From the recent neutrino oscillation experiments [1] it becomes affirmative that neutrinos have masses. The present and near future experiments enter into the stage of precision test for masses and lepton mixing angles. In a series of papers we have discussed the relations among the parameters which appear in the neutrino oscillation experiments and in the neutrinoless double beta decay \((\beta\beta)_{0w}\) experiments. The most recent experimental upper bound for the averaged mass \((m_\nu)_ee\) for Majorana neutrinos from \((\beta\beta)_{0w}\) is given by \((m_\nu)_ee < 0.2 \text{eV} [2]\). The next generation experiment GENIUS \([3]\) is anticipated to reach a considerably more stringent limit \((m_\nu)_ee < 0.01 - 0.001 \text{eV}\). In these situations the investigation of the CP violation effects in the lepton sector has become more and more important. In the previous paper we proposed \([4]\) the graphical method for obtaining the constraints on \(CP\) violation phases which are partly supported by theoretical models \([8]\). We also discuss some matrix models which lead to the above \(CP\) violating factors take ±1 or ±i in the neutrinoless double beta decay for illustrative clearance. We also discuss some mass matrix models which lead to the above \(CP\) violating factors.

The averaged mass \((m_\nu)_{ee}\) obtained from \((\beta\beta)_{0w}\) is given \([9]\) by the absolute values of averaged complex masses for Majorana neutrinos as

\[
(m_\nu)_{ee} = |M_{ee}|. \tag{1}
\]

Here the averaged complex mass \(M_{ee}\) is defined by

\[
M_{ee} \equiv \sum_{j=1}^{3} U_{ej}^2 m_j \quad \equiv |U_{e1}|^2 m_1 + |U_{e2}|^2 e^{2i\beta} m_2 + |U_{e3}|^2 e^{2i(\rho-\phi)} m_3. \tag{3}
\]

We now rewrite the complex mass \(M_{ee}\) in terms of the phases \(\beta, \rho, \phi\) appearing in \(U\) as

\[
M_{ee} = |U_{e1}|^2 m_1 + |U_{e2}|^2 \eta_2 m_2 + |U_{e3}|^2 \eta_3 m_3 \tag{6}
\]

Let us review briefly the graphical representations \([4]\) of the complex mass \(M_{ee}\). We now rewrite the complex mass \(M_{ee}\) as

\[
M_{ee} = |U_{e1}|^2 \tilde{m}_1 + |U_{e2}|^2 \tilde{m}_2 + |U_{e3}|^2 \tilde{m}_3 \tag{7}
\]
Here we have defined the complex masses $\tilde{m}_i (i = 1, 2, 3)$ by

$$
\tilde{m}_1 \equiv m_1, \quad \tilde{m}_2 \equiv m_2 e^{2i\beta} m_2, \\
\tilde{m}_3 \equiv m_3 e^{2i\rho} m_3,
$$

(8)

The $M_{ee}$ is the "averaged" complex mass of the masses $\tilde{m}_i (i = 1, 2, 3)$ weighted by three mixing elements $|U_{ej}|^2 (j = 1, 2, 3)$ with the unitarity constraint $\sum_{j=1}^3 |U_{ej}|^2 = 1$. Therefore, the position of $M_{ee}$ in a complex mass plane is within the triangle formed by the three vertices $\tilde{m}_i (i = 1, 2, 3)$ if the magnitudes of $|U_{ej}|^2 (j = 1, 2, 3)$ are unknown (Fig.1). This triangle is referred to as the complex-mass triangle [4]. The three mixing elements $|U_{ej}|^2 (j = 1, 2, 3)$ indicate the division ratios for the three portions of each side of the triangle which are divided by the parallel lines to the side lines of the triangle passing through the $M_{ee}$ (Fig.2). The $CP$ violating phases $2\beta$ and $2\rho'$ represent the rotation angles of $\tilde{m}_2$ and $\tilde{m}_3$ around the origin, respectively. Since $\langle m_\nu \rangle_{ee} = |M_{ee}|$, the present experimental upper bound on $\langle m_\nu \rangle_{ee}$ (we denote it by $\langle m_\nu \rangle_{max}$) indicates the maximum distance of the point $M_{ee}$ from the origin and forms the circle in the complex plane.

So far we have discussed in the scheme of the three flavour mixings. The sterile neutrino is not ruled out yet [11]. We can easily incorporate the sterile neutrino in our formulation. $M_{ee}$ is written in this case as

$$
M_{ee} = \sum_{j=1}^4 |U_{ej}|^2 \tilde{m}_j = |U_{e1}|^2 \tilde{m}_1 + |U_{e2}|^2 \tilde{m}_2 \\
+ (|U_{e3}|^2 + |U_{e4}|^2) \frac{|U_{e3}|^2 \tilde{m}_3 + |U_{e4}|^2 \tilde{m}_4}{|U_{e3}|^2 + |U_{e4}|^2} \\
\equiv |U_{e1}|^2 \tilde{m}_1 + |U_{e2}|^2 \tilde{m}_2 + |U_{e3}|^2 \tilde{m}_3
$$

(9)

and Fig.2 is modified as Fig.3.
TABLE I. Given the \( \eta_i = \pm 1 \) or \( \pm i \), the constraints on the averaged mass \( \langle m_{\nu} \rangle_{ee} \) are given. Notations are as follows. \( |U_{e3}|^2_{\text{max}} = 0.026 \), \( A_i \equiv m_i + |U_{e3}|^2_{\text{max}}(m_3 - m_i) \), \( B_{ij} \equiv m_i m_j / |U_{e3}|^2_{\text{max}} \), \( C \equiv m_1 - |U_{e3}|^2_{\text{max}}(m_1 + m_3) \), \( D_{ij} \equiv m_i(m_j - |U_{e3}|^2_{\text{max}}(m_j + m_3))/\sqrt{m_i^2 + m_j^2} \) and \( E_{ij} \equiv \sqrt{(1 - |U_{e3}|^2_{\text{max}}) m_i^2 + |U_{e3}|^2_{\text{max}} m_j^2} \). The max \((a, b)^\ast\) indicates the larger value between \( a \) and \( b \). The double signs of \( \eta_2 \) and \( \eta_3 \) are in the same order.

\[
\begin{array}{c|c|c|c}
\eta_2 = 1 & \eta_2 = -1 & \eta_2 = \pm i \\
\hline
\eta_3 = 1 & m_1 \leq \langle m_{\nu} \rangle_{ee} \leq A_2 & 0 \leq \langle m_{\nu} \rangle_{ee} \leq \max\left( \frac{m_2}{A_1} \right) & B_{12} \leq \langle m_{\nu} \rangle_{ee} \leq \max\left( \frac{m_2}{A_1} \right) \\
\eta_3 = -1 & \max\left( \frac{0}{C} \right) \leq \langle m_{\nu} \rangle_{ee} \leq \max\left( \frac{m_2}{C} \right) & 0 \leq \langle m_{\nu} \rangle_{ee} \leq A_2 & \max\left( \frac{0}{D_{21}} \right) \leq \langle m_{\nu} \rangle_{ee} \leq \max\left( \frac{m_2}{D_{21}} \right) \\
\eta_3 = \pm i & (i) |U_{e3}|^2_{\text{max}} < m_1^2/(m_1^2 + m_2^2), & 0 \leq \langle m_{\nu} \rangle_{ee} \leq \max\left( \frac{m_2}{E_{23}} \right) & B_{12} \leq \langle m_{\nu} \rangle_{ee} \leq A_2 \\
& \quad E_{13} \leq \langle m_{\nu} \rangle_{ee} \leq \max\left( \frac{m_2}{E_{23}} \right) & & \\
& (ii) |U_{e3}|^2_{\text{max}} \geq m_1^2/(m_1^2 + m_2^2), & & \\
& \quad B_{13} \leq \langle m_{\nu} \rangle_{ee} \leq \max\left( \frac{m_2}{E_{23}} \right) & & \\
\hline
\eta_3 = \mp i & & & \max\left( \frac{0}{D_{12}} \right) \leq \langle m_{\nu} \rangle_{ee} \leq \max\left( \frac{m_2}{E_{23}} \right) \\
\end{array}
\]

FIG.5 The graph shows the allowed region of \( |U_{e2}|^2 \) and \( |U_{e3}|^2 \). The mixing elements \( |U_{e\ell}|^2 \) are expressed as the ratios of the heights from vertices \( m_i \) separated by \( M_{ee} \) (Fig.2). The mass triangle (left diagrams) is deformed to the isosceles right triangle (right diagrams) retaining these ratios. This figure is depicted setting \( m_1 : m_2 : m_3 : \langle m_{\nu} \rangle_{ee} = 6 : 8 : 10 : 5 \) and \( 2\beta = 5\pi/8 \) as an example. \( 2\rho^\prime \) is changed every \( \pi/3 \). There is no allowed region in upper-right half, because \( |U_{e2}|^2 + |U_{e3}|^2 = 1 - |U_{e1}|^2 \leq 1 \) from unitarity conditions.

We proceed to discuss the main theme to give the constraints on \( \langle m_{\nu} \rangle_{ee} \), given \( \eta_i = \pm 1 \) or \( \pm i \). We argue here in the three generations, though the arguments can be extended to the four generation case. We list up the constraints on \( \langle m_{\nu} \rangle_{ee} \) for the various combinations of given \( CP \) violating factors \( \eta_i \) and explain how our formulation works. (TABLE I)

We demonstrate the graphical method for \( \eta_2 = i \) and \( \eta_3 = 1 \) case. In this case Fig.1 becomes Fig.4. Here we impose the constraint on \( U_{e3} \) from the oscillation experiments of CHOOZ and SuperKamiokande [12], \( |U_{e3}|^2 < 0.026 \). The shaded region is allowed. \( \langle m_{\nu} \rangle_{ee} \) is the distance from the origin to the shaded region. So the minimum and maximum of \( \langle m_{\nu} \rangle_{ee} \) is easily estimated from Fig.4. Obviously, the minimum is \( OA \). The maximum value depends on whether the line \( PQ \) crosses the horizontal axis at a point larger than \( m_2 \) or not. If \( OP > m_2 \), the maximum is \( OP \). If \( OP < m_2 \), the maximum becomes \( m_2 \).

Of course, our method is applicable for arbitrary
Then, the mass matrix with the maximal mixing for \( \theta_1 \) by changing the complex-mass triangle to the isosceles right triangle (Fig. 5).

Next let us discuss a mass matrix model which gives the typical \( CP \) violating phases demonstrated above. In order to see the meaning of the typical \( CP \) violating factor \( \eta_2 = i \). Here the matrix elements \( M^* \) is the complex conjugate of \( M \). This matrix is diagonalized by a maximal mixing matrix and two mass eigen values remain to be independent free parameters as is shown

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
M & M^* \\
M^* & M
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
= \begin{pmatrix}
m_1 & 0 \\
0 & im_2
\end{pmatrix},
\]

where

\[
m_1 = 2 \text{Re}M, \quad m_2 = 2 \text{Im}M.
\]

This argument is generalized to the three generations as follows. We consider a model in which the mass matrices for the charged leptons and the Majorana neutrinos, \( M_e \) and \( M_\nu \), are given by

\[
M_e = \text{diag}(m_e, m_\mu, m_\tau),
\]

\[
M_\nu = \begin{pmatrix}
c_3 & 0 & s_3 \\
-s_2s_3 & -c_2 & s_2c_3 \\
c_2s_3 & s_2 & c_2c_3
\end{pmatrix}
\times
\begin{pmatrix}
M & M^* & 0 \\
M^* & M & 0 \\
0 & 0 & m_3
\end{pmatrix}^T.
\]

Then, the mass matrix \( M_\nu \) is diagonalized by an orthogonal matrix \( O_\nu \) with a pure imaginary eigenvalue for the second generation neutrino as

\[
O_\nu^T M_\nu O_\nu = \begin{pmatrix}
m_1 & im_2 \\
0 & m_3
\end{pmatrix},
\]

where

\[
O_\nu = \begin{pmatrix}
c_1c_3 & s_1c_3 & s_3 \\
-s_1c_2 - c_1s_2s_3 & c_1c_2 - s_1s_2s_3 & s_2c_3 \\
s_1s_2 - c_1c_2s_3 & -c_1s_2 - s_1c_2s_3 & c_2c_3
\end{pmatrix}.
\]

with the maximal mixing for \( \theta_1 \). Namely the assumption Eq.(14) leads to a pure imaginary eigenvalue \( im_2 \) and a maximal mixing \( c_1 = s_1 = \pi/4 \). Here \( m_1 \) and \( m_2 \) are defined in (12). The mass matrix is finally diagonalized with real diagonalized masses as

\[
U_\nu^T M_\nu U_\nu = \begin{pmatrix}
m_1 & m_2 & 0 \\
m_2 & m_3 & 0 \\
0 & 0 & m_3
\end{pmatrix},
\]

where

\[
U_\nu = O_\nu \begin{pmatrix}
1 & e^{i\pi/4} \\
e^{i\pi/4} & 1 \\
0 & 0
\end{pmatrix}.
\]

This mass matrix model corresponds to the case with \( \eta_2 = i \) and \( \eta_3 = 1 \) discussed before.

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