Dynamical Properties of Euclidean Solutions in a Multidimensional Cosmological Model

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In the framework of the Hartle-Hawking no-boundary proposal, we investigated quantum creation of the multidimensional universe with a cosmological constant ($\Lambda$) but without matter fields. We have found that the classical solutions of the Euclidean Einstein equations in this model have “quasi-attractors”, i.e., most trajectories on the $a$-$b$ plane, where $a$ and $b$ are the scale factors of external and internal spaces, go around a point. It is presumed that the wave function of the universe has a hump near this quasi-attractor point. In the case that both the curvatures of external and internal spaces are positive, and $\Lambda > 0$, there exist Lorentzian solutions which start near the quasi-attractor, the internal space remains microscopic, and the external space evolves into our macroscopic universe.

§1. Introduction

In modern theories of unified physical interactions, spacetime has more than four dimensions. It is wellknown that Kaluza-Klein theory$^1,2$ created interest in investigations in spacetime which has more than four dimensions. Superstring theories (e.g. Ref.$^3$) are at the moment the most promising candidates for a unified description of the basic physical interactions. There are five anomaly-free, perturbative superstring theories. The critical dimensions of spacetime are ten for these theories. There is now evidence that the five superstring theories are related by duality symmetries. Furthermore, they are related to $N = 1$, $D = 11$ supergravity theory. It is conjectured that these theories are the limits of one theory, M-theory, in which spacetime has eleven dimensions.

Since spacetime has four large dimensions and some additional number of small and highly curved spatial dimensions, we can see only the four large dimensions. How does compactification of internal space take place? The first answer is through multidimensional cosmology. In general relativity, the geometry of spacetime is dynamical. The three-dimensional space we observe was once as small as the internal space, and expanded during evolution of the universe, while internal space contracted or has remained small during evolution of the universe. Therefore, internal space is microscopic and is not observable. This explanation is called dynamical compactification. The second explanation involves by spontaneous symmetry breaking. Though spacetime would have $SO(D-1,1)$ for $D > 4$, after symmetry breaking only $SO(3,1)$ is visible, and internal space is static and at the Planck scale. This explanation is called spontaneous compactification. Lately, another possible mechanism of compactification by branes in string theory has been discussed.$^4,5$ It is possible that the Standard Model gauge fields
exist on branes rather than in the bulk of spacetime. If this is the case, the situation in which gravity exists in the bulk of the 10-dimensional spacetime, while the Standard Model particles exist on a 3-brane is possible. It remains unknown which mechanism of compactification explains our universe. In this paper we consider dynamical compactification only in the view of multidimensional cosmology.

The simplest multidimensional cosmological model that is useful for understanding compactification is the type of extended Friedmann-Robertson-Walker universe. The pioneering work for this model was done by Chodos and Detweiler. They found that the 5-dimensional vacuum Einstein equations possess the Kasner solution, which describes a universe in which the internal space shrinks, while the external space expands. Freund investigated the model in the case of $D = 11$ supergravity. Various multidimensional models were investigated, and the manner in which the existence of internal space might impact cosmological issues such as entropy production and inflation (e.g. Refs. 8-10) was discussed.

On the other hand, the creation of the universe is a problem of great interest. In the standard big-bang cosmology, the universe appeared from an initial singularity. When the density is greater than the Planck density, classical general relativity breaks down and quantum gravitational effects are large. Based on quantum gravitational theory, the scenario of quantum creation of the universe was suggested. Vilenkin asserted that the inflationary universe appears through quantum tunneling from nothing. Later Hartle and Hawking suggested the scenario of the creation of the universe through a different method. That method is called the Hartle-Hawking no-boundary proposal. According to this proposal the universe in the quantum era is Euclidean manifold without “boundary”, and through analytic continuation to the Lorentzian manifold, the classical universe is created. The condition of analytic continuation gives the initial condition of the universe. Recently, a pre-big-bang scenario has been suggested. This scenario is based on string theory and has attractive features. However, it has a graceful exit problem. In this paper we consider quantum cosmology from the viewpoint of Hartle and Hawking.

The concept of quantum creation is extended to the multidimensional universe with consistency. Hu and Wu discussed the vacuum model in which the spacetime metric has the form $R \times S^3 \times S^n$ in the framework of the Hartle-Hawking no-boundary proposal, where $R$ is the time, $S^3$ is the external space, and $S^n$ is the internal space. They showed that the universe most probably evolves with an exponentially expanding external space and a static internal space. This solution implies that we observe 4-dimensional spacetime because internal spaces are compact and static at Planckian scales. More complicated multidimensional quantum cosmological models have been investigated (e.g. Refs. 16-20, 25).

Models based on unified theories, such as supergravity theories, which are the low energy limits of superstring theories, would be more realistic than the simplest vacuum model. However, it is thought that the essence of the creation of the universe may be included in the vacuum model, and it is useful for investigating more realistic models. Detailed analysis of the vacuum model is, therefore, important. Hu and Wu discussed only the instanton that nucleates the most probable universe, but the instanton solutions are unstable with respect to perturbations, and the Euclidean classical solutions near the instanton have effects on the wave function of the universe. Therefore
we analyse instantons in more detail, and also investigate dynamical properties of the
Euclidean Einstein equations of this model by numerical calculations. As shown in §3, the
dynamical properties of the system is of interest, and it is found that the system
has a “quasi-attractor”.

The plan of the paper is as follows. In §2, we introduce the multidimensional cosmo-
lological model with cosmological constant ($\Lambda$) but without matter fields. In §3, we
give the constraints on the signatures of the spatial curvature and the cosmological
constant and analyze the Euclidean classical solutions. In §4, we investigate contin-
uation from the Euclidean classical solutions to Lorentzian solutions, and discuss the
scenario quantum creation of the universe. The last section is devoted to conclusions
and remarks.

§2. The model

We consider a $D(= 1 + m + n)$-dimensional vacuum universe with a cosmological
constant ($\Lambda$) but without matter fields. For cosmological purposes, we assume a metric
of the form

$$\tilde{g}_{AB} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & a^2(t)g_{ij} & 0 \\ 0 & 0 & b^2(t)g_{IJ} \end{pmatrix},$$

(2.1)

where $i, j = 1, 2, \ldots, m$ and $I, J = 1, 2, \ldots, n$. The variables $a$ and $b$ are the scale factors
of $m$-dimensional and $n$-dimensional spaces, respectively. The gravitational action is
described as

$$S = \frac{1}{16\pi G} \int d^Dx \sqrt{-\tilde{g}} \left[ \tilde{R} - 2\Lambda \right] + S_{\text{boundary}},$$

(2.2)

where $\tilde{G}$ is the $D$-dimensional gravitational constant, and $\Lambda$ is the cosmological con-
stant. The last term $S_{\text{boundary}}$ is the York-Gibbons-Hawking boundary term.\textsuperscript{21, 22) Sub-
stituting the metric, the action is given by

$$S = \frac{1}{16\pi G} \int dt d^m x d^n x \sqrt{g_{m}g_{n}a^{m}b^{n}}$$

$$\left[ -m(m-1)\left( \frac{\dot{a}}{a} \right)^2 - 2mn \frac{\dot{a}\dot{b}}{ab} - n(n-1)\left( \frac{\dot{b}}{b} \right)^2 \right]$$

$$+ m(m-1)k_{m}a^{2} + n(n-1)k_{n}b^{2} - 2\Lambda \right],$$

(2.3)

where the dots denote derivatives with respect to time $t$, $g_{m}$ and $g_{n}$ are the determinants
of the metrics $g_{ij}$ and $g_{IJ}$, and $k_{m}$ and $k_{n}$ denote the signs of the curvature of external
and internal space (1, 0 or $-1$), respectively. The momenta conjugate to the scale
factors $a$ and $b$ are given by

$$\pi_{a} = \frac{\partial L}{\partial \dot{a}} = -\frac{\sqrt{g_{i}g_{e}}}{16\pi G}a^{m}b^{n}$$

$$\left[ 2m(m-1)\frac{\dot{a}}{a^2} + 2mn \frac{\dot{b}}{ab} \right],$$

(2.4)

$$\pi_{b} = \frac{\partial L}{\partial \dot{b}} = -\frac{\sqrt{g_{i}g_{e}}}{16\pi G}a^{m}b^{n}$$

$$\left[ 2mn \frac{\dot{a}}{ab} + 2n(n-1)\frac{\dot{b}}{b^2} \right].$$

(2.5)
The Hamiltonian $H$ is described as
\[
H = \pi_a \dot{a} + \pi_b \dot{b} - \mathcal{L} \tag{2.6}
\]
\[
= \frac{\sqrt{g_\text{e} g_\text{c}}}{16\pi G} a^m b^n \left[ m(m - 1) \left( \frac{\dot{a}}{a} \right)^2 + 2mn \frac{\dot{a} \dot{b}}{ab} + n(n - 1) \left( \frac{\dot{b}}{b} \right)^2 + V \right]. \tag{2.7}
\]

The potential energy $V$ is defined by
\[
V = m(m - 1) \frac{k_m}{a^2} + n(n - 1) \frac{k_n}{b^2} - 2\Lambda. \tag{2.8}
\]

The Einstein equations are given by
\[
m \ddot{a} a + n \ddot{b} b - \alpha = 0, \tag{2.9}
\]
\[
\frac{\dot{a}}{a} + (m - 1) \left( \frac{\dot{a}}{a} \right)^2 + n \frac{\dot{b}}{ab} + (m - 1) \frac{k_m}{a^2} - \alpha = 0, \tag{2.10}
\]
\[
\frac{\dot{b}}{b} + (n - 1) \left( \frac{\dot{b}}{b} \right)^2 + m \frac{\dot{a}}{ab} + (n - 1) \frac{k_n}{b^2} - \alpha = 0, \tag{2.11}
\]
where
\[
\alpha = \frac{2\Lambda}{m + n - 1}. \tag{2.12}
\]

These equations imply the Hamiltonian constraint
\[
m(m - 1) \left( \frac{\dot{a}}{a} \right)^2 + 2mn \frac{\dot{a} \dot{b}}{ab} + n(n - 1) \left( \frac{\dot{b}}{b} \right)^2 + V = 0. \tag{2.13}
\]

§3. Euclidean solutions

We consider the quantum creation of the universe in the framework of the Hartle-Hawking no-boundary proposal. In the WKB approximation, the Hartle-Hawking wave function can be expressed in the form
\[
\Psi[h_{ij}, \Phi] \sim \Sigma_k A_k \exp(-B_k), \tag{3.14}
\]
where $B_k$ is the Euclidean action for the solutions of the Euclidean field equations, which are compact and have the given 3-metric $h_{ij}$ and matter field $\Phi$ on the boundary. The prefactor $A_k$ represents the fluctuations around these solutions. Since the factor $\exp(-B_k)$ dominates wave function behavior, we analyse the classical solutions of the Euclidean Einstein equations.

The Euclidean Einstein equations are given by
\[
m \dddot{a} a + n \dddot{b} b + \alpha = 0, \tag{3.15}
\]
\[
\frac{a''}{a} + (m - 1) \left( \frac{a'}{a} \right)^2 + n \frac{a'b'}{ab} - (m - 1) \frac{k_m}{a^2} + \alpha = 0, \tag{3.16}
\]
\[
\frac{b''}{b} + (n - 1) \left( \frac{b'}{b} \right)^2 + m \frac{a'b'}{ab} - (n - 1) \frac{k_n}{b^2} + \alpha = 0. \tag{3.17}
\]
where primes denote derivatives with respect to the imaginary time \( \tau \). The Hamiltonian constraint in the Euclidean version is described as

\[
m(m-1)\left(\frac{a'}{a}\right)^2 + 2mn\frac{a'b'}{ab} + n(n-1)\left(\frac{b'}{b}\right)^2 - V = 0. \tag{3.18}
\]

We assume that \( a = 0 \) and/or \( b = 0 \) at \( \tau = 0 \). The Hartle-Hawking no-boundary proposal gives the boundary condition for the Euclidean Einstein equations. According to this proposal, the Euclidean solutions must be analytic. At \( \tau = 0 \), therefore, \( a \) and \( b \) can be expanded in Taylor series. This determines the boundary conditions of the Euclidean solutions. In addition, the external space of the present universe is macroscopic and the scale factor of the internal space is related to the gauge coupling constant. Therefore we do not consider trivial solutions, \( a(\tau) = 0 \) and/or \( b(\tau) = 0 \) for any \( \tau \). These constraints lead to the following boundary conditions:

(a) Boundary Condition 1 (BC1)

\[
k_m = 1, \tag{3.19}
\]

\[
a(0) = 0, a'(0) = 1, \tag{3.20}
\]

\[
b(0) = b_0, b'(0) = 0. \tag{3.21}
\]

(b) Boundary Condition 2 (BC2)

\[
k_n = 1, \tag{3.22}
\]

\[
a(0) = a_0, a'(0) = 0, \tag{3.23}
\]

\[
b(0) = 0, b'(0) = 1. \tag{3.24}
\]

The boundary condition BC2 is the symmetric version of the boundary condition BC1. These boundary conditions imply that either internal space or external space has positive curvature \( (k_m = 1 \text{ or } k_n = 1) \). We can solve the Euclidean Einstein equations numerically under these boundary conditions. Below, the families of Euclidean solutions parameterized by \( b_0 \) and \( a_0 \) are calculated.

We assume that the Euclidean solutions are connected to classical Lorentzian solutions at a finite \( \tau \), where \( a' = b' = 0 \). Due to the Hamiltonian constraint \( (2.13) \), the value of the potential at the connection surface must be

\[
V = \frac{m(m-1)k_m}{a^2} + \frac{n(n-1)k_n}{b^2} - 2\Lambda = 0. \tag{3.25}
\]

Now we discuss the constraints on the signature of the spatial curvatures and the cosmological constant. First, it is easily shown that either internal space or external space must have positive curvature \( (k_m = 1 \text{ or } k_n = 1) \) from the boundary conditions BC1 and BC2. Second, the condition that the connection surface \( V = 0 \) exists gives constraints on the signatures of \( k_m, k_n \) and \( \Lambda \). For example, when each signature of \( k_m \) and \( k_n \) is the same, and \( \Lambda = 0 \), the potential \( V(a, b) \) cannot be zero anywhere. The combinations of signatures of \( k_m, k_n \) and \( \Lambda \) for which the surface \( V = 0 \) exists, are shown in Table I.

Even if these conditions are satisfied, however, the Euclidean solutions cannot be necessarily connected to Lorentzian solutions, since in some classes of solutions, \( a \)
Table I. The signatures of \(k_m, k_n\) and \(\Lambda\) under the condition that the surface \(V = 0\) exists.

| \((m-1)k_m\) | + | + | + | 0 | 0 | - | - | - |
| \((n-1)k_n\) | + | 0 | - | + | 0 | - | + | 0 |

| \(\Lambda = 0\) | no | no | yes | no | yes | no | yes | no |
| \(\Lambda > 0\) | yes | yes | yes | yes | no | yes | no | yes |
| \(\Lambda < 0\) | no | no | yes | no | yes | yes | yes | yes |

\(a\) and \(b\) increase monotonically and have no velocity-zero points, i.e., points for which \(\dot{a} = \dot{b} = 0\). We calculated Euclidean solutions numerically, changing the initial values, and investigated whether these Euclidean solutions can be connected to Lorentzian solutions or not. Except for the case in which \((m-1)k_m > 0\), \((n-1)k_n > 0\) and \(\Lambda > 0\), the scale factors \(a\) and \(b\) vary monotonically, and the condition \(a' = b' = 0\) cannot be satisfied at any \(\tau\). Then, we found that Euclidean solutions can be connected to the Lorentzian region only for the case in which \((m-1)k_m > 0\), \((n-1)k_n > 0\) and \(\Lambda > 0\).

After all, only the case in which \(k_m = k_n = 1\) and \(\Lambda > 0\) is possible in our model. This implies that the internal space cannot be 1-dimensional, because the curvature term of the Einstein equations is zero for 1-dimensional space. Therefore the dimensions of space \(m\) and \(n\) must be greater than 2.

We now discuss the Euclidean solutions in the case that \(k_m = k_n = 1\) and \(\Lambda > 0\) in detail. In this case, there are following three exact solutions:

(1) \[a = \sqrt{\frac{m(m+n-1)}{2\Lambda}} \sin\left(\sqrt{\frac{2\Lambda}{m(m+n-1)}} \tau\right), \quad (3.26)\]
\[b = \left[\frac{(n-1)(n+m-1)}{2\Lambda}\right]^{1/2} \equiv b_A. \quad (3.27)\]

(2) \[a = \sqrt{\frac{(m+n-1)(m-1)}{2\Lambda}} \sin\left(\sqrt{\frac{2\Lambda}{(m+n)(m+n-1)}} \tau\right), \quad (3.28)\]
\[b = \sqrt{\frac{(m+n-1)}{2\Lambda}} \sin\left(\sqrt{\frac{2\Lambda}{(m+n)(m+n-1)}} \tau\right). \quad (3.29)\]

(3) \[a = \left[\frac{(m-1)(n+m-1)}{2\Lambda}\right]^{1/2} \equiv a_A, \quad (3.30)\]
\[b = \sqrt{\frac{n(m+n-1)}{2\Lambda}} \sin\left(\sqrt{\frac{2\Lambda}{n(m+n-1)}} \tau\right). \quad (3.31)\]

The instanton solutions (1) and (3) were discussed by Hu and Wu.\(^{14}\) In addition we have found the instanton solution (2).

In Figs. 1∼3, the paths in the \(a - b\) plane are shown. Here we set \(\Lambda = 1\) since we can always set \(\Lambda = 1\) by rescaling the cosmic time. Figs. 1 and 2 are for the case of the
Fig. 1. Euclidean solutions in the case that \( m = 3, n = 2, k_m = k_n = 1, \Lambda = 1 \) and with the boundary condition BC1 (Eqs. (3.3) and (3.4)). The critical value \( b_{cr} \) is 3.16.

Fig. 2. Euclidean solutions in the case that \( m = 3, n = 7, k_m = k_n = 1, \Lambda = 1 \) and with the boundary condition BC1 (Eqs. (3.3) and (3.4)). The critical value \( b_{cr} \) is 6.71.

boundary condition BC1 (Eqs. (3.3) and (3.4)). Fig. 3 is for the case of the boundary condition BC2 (Eqs. (3.6) and (3.7)). The essential property of the dynamical system is independent of the dimension of the \( g_{ij} \) or \( g_{IJ} \) spaces, as shown in Figs. 1 and 2.
Fig. 3. Euclidean solutions in the case that \( m = 3, n = 7, k_m = k_n = 1, A = 1 \) and with the boundary condition BC2 (Eqs.(3.6) and (3.7)). The critical value \( a_{cr} \) is 6.71.

Note that this dynamical system has a characteristic point \( A \):

\[
(a_A, b_A) = \left( \left( \frac{(m-1)(n+m-1)}{2A} \right)^{1/2}, \left( \frac{(n-1)(n+m-1)}{2A} \right)^{1/2} \right). \tag{3.32}
\]

The trajectories of the Euclidean solutions on the scale factor plane \( a-b \) can be classified into following three classes. First, if \( b_0 \) (or \( a_0 \)) is greater than a critical value \( b_{cr} \) (or \( a_{cr} \)),

\[
b_{cr}, a_{cr} = \sqrt{\frac{(D - 1)(D - 2)}{2A}}, \tag{3.33}
\]

the trajectories flow to \( (a, b) \rightarrow (\infty, 0) \) (or \( (0, \infty) \)) directly. Second, if \( b_0 \) (or \( a_0 \)) is less than a critical value \( b_{cr} \) (or \( a_{cr} \)) but greater than \( b_A \) (or \( a_A \)), the trajectories go around the point \( (a_A, b_A) \) and flow to \( (a, b) \rightarrow (0, \infty) \) (or \( (\infty, 0) \)). When \( b_0 \) (or \( a_0 \)) is just equal to \( b_A \) (or \( a_A \)), \( b \) (or \( a \)) is constant (the solution (1) (or (3))) and the trajectory is a straight line. Third, if \( b_0 \) (or \( a_0 \)) is lower than \( b_A \) (or \( a_A \)), the trajectories go around the point \( (a_A, b_A) \) and flow to \( (a, b) \rightarrow (\infty, 0) \) (or \( (0, \infty) \)).

Note that the critical values \( b_{cr} \) and \( a_{cr} \) depend only on the dimension of the spacetime \( D \) as described by (3.20). We call the point \( (a_A, b_A) \), around which the trajectories circulate, the “quasi-attractor.” The “quasi-attractor” plays a role in gathering the trajectories of the classical Euclidean solutions to the instanton solutions (Eqs. (3.9)-(3.14)), and approximately connecting to the Lorentzian solutions. In the next section we discuss the properties of the Lorentzian solutions.

§4. Lorentzian solutions
In this section we investigate the Lorentzian solutions that are continued from the Euclidean solutions we have presented in the preceding section in the case that the curvatures of both external and internal spaces are positive and $\Lambda > 0$. The Euclidean manifold is connected to the Lorentzian manifold, where the velocities of the scale factors are vanishing, and it can be shown that at the connection surface, the value of the potential $V$ must be vanishing.

There are three following exact solutions which are analytically continued from instantons:

(1)

$$a = \sqrt{\frac{m(m+n-1)}{2\Lambda}} \cosh\left(\sqrt{\frac{2\Lambda}{m(m+n-1)}} t\right),$$

$$b = \left[\frac{(n-1)(n+m-1)}{2\Lambda}\right]^{1/2}. \quad (4.34)$$

(2)

$$a = \sqrt{\frac{(m+n)(m-1)}{2\Lambda}} \cosh\left(\sqrt{\frac{2\Lambda}{(m+n)(m+n-1)}} t\right),$$

$$b = \sqrt{\frac{(m+n)(n-1)}{2\Lambda}} \cosh\left(\sqrt{\frac{2\Lambda}{(m+n)(m+n-1)}} t\right). \quad (4.35)$$

(3)

$$a = \left[\frac{(m-1)(n+m-1)}{2\Lambda}\right]^{1/2}, \quad (4.36)$$

$$b = \sqrt{\frac{n(m+n-1)}{2\Lambda}} \cosh\left(\sqrt{\frac{2\Lambda}{n(m+n-1)}} t\right). \quad (4.37)$$

In the preceding section we found that the “quasi-attractor” plays a role in gathering the paths of the classical Euclidean solutions to the instanton solutions (Eqs. (3.9) ~ (3.14)), and approximately connecting to Lorentzian solutions. Strictly speaking, the Euclidean solutions near these instanton solutions cannot analytically continue to Lorentzian solutions. Since these nearby solutions exists very densely, their effects are necessary for estimating the wave function of the universe. Therefore we make detailed analysis of these solutions. The Lorentzian solutions under the initial conditions that the velocities of the scale factors on the connection surface $V(a, b) = 0$ are zero are approximately continued from these solutions. Thus we calculate the Lorentzian solutions continued approximately from these solutions. In Fig. 4 the Lorentzian solutions in the $a$-$b$ plane are shown. The solutions between the solutions (1) and (3) show that both external and internal spaces expand monotonically. On the other hand, the outer solutions of the exact solution (3) show that internal space expands and external space contracts. The outer solutions of the exact solution (1), which are nothing but the symmetric version of the former solutions, show that external space expands and the internal space contracts (Fig. 5). The last solutions imply that external space
of the observed universe is macroscopic and internal space of the observed universe is microscopic.

Fig. 4. Lorentzian solutions in the case $m = 3$, $n = 7$, $k_m = k_n = 1$ and $\Lambda = 1$. The number in this figure represents that of the Lorentzian analytic solutions (Eqs. (4.2)∼(4.7)). The dotted lines represent the instantons (Eqs. (3.9)∼(3.14))

Fig. 5. Lorentzian solution in the case $m = 3$, $n = 7$, $k_m = k_n = 1$, $\Lambda = 1$ and the initial condition are $a = 4$, $\dot{a} = 0$, $b = [21/(1 - 3/4^2)]^{1/2}$, $\dot{b} = 0$

§5. Summary and discussion

In this paper we have investigated the quantum creation of a multidimensional universe with a cosmological constant but without matter fields in the framework of the Hartle-Hawking no-boundary proposal. The most interesting result is that there are "quasi-attractor," i.e., most trajectories of the classical solutions of the Euclidean Einstein equation go around on the $a$-$b$ plane, independently of the initial values, provided
that the initial values of the scale factors are smaller than a critical value in the case
that the curvatures of both the external and internal spaces are positive and \( \Lambda > 0 \).
This characteristic behavior of the evolution of the scale factors in Euclidean time
is essentially independent of the numbers of external and internal spaces dimensions,
provided that \( m, n \geq 2 \).

Since the trajectories approach the quasi-attractor at the first step, it plays a
role in gathering the paths of the classical Euclidean solutions to the instantons ((3.9)
\sim (3.14)) independently of the initial values. Then, it is very natural to assume that the
wave function has a hump near this quasi-attractor point. There Lorentzian solutions
which start from the points on the continuation surface near the quasi-attractor give a
dominant contribution to the wave function in the Lorentzian region. As the example
is in Fig. 5 shown, there are solutions for which we can interpret that dynamical
compactification takes place; i.e. the external space evolves to the macroscopic realm
and the internal space remains microscopic.

In order to understand the role of this quasi-attractor in the wave function and the
creation of the universe more deeply, and to confirm the above conjecture, it is necessary
to analyze the Wheeler-De Witt equations and to calculate the wave function of the
universe. The Wheeler-De Witt equations of the multi-dimensional vacuum universe
have been analysed by Chmielowski,\(^{23}\) but only for models with zero cosmological
constant. According to his result, if neither of the subspaces has negative curvature,
then there exists no Lorentzian solution. Obviously, interesting cases are those of the
positive cosmological constant, as we have discussed in the present paper. Analysis of
the Wheeler-De Witt equations in this model is under progress.

In order to make more realistic models, we must consider more complicated models
on the basis of fundamental theories. As shown in Fig. 5, the solutions which show that
the external space expands and the internal spaces collapses in finite time are adequate
models for our universe, provided that the collapse of inner space is halted. Furthermore,
because the volume of the internal space is related to the constants of nature, the
present internal space must be stable at the order of the Planck scale. Fortunately, it is
considered that the collapse of the internal space is halted by the pressure of the gauge
fields or by a quantum effects, and the internal space is stabilized. It was shown that
in \( D = 6, N = 2 \) supergravity theories, the solution of (the 4-dimensional Friedmann
universe) \times (a constant \( S^2 \)) is the attractor; i.e., all cosmological solutions starting from
arbitrary initial conditions approach to the above spacetime asymptotically.\(^{24}\) Moreover,
stable compactification of the extra dimensions by quantum effects was studied.
The quantum corrections to the effective potential are attributed to the Casimir effects
in many works (e.g. Refs.\(^{26, 25}\) and references therein). In Ezawa et al,\(^{27}\) the quantum
effects of higher curvature gravity theories are investigated. From these investigations,
if is known that an effective potential of the model obtained under dimensional reduc-
tion to a 4-dimensional effective theory has minima at the Planck scale for the scale
factors of internal spaces. Therefore the internal spaces can be stable.

Our work is simply based on canonical quantum gravity in general relativity. It is,
however, thought that general relativity is a low-energy limit of the ultimate unified
theory. Although the Einstein equation would be necessarily modified at high energy, it
must include essential points of the ultimate theory. It is natural to conjecture that the
ultimate theory may include the present simple model as an approximate description.
It seems that analysis of the simple model given in the present paper may be useful for further investigations of the basis of fundamental theories such as superstring theories and supergravity theories.

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