Fractional Supersymmetry and Lie Algebras

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Abstract

Supersymmetry and super-Lie algebras have been consistently generalized previously. The so-called fractional supersymmetry and $F$–Lie algebras could be constructed starting from any representation $\mathcal{D}$ of any Lie algebra $\mathfrak{g}$. This involves taking the $F$th root of $\mathcal{D}$ in some sense. We show, after having constructed differential realization(s) of any Lie algebra, how fractional supersymmetry can be explicitly realized in terms of appropriate homogeneous monomials.

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1 Introduction

Describing the laws of physics in terms of underlying symmetries has always been a powerful tool. In this respect, it is interesting to study the kind of symmetries which are allowed in space-time. Within the framework of Quantum Field Theory (unitarity of the $S$ matrix etc) it is generally admitted that we cannot go beyond supersymmetry (SUSY). However, the no-go theorem stating that supersymmetry is the only non-trivial extension beyond the Poincaré algebra is valid only if one considers Lie or Super-Lie algebras. Indeed, if one considers Lie algebras, the Coleman and Mandula theorem [1] allows only trivial extensions of the Poincaré symmetry, i.e. extra symmetries must commute with the Poincaré generators. In contrast, if we consider superalgebras, the theorem of Haag, Lopuszanski and Sohnius [2] shows that we can construct a unique (up to the number of supercharges) superalgebra extending the Poincaré Lie algebra non-trivially. It may seem that these two theorems encompass all possible symmetries of space-time. But, if one examines the hypotheses of the above theorems, one sees that it is possible to imagine symmetries which go beyond supersymmetry. Several possibilities have been considered in the literature [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], the intuitive idea being that the generators of the Poincaré algebra are obtained as an appropriate product of more fundamental additional symmetries. These new generators are in a representation of the Lorentz group which can be neither bosonic nor fermionic (bosonic charges close under commutators and generate a Lie algebra, whilst fermionic charges close under anticommutators and induce super-Lie algebras). In an earlier work we proposed an algebraic structure, called an $F$–Lie algebra, which makes this idea precise in the context of fractional supersymmetry (FSUSY) of order $F$ [14]. Of course, when $F = 1$ this is a Lie algebra, and when $F = 2$ this is a Super-Lie algebra. Within the framework of this algebraic structure, we showed that starting from any representation $\mathcal{D}$ of any Lie algebra $g$ it is possible to take in some sense the $F$th–root of $\mathcal{D}$. This means that we were able to consider a representation $\mathcal{D}'$ such that the symmetric tensorial product of order $F$ $S^F(\mathcal{D}')$ is related to $\mathcal{D}$ [14]. The representation $\mathcal{D}'$ is in general an infinite dimensional representation of $g$, i.e. a Verma module [15]. The purpose of this note is to give an explicit way to realize this construction in terms of monomials, i.e. using appropriate differential realizations of $g$.

The content of this paper is as follow. In section 2, we summarize the results of [14] about $F$–Lie algebras. In section 3 we show how for any Lie algebra we can realize the representations in terms of homogeneous monomials constructing differential realization(s) of the algebra $g$. Finally, in section 4 FSUSY is realized for the algebra $g = su(3)$. Of course this construction works along the same lines for any Lie algebra.
2 Algebraic Structure of Fractional Supersymmetry

In this section, we recall the abstract mathematical structure which generalizes the theory of Lie super-algebras and their (unitary) representations. Let $F$ be a positive integer and $q = \exp \left( \frac{2i\pi}{F} \right)$. We consider a complex vector space $S$ together with a linear map $\varepsilon$ from $S$ into itself satisfying $\varepsilon^F = 1$. We set $A_k = S_{q^k}$ and $B = S_1$ (where $S_\lambda$ is the eigenspace corresponding to the eigenvalue $\lambda$ of $\varepsilon$) so that $S = B \oplus \bigoplus_{k=1}^{F-1} A_k$. The map $\varepsilon$ is called the grading. If $S$ is endowed with the following structures we will say that $S$ is a fractional super Lie algebra ($F$-Lie algebra for short):

1. $B$ is a Lie algebra and $A_k$ is a representation of $B$.

2. There are multilinear, $B$–equivariant (i.e. which respect the action of $B$) maps $\{ \ldots \} : S^F(A_k) \to B$ from $S^F(A_k)$ into $B$. In other words, we assume that some of the elements of the Lie algebra $B$ can be expressed as $F$–th order symmetric products of “more fundamental generators”. Here $S^F(D)$ denotes the $F$–fold symmetric product of $D$. It is then easy to see that:

$$\{\varepsilon(a_1), \ldots, \varepsilon(a_F)\} = \varepsilon(\{a_1, \ldots, a_F\}), \forall a_i \in A_k. \quad (2.1)$$

3. For $b_i \in B$ and $a_j \in A_k$ the following “Jacobi identities” hold:

$$[[b_1, b_2], b_3] + [[b_2, b_3], b_1] + [[b_3, b_1], b_2] = 0$$
$$[[b_1, b_2], a_3] + [[b_2, a_3], b_1] + [[a_3, b_1], b_2] = 0$$
$$[b, \{a_1, \ldots, a_F\}] = \{[b, a_1], \ldots, a_F\} + \ldots + \{a_1, \ldots, [b, a_F]\} \quad (2.2)$$
$$\sum_{i=1}^{F+1} [a_i, \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{F+1}\}] = 0.$$

The first identity is the usual Jacobi identity for Lie algebras, the second says that the $A_k$ are representation spaces of $B$ and the third is just the Leibniz rule (or the equivariance of $\{ \ldots \}$). The fourth identity is the analogue of the graded Leibniz rule of Super-Lie algebras for $F$–Lie algebras.

If we want to be able to talk about unitarity, we also require the following additional structure and in this case, $S$ is called an $F$–Lie algebra with adjoint.
4. A conjugate linear map $\dagger$ from $S$ into itself such that:

- (a) $(s^\dagger)^\dagger = s, \forall s \in S$
- (b) $[a, b]^\dagger = [b^\dagger, a^\dagger]$
- (c) $\varepsilon(s^\dagger) = \varepsilon(s)^\dagger$
- (d) $\{a_1, \cdots, a_F\}^\dagger = \{(a_1)^\dagger, \cdots, (a_F)^\dagger\}, \forall a \in A_k.$

(2.3)

From a) and c) we see that for $X \in B$ we have $X^\dagger \in B$, and that for $X \in A_k$, we have $X^\dagger \in A_{F-k}$.

A unitary representation of an $F$–Lie algebra with adjoint $S$ is a linear map $\rho : S \to \text{End}(H)$, (where $H$ is a Hilbert space and $\text{End}(H)$ the space of linear operators acting on $H$) and a unitary endomorphism $\hat{\varepsilon}$ such that $\hat{\varepsilon}^F = 1$ which satisfy

- (a) $\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$
- (b) $\rho\{a_1, \cdots, a_F\} = \frac{1}{F!} \sum_{\sigma \in S_F} \rho(a_{\sigma(1)}) \cdots \rho(a_{\sigma(F)})$
- (c) $\rho(s)^\dagger = \rho(s^\dagger)$
- (d) $\hat{\varepsilon}\rho(s)^{-1} = \rho(\varepsilon(s))$

(2.4)

$(S_F$ being the group of permutations of $F$ elements). As a consequence of these properties, since the eigenvalues of $\hat{\varepsilon}$ are $F^{th}$ roots of unity, we have the following decomposition of the Hilbert space

$$H = \bigoplus_{k=0}^{F-1} H_k,$$

where $H_k = \{|h\rangle \in H : \hat{\varepsilon}|h\rangle = q^k |h\rangle\}$. The operator $N \in \text{End}(H)$ defined by $N |h\rangle = k |h\rangle$ if $|h\rangle \in H_k$ is the “number operator” (obviously $q^N = \hat{\varepsilon}$). Since $\hat{\varepsilon}\rho(b) = \rho(b)\hat{\varepsilon}, \forall b \in B$ each $H_k$ provides a representation of the Lie algebra $B$. Furthermore, for $a \in A_\ell$, $\hat{\varepsilon}\rho(a) = q^\ell \rho(a)\hat{\varepsilon}$ and so we have $\rho(a).H_k \subseteq H_{k+\ell(\text{mod } F)}$

Several remarks can be made at that point Firstly, for all $k = 1, \cdots, F - 1$ it is clear that the subspace $B \oplus A_k$ of $S$ satisfies (2.1-2.2) and the subspace $B \oplus A_k \oplus A_{-k}$ satisfies (2.1-2.3) (when $S$ has an adjoint). Secondly, it is important to notice that bracket $\{\cdots\}$ is a priori not defined for elements in different gradings.

The basic idea to define a fractional supersymmetry is the following. Let $g$ be a Lie algebra and let $\mathcal{D}, \mathcal{D}'$ be representations of $g$. The representation $\mathcal{D}'$ is chosen in such a
way that $S^F(D')$ is related to $D$ ($S^F(D') \sim D$) in a sense specified later on. Then we consider $B = g \oplus D$, a Lie algebra as the semi-direct product of $g$ and $A_1 = D'$. The relation $S^F(D') \sim D$ will be the fundamental one to define an $F$–Lie algebra.

### 3 Differential realization(s) of Lie algebras

We consider now $g$ a semi-simple Lie algebra of rank $r$. Let $h$ be a Cartan sub-algebra of $g$, let $\Phi \subset h^*$ (the dual of $h$) be the corresponding set of roots and let $f_\alpha$ be the one dimensional root space associated to $\alpha \in \Phi$. We chose a basis $\{H_i, i = 1, \cdots, r\}$ of $h$ and elements $E^\alpha \in f_\alpha$ such that the commutation relations become

\[
[H_i, H_j] = 0 \\
[H_i, E^\alpha] = \alpha^i E^\alpha \\
[E^\alpha, E^\beta] = \begin{cases} 
\epsilon(\alpha, \beta) E^{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi \\
\frac{2\alpha \cdot H}{\alpha \cdot \alpha} & \text{if } \alpha + \beta = 0 \\
0 & \text{otherwise}
\end{cases}
\]

(3.1)

We now introduce $\{\alpha(1), \cdots, \alpha(r)\}$ a basis of simple roots. The weight lattice $\Lambda_W(g) \subset h^*$ is the set of vectors $\mu$ such that $\frac{2\alpha \cdot \mu}{\alpha \cdot \alpha} \in \mathbb{Z}$ and, as is well known, there is a basis of the weight lattice consisting of the fundamental weights $\{\mu(1), \cdots, \mu(r)\}$ defined by $\frac{2\mu(i) \cdot \alpha(j)}{\alpha(i) \cdot \alpha(i)} = \delta_{ij}$. A weight $\mu = \sum_{i=1}^{r} n_i \mu(i)$ is called dominant if all the $n_i \geq 0$ and it is well known that the set of dominant weights is in one to one correspondence with the set of (equivalence classes of) irreducible finite dimensional representations of $g$.

Recall briefly how one can associate a representation of $g$ to $\mu \in h^*$. A given representation of $g$ can be defined from a highest weight state $|\mu\rangle (E^\alpha|\mu\rangle = 0, \alpha > 0, 2\frac{\alpha \cdot H}{\alpha \cdot \alpha}|\mu\rangle = n_i|\mu\rangle, i = 1, \cdots, r)$. We denote $h_i = 2\frac{\alpha(i) \cdot H}{\alpha(i) \cdot \alpha(i)}$. The space obtained from $|\mu\rangle$ by the action of the element of $g$: $E^{-\alpha(i_1)} \cdots E^{-\alpha(i_k)}|\mu\rangle$ clearly define a representation of $g$. This construction can be made more precise using the language of Verma module [14, 15]. This representation is denoted $D_\mu$. To come back to our original problem, consider a finite dimensional irreducible representation $D_\mu$. The basic idea to define an $F$–Lie algebra associated to $g \oplus D_\mu$ is to consider the infinite dimensional representation associated to the weight $\mu/F$. In [14], we defined an $F$–Lie algebra associated to $g, D_\mu$ and $D_{\mu/F}$. Here, we reproduce these results in an explicit way using the differential realization of $g$.

As we have recalled the representations of $g$ are just specified by the weight $\mu = \sum_{i=1}^{r} n_i \mu(i)$. But, among the representations of $g$ there are $r$ basic representations. These representations
are associated to the fundamental weights \( \mu = \mu_{(i)} \), and all representations can be obtained from (symmetric) tensorial product of these basic representations. Furthermore, all basic representations can be derived from the antisymmetric product of the elementary representations, which are associated to the weight with terminal point in the Dynkin diagram [16]. Consequently, if one is able to obtain a differential realization of the elementary representations of \( g \) (at maximum 3), one is able to construct all representations easily as we will see. Moreover, for \( su(n+1) = a_n, sp(2n) = c_n \), we need to consider only the fundamental representation (related to \( \mu = \mu_{(1)} \)). Although for \( so(2n+1) = b_n \) the vector \( (\mu = \mu_{(1)}) \) and the spinorial \( (\mu = \mu_{(n)}) \) representations and for \( so(2n) = d_n \) the vector \( (\mu = \mu_{(1)}) \) and the two spinorial representations \( (\mu = \mu_{(n-1)}), (\mu = \mu_{(n)}) \) reproduce all representations. For the exceptional Lie algebras some simplifications may happen from the embedding properties \( e_8 \subset so(16), e_7 \subset so(12) \oplus su(2), e_6 \subset so(10) \oplus u(1) \), and \( f_4, g_2 \) being of small rank calculations can be done easily. In the next subsection we just consider the series \( a_n, b_n, c_n \) and \( d_n \) and construct finite and infinite dimensional representations. It is important to emphasize that all the highest weight representations, finite and infinite dimensional, can be obtained in terms of homogeneous monomial of appropriate variables as a consequence of the differential realization of \( g \). In the next section we apply these realizations to construct explicitly FSUSY.

### 3.1 \( su(n) \)

\( su(n) \) is a rank \( n - 1 \) Lie algebra, but it is more convenient for our purpose to define roots as vectors of \( IR^n \). Introduce \( e_i, i = 1, \cdots, n \) an orthonormal basis of \( IR^n \) the simples roots of \( su(n) \) reads

\[
\alpha_{(i)} = e_i - e_{i+1}, \quad i = 1, \cdots, n - 1, \tag{3.2}
\]

and the positive roots

\[
\alpha = e_i - e_j, \quad 1 \leq i < j \leq n. \tag{3.3}
\]

Now if we introduce\(^2\) \( (x_1, \cdots, x_n) \in (IR^+ - \{0\})^n \) we can define the explicit realization of \( su(n) \):

\[
E^\alpha = E^{e_i - e_j} = x_i \partial_{x_j}, \quad 1 \leq i < j \leq n \tag{3.4}
\]

\[
h_i = x_i \partial_{x_i} - x_{i+1} \partial_{x_{i+1}}, \quad i = 1, \cdots, n - 1.
\]

\(^2\)From now on, all the variables, whatever \( su(n), so(n) \) or \( sp(2n) \) are concerned, are positive and different from zero.
Furthermore, all the highest weight representations of \( su(n) \) can be obtained from (3.4). Indeed, we can check using (3.4) that all the primitive vectors of the basic representations can be realized in terms of antisymmetric products of the \( x \)'s (in this notation we have \(|e_i > = x_i|\))

\[
\begin{align*}
|\mu(1) > & = |e_1 > = x_1, \\
|\mu(2) > & = |e_1 + e_2 > = x_1 \wedge x_2, \\
& \vdots \\
|\mu(n-1) > & = |e_1 + \cdots + e_{n-1} > = x_1 \wedge \cdots \wedge x_{n-1}.
\end{align*}
\]

(3.5)

And correspondingly the representation \( D_{\mu} \) associated to the weight \( \mu = \sum_{i=1}^{n-1} p_i \mu(i) \) with \( p_i \in IR \) can be obtained from the highest weight \(|\mu > = (x_1)^{p_1} \cdots (x_1 \wedge \cdots \wedge x_{n-1})^{p_{n-1}}| \) acting with the operators \( E^{-\alpha(i)} \) given in (3.4). When all the \( p_i \in IN \) we obtain a finite dimensional representation of \( su(n) \) otherwise the representation is infinite dimensional. Of course for \( su(n) \) only the finite dimensional representations are unitary. Moreover, when \( p_i \in IR \) there is no guarantee that the representation can be exponentiated, namely that they are representation of the Lie group \( SU(n) \). Finally, notice that the normalizations in (3.4) and (3.5) are not the usual ones, but they are useful for to construct \( F \)-Lie algebras. These last properties are also valid for all compact Lie algebras.

### 3.2 \( sp(2n) \)

Using, as for \( su(n) \), the canonical basis of \( IR^n, e_i, i = 1, \ldots, n \) the simple roots of \( sp(2n) \) are

\[
\alpha(i) = e_i - e_{i+1}, \quad i = 1, \ldots, n-1, \quad \alpha(n) = 2e_n.
\]

(3.6)

\( sp(2n) \) is not a simply-laced algebra (all roots do not have the same length). The positive roots are given by

\[
e_i \pm e_j, \quad 1 \leq i < j \leq n, \quad 2e_i, \quad 1 \leq i \leq n.
\]

(3.7)

Constructing the \( 2n \)-dimensional representation associated to \( \mu(1) \) and introducing \( 2n \) variables corresponding to the weight of the representation \( D_{\mu(1)} \) \( x_i = |e_i >, x_{-i} = | - e_i > \) is not difficult to define the operators \( E^{\pm \alpha(i)} \) and \( n_i \):

\[
\begin{align*}
E^{\alpha(i)} & = E^{e_i-e_{i+1}} = x_i \partial_{x_{i+1}} + x_{-(i+1)} \partial_{x_{-i}}, \quad 1 \leq i \leq n-1 \\
E^{\alpha(n)} & = E^{2e_n} = x_n \partial_{x_{-n}}
\end{align*}
\]
\[
E^{-\alpha(i)} = E^{-e_i+e_{i+1}} = x_{i+1} \partial x_i + x_{i-1} \partial x_{i+1}, \quad 1 \leq i \leq n-1
\]
\[
E^{-\alpha(n)} = E^{-2e_n} = x_{-n} \partial x_n
\]
\[
h_i = x_i \partial x_i - x_{i+1} \partial x_{i+1} - x_i \partial x_{i-1} + x_{-(i+1)} \partial x_{-(i+1)}, \quad i = 1, \ldots, n-1.
\]
\[
h_n = x_n \partial x_n - x_{-n} \partial x_{-n}.
\]

The notations has been chosen in such a way that comparing the weights of the variables \(x > 0\) with the weights of the generators of \(sp(2n)\), the expression of the \(E^\alpha\) can directly be read. This holds equally for \(su(n)\) and \(so(n)\). Then, the remaining generators can be calculated using the generators associated to the primitives roots. Noticing that \(E^{\alpha_i-e_j}\) maps \(|-e_i \mapsto | -e_j >\) and \(|e_j > \mapsto |e_i >\) we get \(E^{\alpha_i-e_j} = ax_i \partial x_j + bx_j \partial x_i\). The coefficients \(a, b\) can be determined by multiple commutators. Indeed, writing \((j > i) \ j = i + k\), we get \(e_i - e_{i+k} = \alpha(i) + \cdots + \alpha(i+k-1)\) and \(E^{e_i-e_j} = [E^{\alpha(i+k-1)}, \ldots [E^{\alpha(i+1)}, E^{\alpha(i)}] \cdots]\). And similarly for the other generators. Now as for \(su(n)\), all representation can be obtained from (3.8):

\[
|\mu(1) > = |e_1 >= x_1,
\]
\[
|\mu(2) > = |e_1 + e_2 >= x_1 \wedge x_2,
\]
\[
\vdots
\]
\[
|\mu(n-1) > = |e_1 + \cdots + e_{n-1} >= x_1 \wedge \cdots \wedge x_{n-1},
\]
\[
|\mu(n) > = |e_1 + \cdots + e_n >= x_1 \wedge \cdots \wedge x_n.
\]

To obtain all representations we proceed as for \(su(n)\). Some remarks can be done at that point. For \(sp(2n)\), \(\left(D_{\mu(1)} \otimes D_{\mu(1)}\right)_{\text{anti sym}} = D_{\mu(2)} \oplus D_0\) (\(D_0\) being the scalar representation) as can be seen directly from (3.9) and (3.8). Similar results hold for \(D_{\mu(1)}\).

### 3.3 \(so(2n+1)\)

This Lie algebra is the algebra dual of \(sp(2n)\) and his simple roots are obtained from the simple roots of \(sp(2n)\) \((\alpha(i))_{so(2n+1)} = 2(\alpha(i))_{sp(2n)}/((\alpha(i))_{sp(2n)})^2\). With the same notations as for \(su(n)\) and \(sp(2n)\) we have the simple roots

\[
\alpha(i) = e_i - e_{i+1}, \quad i = 1, \ldots, n-1, \quad \alpha(n) = e_n.
\]

The positive roots are given by

\[
e_i \pm e_j, \quad 1 \leq i < j \leq n, \quad e_i, \quad 1 \leq i \leq n.
\]

The vectorial representation (of dimension \(2n + 1\)) allows to define the set of variables \(x_i = |e_i >, x_0 = |0 >\) and \(x_{-i} = |-e_i >\). Constructing explicitly the vectorial representation we obtain

\[
\text{...}
\]
\[ E^{\alpha(i)} = E^{\varepsilon_i - \varepsilon_{i+1}} = x_i \partial_{x_{i+1}} + x_{-(i+1)} \partial_{x_{-i}}, \quad 1 \leq i \leq n - 1 \]
\[ E^{\alpha(n)} = E^{\varepsilon_n} = x_n \partial_{x_0} + x_0 \partial_{x_n} \]
\[ E^{-\alpha(i)} = E^{-\varepsilon_i + \varepsilon_{i+1}} = x_{i+1} \partial_{x_{i}} + x_{-(i+1)} \partial_{x_{-(i+1)}}, \quad 1 \leq i \leq n - 1 \]
\[ E^{-\alpha(n)} = E^{-\varepsilon_n} = x_0 \partial_{x_n} + x_n \partial_{x_0} \]
\[ h_i = x_i \partial_{x_i} - x_{i+1} \partial_{x_{i+1}} - x_i \partial_{x_{-i}} + x_{-(i+1)} \partial_{x_{-(i+1)}}, \quad i = 1, \ldots, n - 1. \]
\[ h_n = x_n \partial_{x_n} - x_{n-1} \partial_{x_{n-1}}. \]

If consider the $p$-forms as for $su(n)$ and $sp(2n)$ we observe that

\[ |\mu(1) \rangle = |e_1 > = x_1, \]
\[ |\mu(2) \rangle = |e_1 + e_2 > = x_1 \wedge x_2, \]
\[ \vdots \]
\[ |\mu(n-1) \rangle = |e_1 + \cdots + e_{n-1} > = x_1 \wedge \cdots \wedge x_{n-1}, \]
\[ |2\mu(n) \rangle = |e_1 + \cdots + e_n > = x_1 \wedge \cdots \wedge x_n, \]

and all representations, except the spinorial one $D_{\mu(n)}$ can be obtained form (3.13) and (3.12). Then if we want to obtain all representations in terms of appropriate monomials another differential realization, associated to $D_{\mu(n)}$ has to be constructed along the same lines (constructing explicitly the spinorial representation). However, if one consider the highest weight states $|\mu_n \rangle = (x_1 \wedge \cdots \wedge x_n)^{1/2}$ one is able to define a primitive vector having all properties of the primitive vector of the spinorial representation. But, in such a representation the operators $E^\alpha$ are not nilpotent. This means that this representation is precisely a Verma module $V_{\mu(n)}$. Then $V_{\mu(n)}$ has a unique maximal proper sub-representation $M_{\mu(n)}$ and the quotient $V_{\mu(n)}/M_{\mu(n)}$ is $D_{\mu(n)}$ (see e.g. [14, 15]) \(^3\). But, constructing explicitly the representation $V_{\mu(n)}$ and introducing at each step a new variable, $y_1, \ldots, y_{2n}$, one is able to construct the differential realization of the spinorial representation. Indeed, for any representation $D_\mu$ such a process can be equally applied and the differential realization of $D_\mu$ can be reached straightforwardly. Of course this is also possible for $su(n), sp(2n)$ and $so(2n)$.

\(^3\)If we use (3.12) for $n = 1$ i.e. for $so(3)$ and we consider $(x_1)^{1/2}$ as a primitive vector of the spinorial representation we can easily see that for any $p > 0$ \( (E^{-\alpha(1)})^p (x_1)^{1/2} \neq 0 \) but \( E^{\alpha(1)} (E^{-\alpha(1)})^2 (x_1)^{1/2} = 0. \) So the representation $V = \left\{ (E^{-\alpha(1)})^p (x_1)^{1/2}, p \geq 0 \right\}$ is precisely a Verma module. But it can be easily seen that $M = \left\{ (E^{-\alpha(1)})^p (x_1)^{1/2}, p \geq 2 \right\}$ is the maximal sub-representation of $V$ (\( \forall m \in M, E^{\alpha(1)}(m), E^{-\alpha(1)}(m), h_1(m) \in M \)) and then $D = V/M$ is the two-dimensional spinorial representation.
3.4 \( so(2n) \)

The case of \( so(2n) \) is similar to \( so(2n + 1) \) but in this case we have two spinorial representations. With the same notations as before we introduce the set of primitive roots

\[
\alpha_{(i)} = e_i - e_{i+1}, \quad i = 1, \ldots, n-1, \quad \alpha_{(n)} = e_{n-1} + e_n,
\]

and the positive roots are given by

\[
e_i \pm e_j, \quad 1 \leq i < j \leq n.
\]

From the \( 2n \)-dimensional vector representation we obtain \( (x_i = |e_i >, x_{-i} = |-e_i >) \)

\[
E_{\alpha_{(i)}} = E^{e_i - e_{i+1}} = x_i \partial_{x_{i+1}} + x_{-(i+1)} \partial_{x_{-i}}, \quad 1 \leq i \leq n-1
\]
\[
E_{\alpha_{(n)}} = E^{e_{n-1} + e_n} = x_{n-1} \partial_{x_{-n}} + x_n \partial_{x_{-(n-1)}}
\]
\[
E^{-\alpha_{(i)}} = E^{-e_i + e_{i+1}} = x_{i+1} \partial_{x_i} + x_{-i} \partial_{x_{-(i+1)}}, \quad 1 \leq i \leq n-1
\]
\[
E^{-\alpha_{(n)}} = E^{-e_{n-1} - e_n} = x_{n-1} \partial_{x_{n-1}} + x_{-(n-1)} \partial_{x_n}
\]
\[
h_i = x_i \partial_{x_i} - x_{i+1} \partial_{x_{i+1}} - x_{-i} \partial_{x_{-i}} + x_{-(i+1)} \partial_{x_{-(i+1)}}, \quad i = 1, \ldots, n-1.
\]
\[
h_n = x_{n-1} \partial_{x_{n-1}} + x_n \partial_{x_n} - x_{n-1} \partial_{x_{n-1}} - x_{-(n+1)} \partial_{x_{-(n+1)}}.
\]

And, as for \( so(2n + 1) \) we observe that the \( p \)-forms are

\[
|\mu_{(1)} > = |e_1 > = x_1,
\]
\[
|\mu_{(2)} > = |e_1 + e_2 > = x_1 \wedge x_2,
\]
\[
\vdots
\]
\[
|\mu_{(n-1)} + \mu_{(n)} > = |e_1 + \cdots + e_{n-1} > = x_1 \wedge \cdots \wedge x_{n-1},
\]
\[
|2\mu_{(n)} > = |e_1 + \cdots + e_n > = x_1 \wedge \cdots \wedge x_n.
\]

Then two more differential realizations (the two spinorial) allow to construct all representations of \( so(2n) \). However, if we use (3.16) for all representations of \( so(2n) \), the spinorial representations will be infinite dimensional Verma modules.

To conclude this section recall once again that on the one hand the differential realization(s) considered for \( su(n) \), \( sp(2n) \), \( so(2n + 1) \) and \( so(2n) \) are just a convenience and simply reproduce known results on Lie algebras. Of course similar expressions (in terms of bosonic or fermionic oscillators) are given in standard text-books (see e.g. [17]) but to my knowledge they were not used to construct representations in a systematic way from the fundamental one \( D_{\mu_{(1)}} \), except for \( su(2) \), \( sl(2, IR) \) and \( sl(2, \mathbb{C}) \) (see [16, 18]). Indeed, the advantage of
the differential realization (3.4), (3.8), (3.12) and (3.16) is to permit constructing explicitly the representations in terms of appropriate monomials. For such representations, finite or infinite dimensional, one, two or at maximum three realizations of the Lie algebra \( g \) are just needed. In addition, acting with (3.4), (3.8), (3.12) or (3.16), on the states of the corresponding representation \( \mathcal{D}_\mu \) (of dimension \( d \)), and introducing new variables \( y_1, \ldots, y_d \) at each step, permits the construction of the differential realization of \( \mathcal{D}_\mu \) directly form \( \mathcal{D}_\mu(1) \).

These differential realizations are also convenient to obtain representations neither bounded from below nor from above (for instance, in the case of \( su(n) \), the representation obtained with \( x_1^{p_1} \cdots x_n^{p_n}, p_i \notin \mathbb{N} \) has no primitive vector). On the other hand, they are very useful to construct FSUSY as shown in the next section.

4 Fractional supersymmetry and Lie algebras

For any Lie algebra \( g \) and any representation \( \mathcal{D}_\mu \), one is able to construct an associated FSUSY as it has been established in [14]. One possible solution is to consider the infinite dimensional representation \( \mathcal{D}_{\mu/F} \). At that point, the results of the previous section apply and allow to define an \( F \)-Lie algebra associated to \( g, \mathcal{D}_\mu \) and \( \mathcal{D}_{\mu/F} \). Indeed, this explicit procedure has only be done for \( so(1,2) \) [12, 14]. For higher rank Lie algebras this construction was obtained abstractly in terms of Verma modules [14]. But now, we observe that the differential realization of \( so(1,2) \) extends along the same lines for any Lie algebra. Then, it becomes possible to realize, as it has been done for \( so(1,2) \), an \( F \)-Lie algebra for any Lie algebra. This explicit construction being analogous for any \( g \) we just reproduce it for the rank two Lie algebra \( su(3) \) (weight diagrams and representations can be represented graphically) but we have to keep in mind that this procedure, to construct an \( F \)-Lie algebra, is equally valid for any Lie algebra \( g \).

Denote \( \alpha, \beta \) the simple roots of \( su(3) \) and \( \gamma = \alpha + \beta \) the third positive root. Introduce, as in section 3, \( x_1, x_2, x_3 > 0 \) of weight \( \vert \mu(1) \rangle = x_1, \vert \mu(1) - \alpha \rangle = x_2, \vert \mu(1) - \alpha - \beta \rangle = x_3 \). The generators takes the form

\[
\begin{align*}
E^\alpha &= x_1 \partial_{x_2} \\
E^{-\alpha} &= x_2 \partial_{x_1} \\
E^\beta &= x_2 \partial_{x_3} \\
E^{-\beta} &= x_3 \partial_{x_2} \\
E^\gamma &= x_1 \partial_{x_3} \\
E^{-\gamma} &= x_3 \partial_{x_1} \\
h_1 &= x_1 \partial_{x_1} - x_2 \partial_{x_2} \\
h_2 &= x_2 \partial_{x_2} - x_3 \partial_{x_3},
\end{align*}
\]

We would like to construct an \( F \)-Lie algebra associated to the three dimensional repre-
presentation $\mathcal{D}_{\mu(1)}$ (this could have been done for any representation of $su(3)$). In the realization (4.1) the vectorial representation writes

$$
\mathcal{D}_{\mu(1)} = \begin{cases} 
    x_1 = |\mu(1)>, \\
    x_2 = |\mu(1) - \alpha > = E^{-\alpha} |\mu(1)>, \\
    x_3 = |\mu(1) - \alpha - \beta > = E^{-\beta} E^{-\alpha} |\mu(1)>.
\end{cases}
$$

(4.2)

So, we consider

$$
B = su(3) \oplus \mathcal{D}_{\mu(1)}
$$

(4.3)

for the bosonic (graded zero part) of the $F$–Lie algebra. The natural representation to define the “$F^\text{th}$–root” of $\mathcal{D}_{\mu(1)}$ is $\mathcal{D}_{\mu(1)}/F$. So, we take for the graded one part

$$
A_1 = \mathcal{D}_{\mu(1)}/F
$$

(4.4)

In the realization (4.1) this representation writes

$$
\mathcal{D}_{\mu(1)}/F = \left\{ \frac{\mu(1)}{F} - n\alpha - p\beta > = (x_1)^{1/F} \left(\frac{x_1}{x_2}\right)^n \left(\frac{x_1}{x_2}\right)^p \sim \right.

\left. \left( E^{-\beta} \right)^p \left( E^{-\alpha} \right)^n |\frac{\mu(1)}{F}> , n \in \mathbb{N}, 0 \leq p \leq n \right\},
$$

(4.5)

leading to the following weight diagram

```
Figure 1: Weight diagram for the $\mathcal{D}_{\mu(1)}/F$ representation of $su(3)$. The down-left corner represents the positive roots of $su(3)$. The representations are infinite dimensional in the direction of $\alpha$ and $\gamma$, but finite dimensional in the direction $\beta$.
```
To define an $F$–Lie algebra associated to $\text{su}(3) \oplus \mathcal{D}_{\mu(1)} \oplus \mathcal{D}_{\mu(1)}/F$ we consider the representation (reducible)

$$\mathcal{S}^F \left( \mathcal{D}_{\mu(1)}/F \right) = \left\{ \bigotimes_{i=1}^F (x_1)^{1/F} \left( \frac{x_1}{x_2} \right)^{n_i} \left( \frac{x_3}{x_4} \right)^{p_i}, \ n_i \in \mathbb{N}, 0 \leq p_i \leq n_i \right\}, \quad (4.6)$$

with $\otimes$ the symmetric tensorial product.

In we compare $x_1$, the primitive vector of $\mathcal{D}_{\mu(1)}$ with $\otimes^F (x_1)^{1/F}$ we observe that these two vectors, as primitive vectors, satisfy the same properties

$$h_1(x_1) = x_1, \quad h_1(\otimes^F (x_1)^{1/F}) = \otimes^F (x_1)^{1/F},$$

$$h_2(x_1) = 0, \quad h_2(\otimes^F (x_1)^{1/F}) = 0,$$

$$E^{a,b,c}(x_1) = 0, \quad E^{a,b,c}(\otimes^F (x_1)^{1/F}) = 0. \quad (4.7)$$

But now, if we construct the representation from these primitives vectors, on the one hand we get $\mathcal{D}_{\mu(1)}$ and on the other hand the infinite dimensional representation

$$\langle \otimes^F (x_1)^{1/F} \rangle = \left\{ \left| \mu(1) - n\alpha - p\beta \right| =\right.$$

$$\left( E^{-\beta} \right)^p \left( E^{-\alpha} \right)^n \left( \otimes^F (x_1)^{1/F} \right), n \in \mathbb{N}, 0 \leq p \leq n \}. \quad (4.8)$$

But a direct calculation show that the following relations hold

$$E^\alpha |\mu(1) - 2\alpha >= 0$$

$$E^\alpha |\mu(1) - 2\alpha - \beta >= 0$$

$$E^\gamma |\mu(1) - 2\alpha - 2\beta >= 0. \quad (4.9)$$

This means that $\mathcal{V}_{\mu(1)} = \langle \otimes^F (x_1)^{1/F} \rangle$ is a Verma module (the operator $E^\alpha$ is not nilpotent) and $M = \left\{ \left( E^{-\beta} \right)^p \left( E^{-\alpha} \right)^n \left( \otimes^F (x_1)^{1/F} \right), n \in \mathbb{N}, 0 \leq p \leq n, (n, p) \neq (0, 0), (1, 0), (1, 1) \right\}$ is the maximal proper sub-representation of $\mathcal{V}_{\mu(1)}$ ($\forall T \in \text{su}(3), \forall m \in M, T(m) \in M$). Then $\mathcal{V}_{\mu(1)}/M$ and $\mathcal{D}_{\mu(1)}$ are isomorphic. Then, from $\mathcal{V}_{\mu(1)} \subseteq \mathcal{S}^F \left( \mathcal{D}_{\mu(1)}/F \right)$ one can find an injection $i: \mathcal{V}_{\mu(1)} \longrightarrow \mathcal{S}^F \left( \mathcal{D}_{\mu(1)}/F \right)$. Conversely, from $\mathcal{D}_{\mu(1)} \equiv \mathcal{V}_{\mu(1)}/M$ we can define a surjection $\pi: \mathcal{V}_{\mu(1)} \longrightarrow \mathcal{D}_{\mu(1)}$. Consequently the following diagram

$$\mathcal{S}^F \left( \mathcal{D}_{\mu(1)/F} \right) \xleftarrow{i} \mathcal{V}_{\mu(1)} \xrightarrow{\pi} \mathcal{D}_{\mu(1)},$$

shows that we cannot define an $F$–Lie algebra in such a way (because we cannot find a mapping from $\mathcal{S}^F \left( \mathcal{D}_{\mu(1)/F} \right)$ into $\mathcal{D}_{\mu(1)}$ as stated in property 2 of $F$–Lie algebras, see section
2). Indeed, in [12] FSUSY in $1 + 2$ dimensions was constructed along these lines but we did not have the structure of $F$–Lie algebra.

To obtain an $F$–Lie algebra some constraints have to be introduced. Following [14], we define $\mathcal{F}$ the vector space of functions on $x_1, x_2, x_3 > 0$. The multiplication map $m_n : \mathcal{F} \times \cdots \times \mathcal{F} \rightarrow \mathcal{F}$ given by $m_n(f_1, \cdots, f_n) = f_1 \cdots f_n$ is multilinear and totally symmetric. Hence, it induces a map $\mu^F$ from $\mathcal{S}^F(\mathcal{D}_{\mu})$ into $\mathcal{F}$. Restricting to $\mathcal{S}^F(\mathcal{D}_{\mu(1)})$ one gets

\[
\mu^F : \mathcal{S}^F(\mathcal{D}_{\mu(1)}) \rightarrow \mathcal{S}^F_{\text{red}}(\mathcal{D}_{\mu(1)})
\]

\[
\bigotimes_{i=1}^F (x_1)^{1/F} \left( \frac{x_1}{x_2} \right)^{n_i} \left( \frac{x_2}{x_3} \right)^{p_i} \rightarrow x_1 \left( \sum_{i=1}^F x_1^{n_i} x_2^{p_i} \right).
\]

(4.10)

We observe that $\mathcal{S}^F_{\text{red}}(\mathcal{D}_{\mu(1)}) = \left\{ x_1 \left( \frac{x_2}{x_1} \right)^n \left( \frac{x_3}{x_2} \right)^p, n \in \mathbb{N}, 0 \leq p \leq n \right\} \supset \mathcal{D}_{\mu(1)}$, meaning that we can find an injection $\iota'$ from $\mathcal{D}_{\mu(1)}$ into $\mathcal{S}^F_{\text{red}}(\mathcal{D}_{\mu(1)})$. This representation is reducible but indecomposable. Namely, we cannot find a complement of $\mathcal{D}_{\mu(1)}$ in $\mathcal{S}^F_{\text{red}}(\mathcal{D}_{\mu(1)})$ stable under $su(3)$. For instance $x_1 \left( \frac{x_2}{x_1} \right)^2$ is such that $E^\alpha \left( x_1 \left( \frac{x_2}{x_1} \right)^2 \right) = 2x_2$, but $E^{-\alpha}(x_2) = 0$. As before we observe that the diagram

\[
\mathcal{D}_{\mu(1)} \xrightarrow{\iota'} \mathcal{S}^F(\mathcal{D}_{\mu(1)}/\mathcal{F})_{\text{red}} \xleftarrow{\mu^F} \mathcal{S}^F(\mathcal{D}_{\mu(1)/\mathcal{F}})
\]

leads to the same conclusion on the structure of $F$–Lie algebra. With these simple observations we can conclude as in [14] that we cannot define an $F$–Lie algebra with $B = su(3) \oplus \mathcal{D}_{\mu(1)}$. To obtain such a structure, one possible solution is to extend $\mathcal{D}_{\mu(1)}$ into an infinite dimensional representation. For instance,

\[
(su(3) \oplus \mathcal{S}^F(\mathcal{D}_{\mu(1)/\mathcal{F}}))_{\text{red}} \oplus \mathcal{D}_{\mu(1)/\mathcal{F}}
\]

(4.11)

has a structure of $F$–Lie algebra (a similar structure could have been defined with $(su(3) \oplus \mathcal{V}_{\mu(1)}) \oplus \mathcal{D}_{\mu(1)/\mathcal{F}}$).

The problem to construct an $F$–Lie algebra associated to $su(3), \mathcal{D}_{\mu(1)}, \mathcal{D}_{\mu(1)/\mathcal{F}}$ is basically related to the fact that we would like to relate a finite dimensional representation $\mathcal{D}_{\mu(1)}$ with an infinite dimensional one $\mathcal{D}_{\mu(1)/\mathcal{F}}$. One possible solution, as we just have seen, is to extend $\mathcal{D}_{\mu(1)}$ into an infinite dimensional (reducible but indecomposable) representation $\mathcal{S}^F(\mathcal{D}_{\mu(1)/\mathcal{F}})_{\text{red}}$. This procedure being quite general, works similarly for any Lie algebra $g$. 

13
5 Conclusion

Under some assumptions symmetries beyond supersymmetry can be constructed. Fractional
supersymmetry and \( F \)-Lie algebras are a possible solutions. In this note, we have shown
that differential realization(s) of Lie algebras is(are) an useful tool(s) to construct explicitly
a structure of \( F \)-Lie algebra associated to any representation \( \mathcal{D}_\mu \) of any Lie algebra \( g \). The
basic point is to consider the infinite dimensional representation \( \mathcal{D}_\mu/F \). We have shown, that
considering an infinite dimensional (reducible but indecomposable) representation extending
\( \mathcal{D}_\mu \) enables us to construct an \( F \)-Lie algebra (this solves the problem related to the fact
that, in general, \( \mathcal{D}_\mu \) is finite dimensional, although \( \mathcal{D}_\mu/F \) is infinite dimensional). Another
possible solution, if to extend the Lie algebra \( g \) into a infinite dimensional Lie algebra.
When \( g = so(1,2) \) this algebra reduces to the centerless Virasoro algebra [14]. Another
advantage of the differential realization of section 3 is the possibility to construct explicitly,
in a differential way, this infinite dimensional algebra (this will be done elsewhere, at least
for \( su(n) \)).

Finally, we would like to conclude that for \( g = so(1,2) \) unitary representations have been
constructed [12]. It has been observed that it is a symmetry which acts on relativistic anyons
[19]. The question of the interpretation of FSUSY in higher dimensional space-time is still
open.

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