -\tilde{X}'_a$$

and all the $\Pi_{X_m}$ and $X_m$ remain unchanged. By construction both $\sigma$-models $S^{conf}_\sigma(e, \varphi)$ and $S^{conf}_\sigma(E, \phi)$ have coinciding energy spectrum and partition functions. The corresponding Lagrangeans are related by the generating function $F$:

$$\mathcal{L}(e, \varphi) = \mathcal{L}(E, \phi) + \frac{dF}{dt}, \quad F = \frac{1}{8\pi\alpha'} \int dx \left( X \cdot \tilde{X}' - X' \cdot \tilde{X} \right),$$

$$\frac{\delta F}{\delta X^\alpha} = \Pi_{X^\alpha}, \quad \frac{\delta F}{\delta \tilde{X}^\alpha} = -\Pi_{\tilde{X}^\alpha}$$

An important feature of the abelian T-duality (3.30)and (3.31) is that it maps the $U(1)^d$ Noether charges $Q^\alpha = \int_{-\infty}^\infty J^\alpha_\ell dx$ of $S^{conf}_\sigma$ into the topological charges $\tilde{Q}^\alpha_{\text{top}} = \int_{-\infty}^\infty dx \partial_x \tilde{X}^\alpha$ of

\footnote{under certain symmetry restrictions on the geometrical data: $e_{MN}(X) = g_{MN}(X) + b_{MN}(X)$ and $\varphi(X)$. In the case of abelian T-duality $e_{MN}(X)$ and $\varphi(X)$ independent of $d \leq D$ of $X_\alpha$, ($\alpha = 1, 2, \cdots d \leq D$) called isometric target-space coordinates.}
the dual model $\tilde{S}^{\text{con}}_{\alpha}$:

\[
J_\mu^\alpha = \frac{1}{2} e^{\alpha \beta} (X_\alpha) \partial_\mu X_\beta + \frac{1}{2} \epsilon^{\alpha \beta \gamma} (X_\alpha) \partial_\mu X_\beta \equiv \frac{1}{2} \epsilon_{\mu \nu} \partial^\nu \tilde{X}^\alpha
\]

\[
\tilde{J}_\mu^\alpha = \frac{1}{2} E^{\alpha \beta} (\tilde{X}_\alpha) \partial_\mu \tilde{X}_\beta + \frac{1}{2} E^{\alpha \beta \gamma} (\tilde{X}_\alpha) \partial_\mu \tilde{X}_\beta \equiv \frac{1}{2} \epsilon_{\mu \nu} \partial^\nu X^\alpha
\]

i.e. $(Q^\alpha, Q_{\text{top}}^\alpha) \rightarrow (\tilde{Q}_{\text{top}}^\alpha, \tilde{Q}^\alpha)$. It is well known [17], [19] the main part of the conformal $\sigma$-models representing relevant string backgrounds can be derived from the axial or vector gauged $G/H$-WZW models. All the models constructed in Sect.2. with vanishing potential term $V = \frac{m^2 k}{2 \pi} Tr (\epsilon_+ B \epsilon_- B^{-1})$ (i.e. $m = 0$) are of this type. They have $d = r$ isometric directions, i.e. $\epsilon_{mn}$ do not depend on $\varphi_i$, $(i = 1, \ldots, r - 1)$ and $\theta = \frac{1}{2} \ln \frac{\tilde{\psi}}{\psi}$. The T-duality group in this case is known to be $O(r, r | Z)$ (see for example [17]). Adding the potential $V$ with $\epsilon_\pm = \sum_{i \neq a} E_{\pm, \alpha_i}^{(0)}$ (see eqs. (2.16) and (2.23)) specific for the conformal NA -Toda theories with one global $U(1)$ symmetry, we are decreasing the number of the isometric coordinates from $d = r$ to $d_0 = 1$. Taking $\epsilon_\pm = \sum_{i \neq a, b} E_{\pm, \alpha_i}^{(0)}$ one can construct NA -Toda theories with $d_0 = 2$, etc.

The problem we are addressing in this section is about the T-duality between $d_0 = 1$ axial and vector integrable (nonconformal) models of Sect.2.1 and Sect.2.2. with potential terms constructed by taking

\[
\epsilon_\pm = \sum_{i=2}^{r} E_{\pm, \alpha_i}^{(0)} + E_{\pm, \alpha_i}^{(1)}
\]

\[
V_a = \frac{m^2 k}{2 \pi} \left( \sum_{i=1}^{r} e^{\varphi_i - \psi_i - 1} + e^{\varphi_r - \psi_i - 1} (1 + \psi \chi e^{-\varphi_i}) \right)
\]

\[
V_{\text{vec}} = \frac{m^2 k}{2 \pi} \left( \sum_{i=2}^{n} e^{\alpha_i - \varphi_i - 1} + 1 \chi e^{-\alpha_i} \right)
\]

for the $A_r^{(1)}$ model of Example 1. For the $B_n^{(1)}$ model of Example 2 (see also Sect.4.2) we have

\[
\epsilon_\pm = \sum_{i=1}^{n-2} E_{\pm, \alpha_i}^{(0)} + E_{\pm, \alpha_i}^{(1)}, \quad \alpha_0 + \alpha_1 + 2 (\alpha_2 + \cdots \alpha_{n-1} + \alpha_n) = 0
\]

with potential given by eq.(4.49). In both cases the axial IM’s isometric coordinate is $X = \theta = \frac{1}{2} \ln \frac{\tilde{\psi}}{\psi}$ ($\tilde{u}^2 = \tilde{\psi}$). For the corresponding vector IM’s we choose $\tilde{X}_{B_{n}^{(1)}} = R_B = -2 \ln E$ and $\tilde{X}_{A_r^{(1)}} = R_A = \frac{r+1}{r-1} \ln E$ as isometric coordinates. It is important to mention that in the case of $A_r^{(1)}$ vector model the canonical transformation (3.31) with $d = 1$ has to be accompanied by the following point transformation:

\[
\phi_k = \phi_k' - \frac{r-k}{2r} R_A, \quad k = 1, 2, \cdots r - 1
\]

Then performing $d_0 = 1$ T-duality transformation (3.30) (together with the $A_r^{(1)}$ fields transformation (3.36)) we realize that $\mathcal{L}_{\text{vec}}$ and $\mathcal{L}_a$ given by eqs. (2.23) and (2.16), with potentials
(3.34) and (4.49), are related by eq.(3.32) with
\[ \frac{dF}{dt} = 2\Gamma (\partial ln\tilde{u}\partial lnE - \bar{\partial}ln\tilde{u}\partial lnE) \]

Notice that the \(B_{n}^{(1)}\) vector and axial Lagrangeans have the same form, i.e. they appear to be \(T\)-selfdual.

An alternative way to perform the T-duality transformation between \(d_{0} = 1\) axial and vector IM's in consideration consists in making the following nonlocal change of the field variables:

a) \(A_{n}^{(1)}\) case
\[ E = e^{\frac{1}{2}R_{A}}, \quad F = e^{\frac{1}{2}R_{A}}(1 + \psi\chi e^{r_{1}}), \quad \phi_{k} = \varphi_{k} + \frac{r - k}{r}R_{A} \]  
(3.37)

b) \(B_{n}^{(1)}\) case
\[ E = e^{-\frac{1}{2}R_{B}}, \quad F = e^{\frac{1}{2}R_{B}}(1 + \psi\chi), \quad \phi_{k} = \varphi_{k} \]  
(3.38)

instead of the canonical transformation (3.31) (resulting in (3.30)). Eqs. (3.37) and (3.38) in fact represents the integrated form of (3.31). Their derivation (see Sect.5 of ref. [16]) is based on the comparison of the \(g_{0}\) (or \(B\)) group elements written in axial and vector parametrizations (2.13) and (2.19), i.e. imposing \(g_{0}^{vec} = g_{0}^{ax}\). An important ingredient of this calculation are the relations (3.33) between the \(U(1)\) currents and the topological currents \(\epsilon^{\mu\nu}\partial_{\nu}R\) (and \(\epsilon^{\mu\nu}\partial_{\nu}\theta\)). Note that \(R = -2\ln E\) is a nonlocal (nonphysical) field in the axial model, but it appears to be physical in the vector model. In the case of the \(B_{n}^{(1)}\) model (2.25) the \(U(1) \leftrightarrow \text{top-currents}\) relations (3.33) take the following explicit form:
\[
\begin{align*}
\partial ln\tilde{u} &= \frac{g}{g-1} \frac{\partial lnE}{g-1} - \frac{1}{2} \frac{\partial g}{g-1} \\
\bar{\partial}ln\tilde{u} &= -\frac{g}{g-1} \frac{\partial lnE}{g-1} + \frac{1}{2} \frac{\partial g}{g-1}
\end{align*}
\]  
(3.39)

where \(g = EF\) and \(\tilde{u}^{2} = \frac{\chi}{\psi}\).

Although the T-duality between the vector and axial integrable models is quite similar to the conformal "free" case (i.e. \(V = 0\)) the off-critical T-duality addresses few new problems specific for the integrable models. In the case of imaginary coupling constant \(\beta^{2} = -\frac{2\pi}{k}\), i.e. \(\beta \rightarrow i\beta_{0}\) and \(\varphi_{k} \rightarrow i\beta_{0}\varphi_{k}, \psi \rightarrow i\beta_{0}\psi, etc.\) one expects that both axial and vector IM's possess soliton solutions. One might wonder what is the relation between the solitons (and breathers) of the T-dual integrable models, whether their soliton spectra coincides (modulo the interchanges \(Q \rightarrow \tilde{Q}_{\text{top}}, \tilde{Q} \rightarrow Q_{\text{top}}\)) and finally about the \(O(1,1|Z)\) symmetry of the solitons energies and massess. Partial answer of all these questions is presented in our recent work [16].

\(\text{3}\)the only new feature is that one should take care about the specific "point" transformations involving the potential \(V\) and that the isometric coordinates are reduced from \(d = r\) to \(d_{0} = 1\).
4 No Torsion Theorem

We now discuss a condition for which the axial gauging generate torsionless models. Consider a finite dimensional lie algebra $G$ with grading operator given by

$$Q_a = \sum_{i \neq a}^{r} \frac{2}{\alpha_i^2} H_{\lambda_i}$$

and consider the most general constant generators of grade $\pm 1$, i.e.,

$$\epsilon_\pm = \sum_{i \neq a}^{r} c_{\pm i} E_{\pm \alpha_i} + b_\pm E_{\pm (a_0 + a_0 + 1)} + d_\pm E_{\pm (a_0 + a_0 - 1)}.$$

(4.40)

It is clear that if $c_{\pm i}, b_\pm, d_\pm \neq 0$, there shall be no $G^0$ commuting with $\epsilon_\pm$, since that require an orthogonal direction to all roots appearing in $\epsilon_\pm$. These are the generalized non-singular NA-Toda models of ref. [13]. The NA-Toda models of singular metric NA-Toda models: originated by the presence of $e^{h_j}$ in $\Delta_a$ and in the kinetic term as well. Since we are removing all dependence in $G^0$, when parameterizing $g^0$, cases (iii) and (iv) may be studied together with

$$g^0 = \exp(\chi E_{-\alpha_a}) \exp(\Phi(H)) \exp(\psi E_{\alpha_a})$$

(4.41)

where $\Phi(H) = \sum_{i=1}^{a-2} \varphi_i h_i + \varphi_{-}(\chi_- \cdot H) + \varphi_{+}(\chi_+ \cdot H) + \sum_{i=0}^{r} \varphi_i h_i$,

$$\chi^{(iii)}_- = \alpha_{a-1} + \alpha_a, \quad \chi^{(iii)}_+ = \alpha_{a+1}.$$  

(4.42)

$$\chi^{(iv)}_- = \alpha_{a-1}, \quad \chi^{(iv)}_+ = \alpha_a + \alpha_{a+1}.$$  

(4.43)

If we leave $G^0$ unconstrained the resulting model belongs again to the non singular NA-Toda class of models [13].
for cases (iii) and (iv) respectively, and $G^0 = Y \cdot H$, such that $Tr(\chi_\pm \cdot HG^0_0) = 0$. Such parametrization of $g^f_0$ yields

$$
\Phi(\alpha_a) = \sum_{i=1}^{a-2} k_{ai} \varphi_i + (\alpha_a \cdot \chi_-) \varphi_- + (\alpha_a \cdot \chi_+) \varphi_+ + \sum_{i=a+2}^{r} k_{ai} \varphi_i
$$

Now, if we consider Lie algebras whose Dynkin diagrams connect only nearest neighbours, i. e.,

$$
\Phi(\alpha_a) = (\alpha_a \cdot \chi_-) \varphi_- + (\alpha_a \cdot \chi_+) \varphi_+,
$$

(4.44)

then the “no-torsion condition” implies $\Phi(\alpha_a) = 0$.

Considering case (iii), we have

$$
\alpha_a \cdot \chi_- = \alpha_a \cdot (\alpha_{a-1} + \alpha_a) = 0,
$$

(4.45)

$$
\alpha_a \cdot \chi_+ = \alpha_a \cdot (\alpha_{a+1}) = 0.
$$

(4.46)

In this case, the only solution for both equations is to take $a = r$ (in such a way that $a_{r+1} = 0$) and $G = B_r$ (so that $\alpha_r^2 = -\alpha_{r-1} \cdot \alpha_r = 1$). This is precisely the case proposed by Leznov and Saveliev [21] and subsequently discussed by Gervais and Saveliev [14] and also by Bilal [7], for the particular case of $B_2$.

For case (iv), the “no-torsion condition” requires that

$$
\alpha_{a-1} \cdot \alpha_a = 0, \quad \alpha_a \cdot (\alpha_a + \alpha_{a+1}) = 0,
$$

which are satisfied by $a = 1$ and also by $G = C_2$, since $\alpha_{a-1} = 0$ and also $\alpha_1^2 = -\alpha_1 \cdot \alpha_2 = 1$, respectively.

In general, the “no-torsion condition”, i. e., $\Phi(\alpha_a) = 0$, may be expressed in terms of the structure of the co-set $G_0/G_0^0 = u(1)^{r-1} \otimes sl(2)/u(1)$. The crucial ingredient for the appearence of $\Phi(\alpha_a)$ arises from the conjugation

$$
Tr(A_0g^f_0 \bar{A}_0(g^f_0)^{-1} + A_0 \bar{A}_0) = 2\lambda_a^2 \left(1 + \frac{2}{\alpha_a^2} \chi \psi \exp(\Phi(\alpha_a))\right).
$$

Henceforth, if all generators belonging to the Cartan subalgebra parameterizing $g^f_0$ commute with $E_{\pm \alpha_a}$, then $\Phi(\alpha_a) = 0$, and therefore the structure of the co-set

$$
\frac{G_0}{G_0^0} = \frac{u(1)^{r-1} \otimes sl(2)}{u(1)} = u(1)^{r-1} \otimes \frac{sl(2)}{u(1)}
$$

(4.47)

is the general condition for the absence of the antisymmetric term in the action.

Summarizing, for finite dimensional Lie algebras, it was shown that the absence of the antisymmetric terms in the action can only occur for $G = B_r, a = r$ and and $\epsilon_\pm = \sum_{i=1}^{r-2} c_\pm E_{\pm \alpha_i} + d_\pm E_{\pm(\alpha_r+\alpha_{r-1})}$. In such case, $G^0_0$ is generated by $Y \cdot H = (\frac{2\lambda_r}{\alpha_r^2} - \frac{2\lambda_{r-1}}{\alpha_{r-1}^2}) \cdot H$.
and $\Phi(H) = \sum_{i=1}^{n-2} \varphi_i h_i + \varphi_-(\alpha_{r-1} + \alpha_r) \cdot H$. Due to the root structure of $B_n$, we verify that $\Phi(\alpha_r) = \alpha_r \cdot (\alpha_{r-1} + \alpha_r) \varphi_- = 0$.

In extending the no torsion theorem to infinite affine Lie algebras let us choose $h' = 1 - \sum_{i \neq r} \frac{2}{\alpha_i} \lambda_i \cdot \alpha_0$ where $-\alpha_0$ is the highest root of $G$ such that $\alpha_0 \cdot (\frac{2\lambda}{\alpha_2} - \frac{2\lambda}{\alpha_{n-1}}) = 0$ and the gradation $Q_a(h')$ preserves the zero grade subalgebra $G_0$, (apart from $\hat{c}$ and $\hat{d}$). We consider $\hat{c} = \epsilon_{\pm} = E_{(1)}$. Since conformal and the affine models differ only by the potential term, the solution for the no torsion condition is also satisfied for infinite dimensional algebras, whose Dynkin diagram possess a $B_n$-“tail like”. An obvious solution is the untwisted $B_n^{(1)}$ model. Two other solutions were found within the twisted affine Kac-Moody algebras $A_2^{(2)}$ and $D_{n+1}^{(2)}$ as we shall describe in detail.

4.1 The $B_n^{(1)}$ Torsionless Affine NA Toda model

Let $Q = 2(n-1)d + \sum_{i=1}^{n-1} \frac{2\lambda_i d}{\alpha_i}$ decomposing $B_n^{(1)}$ into graded subspaces. In particular $G_0 = SL(2) \otimes U(1)^{n-1} \otimes U(1) \otimes U(1) \otimes U(1)$ generated by $\{E_{\pm}^{(0)}, h_1, \ldots, h_n, \hat{c}, \hat{d}\}$. Following the no torsion theorem of ref. [9], we have to choose $\hat{c}_\pm = \sum_{i=1}^{n-2} c_{\pm} + E_{\pm}^{(0)} + E_{\pm}^{(n)} + c_{\pm} E_{\pm}^{(n)}$, where $\alpha_0 + \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{n-1} + \alpha_n) = 0$ is the highest root of $B_n$ and $G_0^{(0)}$ is generated by $Y \cdot H = (\frac{2\lambda_0}{\alpha_n} - \frac{2\lambda_{n-1}}{\alpha_{n-1}}) \cdot H$ such that $[Y \cdot H, \hat{c}_\pm] = 0$. The coset $G_0/G_0^{(0)}$ is then parametrized according to (2.13) with $\Phi(H) = \sum_{i=1}^{n-1} H_i \varphi_i + \eta \hat{h} + \nu \hat{d}$ where $H_i = (\alpha_n + \cdots + \alpha_i) \cdot H$ so that $Tr(H_i H_j) = \delta_{ij}, i, j = 1, \ldots, n - 1$ and the total effective action becomes

$$S = -\frac{k}{4\pi} \int d^2x \left( \frac{1}{4} \sum_{i=1}^{n-1} g^{\mu\nu} \partial_\mu \varphi_i \partial_\nu \varphi_i + g^{\mu\nu} \partial_\mu \psi \partial_\nu \chi + \frac{1}{2} g^{\mu\nu} \partial_\mu \nu \partial_\nu \eta - 2V \right)$$

(4.48)

where the “affine potential” ($n > 2$) is

$$V = \sum_{i=1}^{n-2} |c_i|^2 e^{\varphi_i - \varphi_{i+1}} + |c_{n-1}|^2 (1 + 2 \psi \chi) e^{-\varphi_{n-1}} + |c_n|^2 e^{\varphi_1 + \varphi_2 - \eta}$$

(4.49)

The action (4.48) is invariant under conformal tranformation

$$z \rightarrow f(z), \quad \bar{z} \rightarrow g(\bar{z}), \quad \psi \rightarrow \psi, \quad \chi \rightarrow \chi,$$

$$\varphi_s \rightarrow \varphi + s \ln f' g'; \quad s = 1, 2, \ldots, n - 1; \quad \eta \rightarrow \eta + 2(n - 1) \ln f' g'$$

(4.50)

We should point out that the $\eta$ field plays a crucial role in establishing the conformal invariance of the theory. Integrable deformation of such class of theories can than, be systematicaly obtained by setting $\eta = 0$.

For the case $n = 2$ we choose, $\hat{c}_\pm = E_{(1)}^{(0)} + E_{(n-1)}^{(n-1)}$, $\Phi(\alpha_{n-1}) = \varphi$, i.e. $\hat{G} = \hat{SO}(5)$, is also special in the sense that its complexified theory, i.e.

$$\psi \rightarrow i\psi; \quad \chi \rightarrow i\psi^*; \quad \varphi \rightarrow i\varphi$$

leads to the real action

$$S = -\frac{k}{4\pi} \int d^2x \left( \frac{g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi^*}{(1 - \psi \psi^*)} + \frac{1}{4} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + 8(1 - 2\psi \psi^*) \cos \varphi \right)$$

(4.51)
4.2 The twisted NA Toda Models

The twisted affine Kac-Moody algebras are constructed from a finite dimensional algebra possessing a nontrivial symmetry of their Dynkin diagrams (folding). Such symmetry can be extended to the algebra by an outer automorphism $\sigma$ [15], as

$$\sigma(E_\alpha) = \eta_\alpha E_{\sigma(\alpha)}$$

where $\eta_\alpha = \pm 1$. For the simple roots, $\eta_{\alpha_i} = 1$. The signs can be consistently assigned to all generators since nonsimple roots can be written as sum of two roots other roots.

The no torsion theorem require a $B_n$-“tail like” structure which is fulfilled only by the $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ (see appendix N of ref. [15]). In both cases the automorphism is of order 2 (i.e. $\sigma^2 = 1$).

Let us denote by $\alpha$ the roots of the untwisted algebra $G$. For the $A_{2n}^{(2)}$ case, the automorphism is defined by

$$\sigma(\alpha_1) = \alpha_{2n}, \quad \sigma(\alpha_2) = \alpha_{2n-1} \ldots, \sigma(\alpha_{n-1}) = \alpha_n$$

whilst for the $D_{n+1}^{(2)}$, the automorphism acts only in the “fish tail” of the Dynkin diagram of $D_{n+1}$, i.e.

$$\sigma(E_{\alpha_1}) = E_{\alpha_1}, \quad \ldots, \sigma(E_{\alpha_{n-1}}) = E_{\alpha_{n-1}}, \quad \sigma(E_{\alpha_n}) = E_{\alpha_{n+1}}$$

The automorphism $\sigma$ decomposes the algebra $G = G_{\text{even}} \cup G_{\text{odd}}$. The twisted affine algebra is constructed from $G$ assigning an affine index $m \in \mathbb{Z}$ to the generators in $G_{\text{even}}$ while $m \in \mathbb{Z} + \frac{1}{2}$ to those in $G_{\text{odd}}$ (see appendix N of [15]).

The simple root step operators for $A_{2n}^{(2)}$ are

$$E_{\beta_i} = E_{\alpha_i}^{(0)} + E_{\alpha_{2n-i+1}}^{(0)}, \quad i = 1, \ldots, n \quad E_{\beta_0} = E_{\frac{1}{2}(\alpha_1 + \cdots + \alpha_{2n})}^{(0)}$$

corresponding to the simple and highest roots

$$\beta_i = \frac{1}{2}(\alpha_i + \alpha_{2n-i+1}) \quad i = 1, \ldots, n, \quad -\alpha_0 = \alpha_1 + \cdots + \alpha_{2n} = 2(\beta_1 + \cdots + \beta_n)$$

respectively.

For $D_{n+1}^{(2)}$, simple root step operators are

$$E_{\beta_i} = E_{\alpha_i}^{(0)} \quad i = 1, \ldots, n-1, \quad E_{\beta_n} = E_{\alpha_n}^{(0)} + E_{\alpha_{n+1}}^{(0)}$$

$$E_{\beta_0} = E_{\frac{1}{2}(\alpha_1 + \cdots + \alpha_{n-1} + \alpha_{n+1})}^{(0)} - E_{\frac{1}{2}(\alpha_1 + \cdots + \alpha_{n-2} + \alpha_n)}^{(0)}$$

corresponding to the simple and highest roots

$$\beta_i = \alpha_i \quad i = 1, \ldots, n-1, \quad \beta_n = \frac{1}{2}(\alpha_n + \alpha_{n+1}),$$

$$-\alpha_0 = \alpha_1 + \cdots + \alpha_{n-1} + \frac{1}{2}(\alpha_n + \alpha_{n+1}) = \beta_1 + \cdots + \beta_n$$
where have denoted by $\beta$ the roots of the twisted (folded) algebra.

The torsionless affine NA Toda models are defined by

$$Q = 2(2n - 1)\ddot{d} + \sum_{i \neq n, n+1}^{2n} \frac{2\lambda_i \cdot H}{\alpha_i^2},$$

(4.59)

and

$$Q = (2n - 2)\ddot{d} + \sum_{i=1}^{n} \frac{2\lambda_i \cdot H}{\alpha_i^2}$$

(4.60)

for $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ respectively, where $\lambda_i$ are the fundamental weights of the untwisted algebra $G$, i.e. $\frac{2\lambda_i \cdot \alpha_i}{\alpha_i^2} = \delta_{ij}$.

Both models are specified by the constant grade $\pm 1$ operators $\epsilon_\pm$

$$\epsilon_\pm = \sum_{i=1}^{n-2} c_{\pm i} E_{\pm \beta_i} + c_{\pm(n-1)} E_{\pm(\beta_{n-1} + \beta_n)} + c_{\pm n} E_{\pm \beta_0}$$

(4.61)

where $\beta_i$ are the simple roots of the twisted affine algebra specified in (4.56) and in (4.58).

According to the grading generators (4.59) and (4.60), the zero grade subalgebra is in both cases $G_0 = SL(2) \otimes U(1)^{n-1} \otimes U(1)_c \otimes U(1)_{\dot{d}}$ generated by $\{E_{\pm j}, h_1, \cdots, h_n, \dot{c}, \dot{d}\}$. Hence the zero grade subgroup is parametrized as in (4.41) where we have taken $\eta = 0$, responsible for breaking the conformal invariance. The factor group is given in (2.13), where $G_0$ is generated by $Y \cdot H = (\frac{2\mu}{\beta_i^2} - \frac{2n-1}{\beta_i^2}) \cdot H$ and $\mu_i$ are the fundamental weights of the twisted algebra i.e. $\frac{2n-1}{\beta_i^2} = \delta_{ij}$.

In order to decouple the $\varphi_i$, $i = 1, \cdots, n-1$ we chose an orthonormal basis for the Cartan subalgebra, i.e. $\Phi(H) = \mathcal{H}_i \varphi_i + \eta \dot{h} + \nu \dot{c}$ where

$$\mathcal{H}_i = (\alpha_i + \cdots + \alpha_{2n-1}) \cdot H, \quad Y \cdot H = \mathcal{H}_n, \quad Tr(\mathcal{H}_i \mathcal{H}_j) = 2\delta_{ij}, \quad i, j = 1, \cdots n$$

(4.62)

and

$$\mathcal{H}_i = (\alpha_{n-i+1} + \cdots + \alpha_{n+1}) \cdot H, \quad Y \cdot H = \mathcal{H}_n, \quad Tr(\mathcal{H}_i \mathcal{H}_j) = \delta_{ij}, \quad i, j = 1, \cdots n$$

(4.63)

for $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ respectively.

The Lagrangean density is obtained from (2.16) leading to

$$\mathcal{L}_{A_{2n}^{(2)}} = \frac{\partial \chi \bar{\partial} \psi}{1 + \frac{1}{2} \psi \chi} + \frac{1}{2} \sum_{i=1}^{n-1} \partial \varphi_i \bar{\partial} \varphi_i - V_{A_{2n}^{(2)}}$$

(4.64)

and

$$\mathcal{L}_{D_{n+1}^{(2)}} = \frac{2 \partial \chi \bar{\partial} \psi}{1 + \psi \chi} + \frac{1}{2} \sum_{i=1}^{n-1} \partial \varphi_i \bar{\partial} \varphi_i - V_{D_{n+1}^{(2)}}$$

(4.65)

where

$$V_{A_{2n}^{(2)}} = \sum_{i=1}^{n-2} |c_i|^2 e^{-\varphi_i + \varphi_{i+1}} + \frac{1}{2} |c_n|^2 e^{2\varphi_1} + |c_{n-1}|^2 e^{-\varphi_{n-1}}(1 + \psi \chi)$$

(4.66)

and

$$V_{D_{n+1}^{(2)}} = \sum_{i=1}^{n-2} |c_i|^2 e^{-\varphi_i + \varphi_{i+1}} + \frac{1}{2} |c_n|^2 e^{2\varphi_1} + |c_{n-1}|^2 e^{-\varphi_{n-1}}(1 + 2\psi \chi)$$

(4.67)

The models described by (4.48), (4.64) and (4.65) coincide with those proposed by Fateev in [6].
5 Zero Curvature

The equations of motion for the NA Toda models are known to be of the form [21]

\[ \partial(B^{-1}\partial B) + [\varepsilon_-, B^{-1}\varepsilon_+ B] = 0, \quad \partial(\partial B B^{-1}) - [\varepsilon_-, B\varepsilon_+ B^{-1}] = 0 \] (5.68)

The subsidiary constraint \(J_{Y,H} = Tr(B^{-1}\partial BY \cdot H) = J_{Y,H} = Tr(\partial B B^{-1}Y \cdot H) = 0\) can be consistently imposed since \([Y \cdot H, \varepsilon_+] = 0\) as can be obtained from (5.68) by taking the trace with \(Y,H\). Solving those equations for the nonlocal field \(R\) yields,

\[ \partial R = \left( \frac{Y \cdot \alpha_n}{Y^2} \right) \psi \frac{\partial \chi}{\Delta} e^{\Phi(n)} \nu, \quad \partial R = \left( \frac{Y \cdot \alpha_n}{Y^2} \right) \chi \frac{\partial \psi}{\Delta} e^{\Phi(n)} \] (5.69)

The equations of motion for the fields \(\psi, \chi\) and \(\varphi, i = 1, \ldots, n - 1\) obtained from (5.68) imposing the constraints (5.69) coincide precisely with the Euler-Lagrange equations derived from (4.64) and (4.65). Alternatively, (5.68) admits a zero curvature representation \(\partial A - \partial A + [A, \tilde{A}] = 0\) where

\[ A = \varepsilon_- + B^{-1}\partial B, \quad \tilde{A} = -B^{-1}\varepsilon_+ B \] (5.70)

Whenever the constraints (5.69) are incorporated into \(A\) and \(\tilde{A}\) in (5.70), equations (5.68) yields the zero curvature representation of the NA singular Toda models. Such argument is valid for all NA Toda models, in particular for the torsionless class of models discussed in the previous two sections.

Using the explicit parametrization of \(B\) given in (4.41), the corresponding \(\varepsilon_\pm\) specified in (4.61), (4.56) and (4.58) together with (5.69) where \(Y\) is given in (4.62) and (4.63), we find, in a systematic manner, the following form for \(A\) and \(\tilde{A}\)

\[ A_{A_2} = \sum_{i=1}^{n-2} c_i \left( E_{-\alpha_i}^{(0)} + E_{-\alpha_{2n-i+1}}^{(0)} \right) + c_{n-1} \left( E_{-\alpha_n - \alpha_{n-1}}^{(0)} + E_{-\alpha_{n+1} - \alpha_{n+2}}^{(0)} \right) + c_n E_{\alpha_1 + \ldots + \alpha_{2n}} \psi^{-\frac{1}{2}} R \left( E_{\alpha_n}^{(0)} + E_{\alpha_{n+1}}^{(0)} \right) + \sum_{i=1}^{n-1} \partial \varphi_i \mathcal{H}_i + \frac{\partial \chi}{\Delta} e^{\frac{1}{2} R} \left( E_{-\alpha_n}^{(0)} + E_{-\alpha_{n+1}}^{(0)} \right) \] (5.71)

and

\[ -\tilde{A}_{A_2} = \sum_{i=1}^{n-2} c_i e^{\varphi_i + \varphi_i + 1} \left( E_{\alpha_i}^{(0)} + E_{\alpha_{2n-i+1}}^{(0)} \right) + c_n e^{2\varphi_1} E_{-\alpha_{1} - \ldots - \alpha_{2n}}^{(\frac{1}{2})} + c_{n-1} e^{-\varphi_n - 1} \left( E_{-\alpha_n + \alpha_{n-1}}^{(0)} + E_{-\alpha_{n+1} + \alpha_{n+2}}^{(0)} \right) + c_n e^{-\frac{1}{2} R - \varphi_n - 1} \left( E_{\alpha_n + \alpha_{n+1} - \alpha_{n+2}}^{(0)} - E_{\alpha_n + \alpha_{n+1} + \alpha_{n+2}}^{(0)} \right) + c_{n-1} \psi^R \left( E_{\alpha_n - \alpha_{n+2}}^{(0)} - E_{\alpha_n + \alpha_{n-1}}^{(0)} \right) + c_{n-1} \psi^{-\frac{1}{2} R} \left( E_{\alpha_n + \alpha_{n+1} - \alpha_{n+2}}^{(0)} - E_{\alpha_n + \alpha_{n+1} + \alpha_{n+2}}^{(0)} \right) \] (5.72)
We have constructed a class of affine NA Toda models from the gauged two-loop WZW models. The construction of the previous section the zero curvature representation is obtained from

$$ -A_{D_{n+1}^{(2)}} = \sum_{i=1}^{n-2} c_i e^{-\varphi_i + \varphi_{i+1}} E_{\alpha_i}^{(0)} + c_{n-1} e^{-\varphi_{n-1}} (E_{\alpha_n + \alpha_{n-1}}^{(0)} + E_{\alpha_{n+1} + \alpha_{n-1}}^{(0)}) $$

$$ + 2c_{n-1} \psi e^{\frac{1}{2} R - \varphi_{n-1}} F_{\alpha_{n+1} + \alpha_{n-1}}^{(0)} + 2c_{n-1} \chi e^{\frac{1}{2} R - \varphi_{n-1}} E_{\alpha_{n-1}}^{(0)} $$

$$ + 2c_{n-1} \psi \chi e^{-\varphi_{n-1}} (E_{\alpha_{n+1} + \alpha_{n-1}}^{(0)} + E_{\alpha_{n-1} + \alpha_n}^{(0)}) + c_{n-1} \psi \chi e^{-\frac{1}{2} R - \varphi_{n-1}} E_{\alpha_{n+1} + \alpha_{n-1}}^{(0)} $$

$$ + c_{n+1} e^{\varphi_1} (E_{-2(\alpha_1 + \ldots + \alpha_{n+1})}^{(4)} - E_{-2(\alpha_1 + \ldots + \alpha_{n+1} + 1)}^{(4)}) $$

and

$$ -\tilde{A}_{D_{n+1}^{(2)}} = \sum_{i=1}^{n-2} c_i e^{-\varphi_i + \varphi_{i+1}} E_{\alpha_i}^{(0)} + c_{n-1} e^{-\varphi_{n-1}} (E_{\alpha_n + \alpha_{n-1}}^{(0)} + E_{\alpha_{n+1} + \alpha_{n-1}}^{(0)}) $$

$$ + 2c_{n-1} \psi e^{\frac{1}{2} R - \varphi_{n-1}} F_{\alpha_{n+1} + \alpha_{n-1}}^{(0)} + 2c_{n-1} \chi e^{-\frac{1}{2} R - \varphi_{n-1}} E_{\alpha_{n-1}}^{(0)} $$

For the untwisted affine $B_n^{(1)}$ model of the previous section the zero curvature representation is obtained from

$$ A_{B_n^{(1)}} = \sum_{i=1}^{n-2} c_i E_{\alpha_i}^{(0)} + c_{n-1} E_{-\alpha_{n-1} + \alpha_n}^{(0)} + c_n E_{\alpha_1 + 2(\alpha_2 + \ldots + \alpha_n)}^{(-1)} $$

$$ + \partial \psi e^{-\frac{1}{2} R} E_{\alpha_n}^{(0)} + \sum_{i=1}^{n-1} \partial \varphi_i H_i + \frac{\partial \chi}{\Delta} e^{\frac{1}{2} R} E_{\alpha_n}^{(0)} $$

$$ -\tilde{A}_{B_n^{(1)}} = \sum_{i=1}^{n-2} c_i e^{-\varphi_i + \varphi_{i+1}} E_{\alpha_i}^{(0)} + c_{n-1} e^{-\varphi_{n-1}} 2 e^{\varphi_1 + \varphi_2} E_{-2(\alpha_1 + \ldots + \alpha_n)}^{(1)} + 2 \chi e^{\varphi_{n-1} + \frac{1}{2} R} $$

$$ + c_{n-1} (1 + 2 \psi \chi) e^{-\varphi_{n-1}} E_{\alpha_{n-1} + \alpha_n}^{(0)} - 2c_{n-1} e^{-\varphi_{n-1} - \frac{1}{2} R} \psi (1 + \psi \chi) E_{\alpha_{n-1} + 2 \alpha_n} $$

The zero curvature representation of such subclass of torsionless NA Toda models shows that they are in fact classically integrable field theories. The construction of the previous sections provides a systematic affine Lie algebraic structure underlying those models.

6 Conclusions

We have constructed a class of affine NA Toda models from the gauged two-loop WZW models, in which left and right symmetries are incorporated by a suitable choice of grading operator $Q$. Such framework is specified by grade $\pm 1$ constant generators $\epsilon_{\pm}$ and the pair $(Q, \epsilon_{\pm})$ determines the model in terms of a zero grade subgroup $G_0$. We have shown that for non abelian $G_0$, it is possible to reduce even further the phase space by constraining to zero the currents commuting with $\epsilon_{\pm}$, $(J \in G_0^0)$ to the fields lying in the coset $G_0/G_0^0$ only. There exists two manners to gauge fix $G_0 = U(1)$-the axial and the vector ones. Similarly to the T-duality transformations between the axil and vector gauged G/H WZW models,
one can find off-critical counterpart of the conformal T-duality, relating now the axial and vector families of IM’s constructed in Sect2. We further analize the problem of deriving a Lie algebraic condition which defines a class of T-selfdual torsionless models, for the case \( G_0 = U(1) \). The action for those models were sistematically constructed and shown to coincide with the models proposed by Fateev [6], describing the strong coupling limit of specific 2-d models representing sine-Gordon interacting with Toda-like models. Their weak coupling limit appears to be the Thirring model coupled to certain affine Toda theories [6].

Following the same line of arguments of the previous sections, one can construct more general models, say, \( G_0 / G_0 = \frac{\text{SL}(2) \otimes U(1)^{n-1}}{U(1)^{n-1}} \), \( G_0 / G_0 = \frac{\text{SL}(2) \otimes \text{SL}(2)}{U(1)^{n-2}} \), \( G_0 / G_0 = \frac{\text{SL}(3) \otimes U(1)^{n-2}}{U(1)} \), etc. Those models represent more general NA affine Toda models obtained by considering specific gradations \( Q_{a,b,...} = h_{a,b,...}d + \sum_{i \neq a,b,...} \frac{2\lambda_i H_i}{\alpha_i^2} \). However the important problem of the classification of all integrable models obtained as gauged two loop \( G \)-WZNW models remains open.

The connections constructed above provide soliton solution by using dressing transformation formalism. Specific examples of soliton behaviour will be reported in a separate publication[8],[16].

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