The Cosmic Microwave Background for a Nearly Flat Compact
Hyperbolic Universe

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ABSTRACT
The fluctuations of the cosmic microwave background (CMB) are investigated for a hyperbolic universe with finite volume. Four-component models with radiation, matter, vacuum energy, and an extra dark energy component, which is a particular version of quintessence, are considered. The general solution of the Friedmann equation for the cosmic scale factor $a(t)$ is given for the four-component models in terms of the Weierstrass \( \wp \)-function. The angular power spectra $C_l$ of the CMB anisotropy are computed for nearly flat models with $\Omega_{\text{tot}} \leq 0.95$. It is shown that the particular compact fundamental cell, which is considered in this paper, leads to a suppression in $C_l$ for $l \lesssim 10$ and $\Omega_{\text{tot}} \lesssim 0.9$.

Key words: cosmology:theory – cosmic microwave background – large–scale structure of universe – topology – dark matter – cosmological constant – quintessence

1 INTRODUCTION

The two crucial properties of the universe at large scales are its curvature and its topology. Both properties are encoded in the cosmic microwave background (CMB), see e.g. (??; ??; ??), which is measured with ever increasing resolution. The detection of the first acoustic peak in the CMB angular power spectrum $C_l$ (??) provides evidence for a flat, or nearly flat, cold dark matter universe with a non-vanishing cosmological constant and/or an extra exotic energy component. The recent Boomerang (??) and MAXIMUM-1 (??) measurements yield evidence even for the second acoustic peak which is, however, less pronounced than expected by standard CMB models. These CMB scenarios are based on isentropic initial perturbations in a universe composed of radiation, baryonic matter according to the Big Bang nucleosynthesis, cold dark matter, and a non-vanishing cosmological constant.

Possible explanations for the low second peak and the surprisingly large scale of the first peak are discussed in (??; ??; ??; ??). The constraints obtained from the MAXIMUM-1 experiment are $\Omega_{\text{tot}} = 0.90 \pm 0.15$, $\Omega_{\text{bar}} h_0^2 = 0.025 \pm 0.010$, $\Omega_{\text{cdm}} h_0^2 = 0.13 \pm 0.10$, and a spectral index $n = 0.99 \pm 0.09$ at 95% confidence level (??). Here $\Omega_{\text{cdm}}$ and $\Omega_{\text{bar}}$ denote the ratio of the cold dark matter (cdm) and baryonic (bar) energy densities, respectively, to the critical energy density.

The standard models describe the structure of the acoustic peaks, but fail to match with the low quadrupole moment $C_2$ of the COBE experiment (??). This can be interpreted as a hint for a non-trivial topology of our universe. The standard models suppose a trivial topology implying a universe with infinite volume for negative and zero curvature. Models with a non-trivial topology lead to a finite volume and to a suppression in the angular power spectrum $C_l$ for low multipoles. This motivates the study of non-trivial topologies (??) which leads to multiple images of a single source called topological lensing (??), and the just mentioned suppression in the large scale CMB anisotropy on which we concentrate in this paper. For flat models the topological length scale is constrained to be significantly larger than half the diameter of the observable universe (??) which renders these models unattractive. (See, however (??).) Thus in the following we discuss models with negative curvature.

The computation of the CMB anisotropy for compact hyperbolic universes can be carried out in two different ways. On the one hand one can compute the fluctuations by using the so-called method of images which requires the group elements which define the fundamental cell of the considered non-trivial topology (??; ??; ??; ??). On the other hand one can use a method which requires the eigenmodes of the fundamental cell with respect to the Laplace-Beltrami operator of the considered space. With the latter method the CMB anisotropy is computed for hyperbolic universes with a vanishing cosmological constant $\Lambda$ for a compact orbifold (??) and several compact manifolds (??; ??).

Our main aim in this paper is to incorporate a non-vanishing cosmological constant $\Lambda$ and, in addition, an extra dark energy component $\rho_{\text{quint}}$ in the anisotropy calculations as suggested by current observations. Recent investigations, in particular of the luminosity-redshift surveys of Type Ia supernovae (??; ??), strongly indicate that current observations require apart from a matter density component $\Omega_{\text{mat}} = \Omega_{\text{cdm}} + \Omega_{\text{bar}} \simeq 0.3 \pm 0.1$ an additional unclustered, dark energy component of 60% of the total energy density of the universe with negative pressure (??; ??) corresponding to an accelerated expansion. In the following we will assume that the
new component is a mixture of vacuum energy, $\Omega_{\text{vac}}$, or cosmological constant $\Lambda$, and a dark energy component, $\Omega_{\text{quint}}$, which is a particular version of quintessence. Whereas a cosmological constant $\Lambda > 0$ corresponds to a constant homogeneous energy component $\epsilon_{\text{vac}} > 0$ with negative pressure and equation of state $w_{\text{vac}} = -1$, the particular quintessence component considered by us consists of a dynamical, time-dependent energy density $\epsilon_{\text{quint}} > 0$ with negative pressure and equation of state $w_{\text{quint}} = -\frac{2}{3}$. (Here $w$ denotes the ratio of pressure to energy density.) In a general context, the quintessence component is generated by a slowly evolving scalar field with an exponential or inverse power law potential and $-1 < w_{\text{quint}} \leq 0$ (\cite{2, 2, 2, 2}). The quintessence models and the more recent “tracker field” models (\cite{2}) for the concept of “k-essence” and (\cite{2}) have been introduced to solve the two cosmological constant problems (\cite{2, 2, 2, 2, 2}). The quintessence component is generated by a slowly evolving scalar field with an exponential or inverse power law potential and $-1 < w_{\text{quint}} \leq 0$ (\cite{2, 2, 2}). The quintessence models and the more recent “tracker field” models (\cite{2}) have been introduced to solve the two cosmological constant problems (\cite{2, 2}).

However, since the concept of a scalar quintessence field is a purely classical phenomenological one, we expect the quantum mechanics of such a theory to be plagued with the usual problem of nonrenormalizability (\cite{2, 2}). We therefore adopt in this paper the point of view of an effective quintessence model, where we do not start from a given potential for the scalar field, but rather give the redshift behavior $\epsilon_{\text{quint}} \sim a^{-1}$ and thus the equation of state $w_{\text{quint}} = -\frac{2}{3}$, which is consistent with the above cited experimental bounds ($a$ is the cosmic scale factor, see the next section). Furthermore, we will assume that the quintessence component is spatially constant which can be understood as follows (\cite{2, 2}). In linear perturbation theory spatial gradients in the scalar field act like particles with very low mass that cannot bind to a nonrelativistic gravitational potential well that is much smaller than the Hubble length $H^{-1}$. Thus dynamical studies of groups and clusters of galaxies with size $\ll H^{-1}$ cannot detect concentrations in $\epsilon_{\text{quint}}$, and thus $\epsilon_{\text{quint}}$ can be assumed to be spatially constant.

The fundamental cell which we consider in this paper is the same pentahedron as considered in (\cite{2}) with Weyl’s law for the tetrahedral fundamental cell with Dirichlet boundary conditions (dashed curve, not visible) and Weyl’s law for a manifold with volume $0.25$ (dotted curve), $R = 1$. The considered pentahedron has a volume $\text{vol}(M) \approx 0.7173068R^3$. Since only one of two symmetry classes is taken into account, the following computations correspond to a hyperbolic manifold with $\text{vol}(M) \approx 0.3586534R^3$. However, a volume comparison is complicated by the fact that Weyl’s law has additional terms for orbifolds in comparison to manifolds, in particular the surface term (\cite{2}) being absent in the case of manifolds. The additional surface term leads to a suppression of $\mathcal{N}(E)$ in comparison with manifolds as shown in figure 1 for $E < 3026$. The figure demonstrates that the considered orbifold mimics a manifold with effective volume $\approx 0.25R^3$. The statistical properties of the eigenmodes are expected to be of the same random nature as observed in quantum chaos (\cite{2, 2}).

2 THE BACKGROUND MODEL

The standard cosmological model based on the Friedmann-Le
damtre-Robertson-Walker metric ($c = 1$)

$$\text{ds}^2 = a^2(\eta) \left\{ d\eta^2 - \gamma_{ij} dx^i dx^j \right\}$$

is governed for negative curvature ($K = -1$) by the Friedmann equation

$$a^2 - a^2 = \frac{8\pi G}{3} \tau_0 a^3 \tau'$$

where $a(\eta)$ is the scale factor and $\tau$ the conformal time. The prime denotes differentiation with respect to $\eta$. The energy-momentum tensor for an ideal fluid is given by

$$T^\mu_\nu = (\epsilon + p) u^\mu u_\nu - p \delta^\mu_\nu ,$$

where $u^\mu$ is the four-velocity of the fluid, and $\epsilon = \epsilon(\eta)$ denotes the energy density and $p = p(\eta)$ the pressure. In the following we consider multi-component models containing a matter-energy density $\epsilon_{\text{mat}}$, a radiation density $\epsilon_{\text{rad}}$ as well as a non-vanishing cosmological constant $\Lambda = 8\pi G \epsilon_{\text{vac}}$ and a spatially constant quintessence component $\epsilon_{\text{quint}}$ with an equation of state $p_{\text{quint}} = -\frac{2}{3} \epsilon_{\text{quint}}$. Then

$$\mathcal{N}(E) \sim \frac{\text{vol}(M)}{6\pi^2} k^3 \quad \text{with} \quad k := \sqrt{E - 1} .$$

The first lowest eigenmodes determine the largest scales of the CMB anisotropies. The smaller the volume the stronger is the suppression of the first multipoles in the angular power spectrum $C_l$.
the 00-component of the energy-momentum tensor is given in comoving coordinates by

\[ T^0_{0} = \sum_{k=0}^{4} \epsilon_k \left( \frac{a_0}{a} \right)^k, \]

expressed in terms of the current radiation density \( \epsilon_{4,0} = \epsilon_{\text{rad},0} \), the current matter density \( \epsilon_{3,0} = \epsilon_{\text{mat},0} \), the current quintessence component \( \epsilon_{1,0} = \epsilon_{\text{quint},0} \) and a vacuum energy density \( \epsilon_{0,0} = \epsilon_{\text{vac}} \). Here \( a_0 := a(\eta_0) \) is the scale factor of the present epoch. The present conformal time \( \eta_0 \) is implicitly given by

\[ a(\eta_0) = \frac{1}{H_0} \sqrt{1 - \Omega_{\text{tot}}}, \quad \Omega_{\text{tot}} = \Omega_{\text{rad}} + \Omega_{\text{mat}} + \Omega_{\text{quint}} + \Omega_{\text{vac}}, \]

where \( H_0 = h_0 \times 100 \text{ km s}^{-1} \text{Mpc}^{-1} \) denotes Hubble’s constant and \( \Omega_k := \epsilon_k / \epsilon_{\text{crit}} \) with \( \epsilon_{\text{crit}} = 3H_0^2/(8\pi G) \). With

\[ \Omega_2 := \Omega_{\text{curv}} := \frac{K}{(a_0 H_0)^2} = \frac{1}{(a_0 H_0)^2} = 1 - \Omega_{\text{tot}} > 0 \]

the Friedmann equation reads

\[ a'(\eta) = H_0 \sqrt{\sum_{k=0}^{4} \Omega_k \epsilon_k a^d-4k}. \tag{1} \]

This gives the infinitely far future \( \eta_\infty \) as

\[ \eta_\infty = \frac{1}{H_0} \int_0^\infty \frac{da}{\sqrt{\sum_{k=0}^{4} \Omega_k \epsilon_k a^{-4k}}}, \tag{2} \]

which yields \( \eta_\infty \leq \infty \) for a large class of models, see below. Notice that the various components redshift like \( a^{-4} \) with an associated equation of state \( w_k := \epsilon_k / \epsilon_{\text{crit}} = (k - 3)/3 \).

Let us define the following quantities

\[ A := \frac{1}{2} \Omega_{\text{mat}} H_0^2 a_0^2 = \frac{2a_0}{\eta}, \]

\[ B := \frac{1}{4} \Omega_{\text{quint}} H_0^2 a_0, \]

\[ C := \frac{1}{12} A^2 \eta^2 \Lambda \]

with

\[ \hat{\eta} := 2\sqrt{\frac{\Omega_{\text{rad}}}{H_0^2 \Omega_{\text{mat}}} \sim \left( 1 + \frac{1}{\sqrt{2}} \right) \eta}, \]

where the subscript “eq” marks the epoch of matter-radiation equality, and \( a_{eq} := a(\eta_{eq}) = a_0 (\Omega_{\text{rad}} / \Omega_{\text{mat}}) \). With the initial conditions \( a(0) = 0 \) and \( a'(0) > 0 \), equation (1) has the unique solution

\[ a(\eta) = \frac{A}{2} \left( \frac{\eta}{\eta_{eq}} - \frac{1}{\sqrt{2}} - \hat{\eta} \frac{\eta}{\eta_{eq}} + AB \hat{\eta}^2 \right)^{1/2}. \tag{3} \]

Here \( \mathcal{P}(\eta) \) denotes the Weierstrass \( \mathcal{P} \)-function which can numerically be evaluated very efficiently by

\[ \mathcal{P}(\eta) = \mathcal{P}(\eta; g_2, g_3) = \frac{1}{\eta^2} + \sum_{k=2}^{\infty} c_k \eta^{2k-2}, \tag{4} \]

with

\[ c_2 := \frac{g_2}{20}, \quad c_3 := \frac{g_3}{78} \]

and (?)

\[ c_k = \frac{3}{(2k+1)(k-3)} \sum_{m=2}^{k-2} c_m c_{k-m} \text{ for } k \geq 4. \]

Figure 2. The scale factor is shown for nearly flat models with \( \Omega_{\text{mat}} = 0.9 \) and \( h_0 = 0.7 \). The full curve corresponds to \( \Omega_{\text{mat}} = 0.9, \Omega_{\text{quint}} = \Omega_{\text{vac}} = 0.0 \), the dashed curve to \( \Omega_{\text{mat}} = 0.3, \Omega_{\text{quint}} = 0.6 \) and \( \Omega_{\text{vac}} = 0.0 \), and the dotted curve to \( \Omega_{\text{mat}} = 0.3, \Omega_{\text{quint}} = 0.0 \) and \( \Omega_{\text{vac}} = 0.6 \). The last model has the smallest \( \eta_\infty \). The present scale factor \( a_0 \) is indicated by a dot.

The so-called invariants \( g_2 \) and \( g_3 \) are determined by the cosmological parameters

\[ g_2 = \frac{1}{12} + 4C - 2AB \]

\[ g_3 = \frac{1}{216} - \frac{8C - A^2 \Lambda}{12} + \frac{AB}{6} - A^2 B^2 \hat{\eta}^2 \]

For cosmologically plausible parameter choices the series (4) needs only to be evaluated by taking into account the first twenty terms and thus the explicit solution (3) is much more efficient than the usual integration of the Friedmann equation.

Expanding (3) in a series at \( \eta = 0 \) gives the scale factor at early times

\[ a(\eta) = A \left\{ \hat{\eta} \eta + \frac{\hat{\eta}}{2} + \frac{\eta^3}{3!} + \left( 1 + 12AB \hat{\eta}^2 \right) \frac{\eta^4}{4!} \right\}. \]

This expansion shows that the quintessence term \( B \) influences the scale factor one power in \( \eta \) lower than the cosmological constant term \( C \).

In the case \( \Lambda > 0 \) and/or \( B > 0 \) the conformal time \( \eta \) is restricted to \( 0 \leq \eta < \eta_\infty \), where \( \eta_\infty \), defined in (2), is obtained from the implicit relation

\[ \mathcal{P}(\eta; \eta_\infty; g_2, g_3) = \frac{1}{12} + \sqrt{C}. \]

In the case \( \Lambda > 0 \) the scale factor \( a(\eta) \) has a simple pole at \( \eta = \eta_\infty \) with residue \( -\sqrt{3/\Lambda} \), i.e.

\[ a(\eta) = \sqrt{\frac{3/\Lambda}{\eta_\infty - \eta}} - \frac{3}{\Lambda} B + O(\eta - \eta_\infty), \]

which leads to an exponential expansion of the universe with scale factor \( R(t) = a(\eta(t)) = O(\exp(\sqrt{3/\Lambda} t)) \) for cosmic time \( t \to \infty \).

For \( \Lambda = 0 \) and \( \Omega_{\text{quint}} > 0 \) one has at \( \eta = \eta_\infty \) a double pole, i.e.

\[ a(\eta) = \frac{1}{(\eta - \eta_\infty)^2} - \frac{1}{12B} + O(\eta - \eta_\infty)^2 \]

leading to \( R(t) = Bt^2 + \ldots \) for \( t \to \infty \), which follows from the exact
formula

\[
t = -\frac{1}{12B}\eta(t) + \frac{1}{B} \zeta(\eta_\infty - \eta(t)) - \frac{1}{B} \zeta(\eta_0) ,
\]

where \( \zeta(\eta) := \zeta(\eta; g_2, g_3) \) denotes the Weierstrass zeta function. In this special case formula (3) reduces to \((P(\eta_0) = \frac{1}{\eta_0}) \)

\[
a(\eta) = \frac{1}{B} P(\eta - \eta_0) - \frac{1}{12B} , \quad \text{with} \quad 0 \leq \eta < \eta_0 .
\]

Finally, if both the cosmological constant and the quintessence component vanish, \( B = C = 0 \), the invariants simplify to \( g_2 = \frac{1}{12} \) and \( g_3 = -\frac{1}{126} \), and the \( P \)-function can be expressed in terms of an elementary function (2)

\[
P \left( \eta; \frac{1}{12}, -\frac{1}{216} \right) = \frac{1}{12} + \frac{1}{4 \sinh^2 \frac{\eta}{2} } ,
\]

which, with (3), leads to \( a(\eta) = A(\eta \sinh \eta + \cosh \eta - 1) \), which is the well-known expression for the scale factor of a two-component model consisting of radiation and matter only. This leads to \( R(t) = t + A t n + O(1) \) for \( t \to \infty \).

In table 1 the scale factor \( a_0 \) and the present age of the universe \( t_0 \) as well as several cosmologically important times, i.e. \( \eta_{eq}, \eta_{SLS}, \eta_0 \) and \( \eta_\infty \), are given for each combination of \( \Omega_{\text{quint}} \) and \( \Omega_{\text{vac}} \). The number \( N_{\text{Pentahedron}} \) of pentahedrons within the surface of last scattering is also presented.

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<th>( \Omega_{\text{quint}} )</th>
<th>( \Omega_{\text{vac}} )</th>
<th>( a_0 ) [cm]</th>
<th>( t_0 ) [Gyr]</th>
<th>( \eta_{eq} )</th>
<th>( \eta_{SLS} )</th>
<th>( \eta_0 )</th>
<th>( \eta_\infty )</th>
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Table 1. The present value of the scale factor \( a_0 \) and the present age of the universe \( t_0 \) as well as several important conformal times, i.e. \( \eta_{eq}, \eta_{SLS}, \eta_0 \) and \( \eta_\infty \), are given for each combination of \( \Omega_{\text{quint}} \) and \( \Omega_{\text{vac}} \). The number \( N_{\text{Pentahedron}} \) of pentahedrons within the surface of last scattering is also presented.

The present age of the universe \( t_0 \) is very close to the limit given by globular cluster ages 13.5 ± 2.0 Gyr (?) and \( t_0 \) is close to the limit given by globular cluster ages 13.5 ± 2.0 Gyr (?) and \( t_0 \) is close to the limit given by globular cluster ages 13.5 ± 2.0 Gyr (?) and \( t_0 \) is close to the limit given by globular cluster ages 13.5 ± 2.0 Gyr (?) and 11.29 ± 0.7 Gyr (?). White dwarf cooling rates lead to an age of our galaxy of 9.3 ± 2.0 Gyr (?) or 8.0 ± 1.5 Gyr (?). For a smaller Hubble constant, e.g. \( h_0 = 0.6 \), the age \( t_0 \) becomes larger with 13.08 Gyr \( \leq t_0 \leq 14.45 \) Gyr.

### 3 THE CMB ANISOTROPY

In the following we consider only scalar perturbations and their influence on the CMB. The metric with scalar perturbations is written in the Newton conformal gauge in terms of scalar functions \( \Phi \) and \( \Psi \) as

\[
d s^2 = a^2(\eta) \left( 1 + 2 \Phi(t) \right) d\eta^2 - (1 - 2 \Psi(t) \gamma_{ij} dx^i dx^j ) ,
\]

where \( \Phi = \Psi \) for a diagonal \( T_{\mu \nu} \). Assuming negligible entropy perturbations \( \delta S = 0 \), the evolution of the metric perturbation \( \Phi \) gives in first-order perturbation theory in the Newton conformal gauge (?)

\[
\Phi'' + 3 \dot{H}(1 + c_s^2) \Phi' - c_s^2 \Delta \Phi + \{ 2 \dot{H}' + (1 + 3 c_s^2)(\dot{H}^2 + 1) \} \Phi = 0 ,
\]

where \( \dot{H} = \frac{\dot{a}}{a} \) and \( \ddot{H} = \frac{\ddot{a}}{a} \).
where $\dot{\hat{t}} := a/d, \ddot{\hat{t}} = (\partial^2/p/\partial t^2)|_S$ and $\Delta$ denotes the Laplace-Beltrami operator. Specifying $\Phi$ at $\eta = 0$ such that it corresponds to a scale-invariant (Harrison-Zel'dovich) spectrum, allows the computation of the time-evolution of the metric perturbation $\Phi$. This in turn gives the input to the Sachs-Wolfe formula (7) which reads for isotropic initial conditions

$$\frac{\delta T}{T} = 2\Phi(\eta_{SLS}, \bar{x}(\eta_{SLS})) - \frac{3}{2} \Phi(0, \bar{x}(0))$$

and

$$+ 2\int_{\bar{x}(\eta_{SLS})}^{\bar{x}(\eta)} d\eta \frac{\partial \Phi(\eta, \bar{x}(\eta))}{\partial \eta},$$

from which one obtains the desired temperature fluctuations $\delta T$ of the CMB. The conformal time at recombination, which defines the surface of last scattering, is denoted by $\eta_{SLS}$. For $\eta_{SLS} \gg \eta_{\text{eq}}$ the first two terms on the right-hand side are approximately

$$2\Phi(\eta_{SLS}, \bar{x}(\eta_{SLS})) - \frac{3}{2} \Phi(0, \bar{x}(0)) \simeq \frac{1}{3} \Phi(\eta_{SLS}, \bar{x}(\eta_{SLS})).$$

This is the so-called ordinary or naive Sachs-Wolfe term (NSW), whereas the other term in (5) is called integrated Sachs-Wolfe term (ISW).

The metric perturbation $\Phi$ is expanded with respect to the eigenmodes

$$\Delta \psi_n(\bar{x}) = -E_n \psi_n(\bar{x}), \quad k_n := \sqrt{E_n - 1},$$

of the considered compact orbifold, i.e.

$$\Phi(\eta, \bar{x}) = \sum_{n=1}^\infty f_n(\eta) \psi_n(\bar{x}),$$

which yields for $f_n(\eta)$ the differential equation

$$f_n''(\eta) + 3\dot{\hat{t}}(1 + c_0^2)f_n'(\eta) + \{c_0^2E_n + 2\dot{\hat{t}}^2 + (1 + 3c_0^2)(\dot{\hat{t}}^2 + 1)\}f_n(\eta) = 0,$$

where $\dot{\hat{t}}$ and $c_0^2$ are determined by the background model. The initial conditions are ($\alpha > 0$ is a normalization constant)

$$f_n(0) = \frac{\alpha}{\sqrt{k_n(k_n^2 + 1)}}$$

and

$$f_n'(0) = -\frac{f_n(0)}{\delta \eta},$$

which carry over to a Harrison-Zel'dovich spectrum having a spectral index $n = 1$ and selecting only the non-decaying modes. Using the eigenmodes the perturbation is defined obeying the periodicity condition imposed by the fundamental cell.

The time dependence of $f_n(\eta)$, determined by the background model (3) via (7), is obtained by numerical integration and is shown in figure 3 for three different models. The first model, shown in figure 3a, is completely dominated by matter, $\Omega_{\text{tot}} = \Omega_{\text{mat}} = 0.9$, whereas the other two models belong to $\Omega_{\text{mat}} = 0.3$ and 0.6 for $\Omega_{\text{quint}}$ and $\Omega_{\text{vac}}$, shown in b) and c), respectively. The latter two cases have a finite $\eta_{\text{eq}}$ at which the perturbation vanishes, i.e. ($\eta \to \eta_{\text{eq}}$)

$$f_n(\eta) \propto (\eta_{\text{eq}} - \eta)^{3/2 + \pi/4}$$

for the case $\Omega_{\text{quint}} > 0$ and $\Omega_{\text{vac}} = 0$, and

$$f_n(\eta) \propto \eta_{\text{eq}} - \eta$$

for $\Omega_{\text{quint}} = 0$ and $\Omega_{\text{vac}} > 0$. Furthermore, perturbation modes with wavelength $\lambda = 2\pi/\eta_{\text{eq}}$ will never enter the horizon in models with $\Omega_{\text{quint}} > 0$ and/or $\Omega_{\text{vac}} > 0$. The first decline of $f_n(\eta)$ from 1 to $\frac{\alpha}{\sqrt{k_n(k_n^2 + 1)}}$ for small values of $\eta$ is due to the transition from the radiation- to the matter-dominated epoch (see, e.g. (7)), which leads to the approximation (6).

Figure 3. The dependence of $f_n(\eta)/f_n(0)$ on the eigenvalue $E_n$ is shown for $\Omega_{\text{mat}} = 0.9$ and $\Omega_{\text{vac}} = 0.7$. The upper full curve corresponds to $E_n = 0$ and the lowest one to $E_n = 5000$. For the intermediate curves the energy is increased in steps of 1000. The dashed curve represents the result $f_{\text{mat}}(\eta) = 5(\sinh^2 \eta - 3\eta \sinh \eta + 4\cosh \eta - 4)/(\cosh \eta - 1)^3$ belonging to a pure-matter model with $c_0 = 0$ used in some related works. In a) the matter dominated case $\Omega_{\text{mat}} = 0.9$ is shown, whereas the other two cases shown in b) and c) belong to $\Omega_{\text{mat}} = 0.3$ and $\Omega_{\text{quint}} = 0.6, \Omega_{\text{vac}} = 0.0$, respectively, $\Omega_{\text{vac}} = 0.0, \Omega_{\text{vac}} = 0.6$. 
Figure 4. The angular power spectrum $\delta T_l = \sqrt{l(l+1)C_l/2\pi}$ is shown in $\mu K$ for models with a vanishing quintessence component.

Figure 5. The angular power spectrum $\delta T_l = \sqrt{l(l+1)C_l/2\pi}$ is shown in $\mu K$ for models with a quintessence component.
With the background model (3), the time-evolution (7), and the Sachs-Wolfe formula (5), the angular power spectrum of the CMB anisotropy

\[ C_l = \frac{1}{2l+1} \sum_{m=-l}^{l} |a_{lm}|^2 \]

can be computed, where \( a_{lm} \) are the expansion coefficients of the CMB anisotropy \( \delta T \) with respect to the spherical harmonics \( Y_l^m(\theta, \phi) \).

With the procedure outlined above, the angular power spectra \( \delta T_l := \sqrt{l(l+1)} C_l / 2\pi \) are computed for models with a vanishing and non-vanishing quintessence component and are shown in figures 4 and 5, respectively. The considered compact orbifold as well as the position of the observer is the same as in (7). The spectra in figures 4a) and 4b) for \( \Omega_{\text{tot}} = 0.5 \) and \( \Omega_{\text{tot}} = 0.8 \), respectively, show a plateau being normalized to 30\( \mu \)K. At higher values of \( l \) the angular power spectra \( \delta T_l \) rise again which is not shown here because our calculations do not take into account the necessary processes leading to the acoustic peak. However, the \( l \) values shown in the figure are well below the horizon length at the recombination epoch, and thus are not affected by causal processes. For low values of \( l \) one observes a nearly linear increase of \( \delta T_l \) which is caused by the finite size of the fundamental cell which in turn causes a cut-off in the \( k \)-spectrum. (The straight lines are drawn solely to guide the eyes.) One observes that the “bend” point, where the behavior turns from a linear increase to a plateau, decreases towards smaller values of \( l \) for increasing vacuum energy. If the amount of vacuum energy is replaced by the same energy contribution of a quintessence component one obtains qualitatively analogous angular power spectra because the behavior of \( f_a(\eta) \) shown in figure 3 is similar for vacuum energy and quintessence for \( \eta \lesssim \eta_0 \). The two models shown in figures 4c) and 4d) possess an even larger \( \Omega_{\text{tot}} \), i.e. \( \Omega_{\text{tot}} = 0.9 \) and \( \Omega_{\text{tot}} = 0.95 \), respectively. Here the suppression is much less pronounced than in the cases with \( \Omega_{\text{tot}} \lesssim 0.85 \). In the case \( \Omega_{\text{tot}} = 0.9 \) the quadrupole moment is larger than the other low multipoles which is due to the large integrated Sachs-Wolfe contribution (see below). In the other case \( \Omega_{\text{tot}} = 0.95 \) one observes very large fluctuations for low values of \( l \).

For four models with a quintessence component the angular power spectra \( \delta T_l \) are shown in figure 5. In figure 5a) and 5b) two models with \( \Omega_{\text{tot}} = 0.9 \) are shown, where in the first case the energy density is equally distributed between \( \Omega_{\text{mat}} \), \( \Omega_{\text{quint}} \) and \( \Omega_{\text{vac}} \), and in the second case the quintessence component dominates. One observes similar angular power spectra which is again explained by the similar behavior of \( f_a(\eta) \). In both cases the multipoles with \( l \lesssim 10 \) are suppressed. In figure 5c) and 5d) two models with a vanishing vacuum energy are shown for \( \Omega_{\text{tot}} = 0.9 \) and \( \Omega_{\text{tot}} = 0.95 \), respectively. In the latter case the suppression of low multipoles is blurred by very large fluctuations, which occur as in figure 4d).

The angular power spectrum does not go to zero at the smallest values of \( l \). This is due to the competition of the two contributions to \( \delta T_l \), i.e. the NSW and the ISW term in (5). As shown in figure 6 the NSW term gives a contribution which indeed vanishes for small values of \( l \) because of the cut-off in the eigenmode spectrum. But the ISW term adds an almost constant contribution which even increases towards small values of \( l \). This interplay is responsible for the fact that \( \delta T_l \) does not fall below \( \sim 15 \mu K \), where a plateau of 30\( \mu K \) is assumed. The relative contribution of the two terms is largely determined by the chosen initial conditions at \( \eta = 0 \). The inflationary models naturally suggest isentropic initial conditions and these are imposed in the above calculations. However, imposing isocurvature initial conditions leads to a much smaller ISW contribution relative to the NSW term. This is shown in figure 6, where in figure 6a) isentropic initial conditions according to most inflationary models and in figure 6b) isocurvature initial conditions are chosen. If the observed increase in \( \delta T_l \) would be approximately linear towards zero for nearly flat models, this would imply a small ISW contribution and this would point to isocurvature initial conditions.

To summarize the results, the anomalously low quadrupole moment obtained from the COBE measurements can be taken as a first sign for a universe with a finite volume. The presented calculations demonstrate that low multipoles occur for the considered compact fundamental domain even for nearly flat, but hyperbolic, models with \( \Omega_{\text{tot}} \lesssim 0.9 \). For even larger values of \( \Omega_{\text{tot}} \gtrsim 0.95 \) very large fluctuations occur which may also be an indication for a finite universe. Furthermore, the kind of increase in \( \delta T_l \) gives a clue to the initial conditions. Future experiments which survey the complete CMB sky like MAP and PLANCK, will have the required signal to noise ratio to reveal a possible finite universe.

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