BPS Domain Walls in Models with Flat Directions

Masashi Naganuma* and Muneto Nitta†

Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro, Tokyo 152-8551, Japan

Abstract

We consider BPS domain walls in the four dimensional $\mathcal{N} = 1$ supersymmetric models with continuous global symmetry. Since the BPS equation is covariant under the global transformation, the solutions of the BPS walls also have the global symmetry. The moduli space of the supersymmetric vacua in such models have non-compact flat directions, and the complex BPS walls interpolating between two disjoint flat directions can exist. We examine this possibility in two models with global $O(2)$ symmetry, and construct the solution of such BPS walls.

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*E-mail: naganuma@th.phys.titech.ac.jp.
†E-mail: nitta@th.phys.titech.ac.jp.
1 Introduction

In recent years, there has been much investigation on domain walls which appear in many areas of physics. These domain walls interpolate between degenerate discrete minima of a scalar potential, with dependence of one spatial coordinate. They can occur naturally when a discrete symmetry is spontaneously broken.

Domain walls can also appear in the supersymmetric field theories when the superpotential has more than two critical points corresponding to degenerate minima of the scalar potential. Especially it has been found that domain walls in supersymmetric theories can saturate the Bogomol’nyi bound [1]. Such domain walls are called BPS domain walls and preserve half of the original supersymmetry [2]. The existence of the BPS domain walls correspond to the central extension of $\mathcal{N} = 1$ superalgebra, and the topological charge of the walls becomes a central charge $Z$ of the superalgebra [3][4]. The BPS bound of the wall and supercharges conserved by the BPS walls are determined by this central charge $Z$.

Domain walls in supersymmetric theories have been well studied in the models with degenerate isolated vacua before [5]-[8]. It was essential there that isolated vacua take the different values of the superpotential, since their differences are related to the energy densities saturating the BPS bound. In many supersymmetric theories, however, the vacuum manifold consists of the continuously degenerated moduli space. Since the supersymmetric vacua are the extrema of the superpotential, $W' = 0$, the connected parts of the moduli space take the same values of the superpotential, and then each connected part is mapped to a single point in the superpotential space. Hence we can expect the existence of domain walls in the models with the moduli space composed of the several disjoint parts, because these disjoint vacua (in the field space) are, in general, mapped to the different points in the superpotential space. In short, the moduli spaces of disjoint supersymmetric vacua appear the same as the isolated vacua in the superpotential space.

In this work, we investigate the BPS domain walls in the $\mathcal{N} = 1$ four dimensional supersymmetric field theories with continuous global symmetry. Domain walls in such theories can be expected to connect the pairs of the vacua in the disjoint
moduli spaces. If the models have the vacua with spontaneous breaking of the global symmetry, the domain walls, connecting these vacua, also break the symmetry and there can be the family of the walls, according to the moduli spaces of these vacua.

There is another reason why we study the BPS walls in the models with continuous global symmetry. It has been known that, when a supersymmetric model has global symmetry, the superpotential has larger symmetry, or the complexification of the original global symmetry, because of the holomorphy of the superpotential [10]. The vacuum manifold has non-compact flat directions spreading over corresponding to the imaginary parts of the vacuum expectation values of the fields. Since the complexified group is not the symmetry of the whole model but that of the superpotential, it is a nontrivial problem whether there exist BPS walls interpolating between two vacua in the disjoint flat directions. Hence we examine this problem by using two supersymmetric models with global $O(2)$ symmetry, including two chiral superfields.

In sect. 2, we discuss the general properties of BPS domain walls in the model with continuous global symmetry. In sect. 3, we introduce our two model with $O(2)$ symmetry. We examine the existence of the complex BPS walls in both the models. In sect. 4, we get the conclusions together and discuss the general features of the complex BPS walls interpolating between flat directions.

2 BPS walls and continuous symmetry

We consider the supersymmetric field theories with only chiral superfields, and the Kähler potential is assumed to be linear: $K = \phi^i \phi$. The supersymmetric vacua are given as the extrema of the superpotential $W(\phi_k)$,

$$\frac{\partial W}{\partial \phi_k} = 0, \quad k = 1, \cdots K,$$

where $\phi_k$ are the scalar components of the chiral superfields. It has been known that, if we denote the two solutions of the equation (2.1) as $\{\phi_k\}_I$ and $\{\phi_k\}_J$, and the corresponding values of the superpotential as $W_I$ and $W_J$, there exists the lower bound of the surface energy density, or tension, for the wall connecting these two
The BPS wall saturating this lower bound satisfy the equation [7],

$$\partial_z \phi_k = e^{i\alpha} \frac{\partial W^*}{\partial \phi_k^*},$$

(2.3)

where $\alpha = \arg(W_J - W_I)$. Hence we have considered the BPS wall depending on the coordinate $z$. This equation (2.3) is called the BPS equation.

If the superpotential $W$ is invariant under global symmetry $G$,

$$W(\phi) \rightarrow W(g\phi) = W(\phi), \quad \phi \rightarrow g\phi, \quad g \in G,$$

(2.4)

where $\phi$ belongs to an unitary representation of $G$, the equation (2.1) is also invariant under $G$:

$$\frac{\partial W(\phi)}{\partial \phi_i} \rightarrow g^{-1}g_{ij} \frac{\partial W(\phi)}{\partial \phi_j}.$$  

(2.5)

Since the superpotential includes only chiral superfields, the invariant group $G$ is enlarged to its complexification, $G^C$. Therefore, when the model has the global symmetry $G$, its superpotential has the complexified $G^C$ symmetry, and in the moduli space of its supersymmetric vacua, there exists a non-compact flat direction along the direction of the imaginary part of the scalar fields.

We can see that the BPS equation (2.3) is covariant under the transformation of the global symmetry $G$, but not covariant under the transformation of $G^C$, since the BPS equation includes both the holomorphic and anti-holomorphic fields. Then, if we can find a solution of eq. (2.3), the configuration transformed from the solution by elements of $G$ are also the solutions of the BPS equation; on the other hand the configuration transformed by elements of $G^C$ do not become the solutions of the BPS equation. Therefore, if the model has more than two disjoint flat directions, it is a nontrivial problem whether there exist the BPS walls interpolating between them. We will examine this problem in the two supersymmetric models.
3 BPS walls in the model with flat directions

3.1 Moduli spaces of our models with flat directions

In this paper, we consider the following two supersymmetric models with flat directions.\footnote{Two models we consider in this paper are not renormalizable. Therefore these models must be interpreted as effective theories.}

At first we consider the model with one flat direction. Its superpotential is

\[ W(\tilde{\phi}) = \frac{1}{4}(\tilde{\phi}^2 - a^2)^2, \quad \tilde{\phi} = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}, \]

where \( \phi^1 \) and \( \phi^2 \) are chiral superfields composing the doublet of \( O(2) \), \( \tilde{\phi} \), and \( a \) is a constant parameter. By a field redefinition, we can take this parameter \( a \) to be real positive without loss of generality. This model has two disjoint vacua:

\[ \text{Vac. I} \quad \tilde{\phi} = 0, \quad W = \frac{a^4}{4}, \]
\[ \text{Vac. II} \quad \tilde{\phi}^2 = a^2, \quad W = 0. \]  

(3.2)

Let us note that \( \phi^i \) are the scalar components of chiral superfields here: we denote the chiral superfields and their scalar components by the same letter. Vac. I is \( O(2) \) symmetric but Vac. II spontaneously breaks \( O(2) \) symmetry. Expectation value for Vac. II can be labeled as

\[ \phi^1 = a \cos \theta, \quad \phi^2 = a \sin \theta. \]

(3.3)

Now fields \( \phi^1 \) and \( \phi^2 \) can take complex value, and we can regard \( \theta \) as a complex parameter. Therefore vacuum manifold of this model is enlarged to the \( O(2)^\mathbb{C} \)-orbit: if we set \( \tilde{\phi} = \tilde{x} + i\tilde{y} \), two disjoint vacua in eq.(3.2) become

\[ \text{Vac. I} \quad \tilde{x} = \tilde{y} = 0, \]
\[ \text{Vac. II} \quad \tilde{x}^2 - \tilde{y}^2 = a^2, \quad \tilde{x} \cdot \tilde{y} = 0. \]

(3.4)

Hence Vac. II can be rewritten as two dimensional surface in the three dimensional linear space \( (x^1, x^2, y^1) \) (see figure 1). Vac. II breaks this \( O(2)^\mathbb{C} \) symmetry.
Figure 1: (a) The moduli space of the first model is composed of two disjoint parts: the origin and one hyperboloid. The hyperboloid has one compact direction represented by the broken circles, and one non-compact direction represented by the hyperbola. (b) The moduli space of the second model is composed of three disjoint parts: the origin and two hyperboloids with different sizes. In both figures (a) and (b), the horizontal axis is \( y_1 \), the vertical axis is \( x_1 \), and the axis orthogonal to them is \( x_2 \). The smallest circle corresponds to the real moduli space of vacua, \( \vec{y} = 0 \), in both figures.

We consider the BPS wall connecting \( O(2)^C \) symmetric and \( O(2)^C \) broken vacua, and show that no BPS wall can connect the complex vacuum - the vacuum with a complex value of fields shifting along the flat direction in this model (see figure 1).

Next we consider the model with two flat directions. Its superpotential is

\[
W(\phi) = \frac{1}{3} \vec{\phi}^2 (\vec{\phi}^2 - a^2)^2, \tag{3.5}
\]

where \( \vec{\phi} \) is an \( O(2) \) doublet composed of chiral superfields \( \phi^1 \) and \( \phi^2 \), and parameter

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\(^2\)It is known that, besides the ordinary Nambu-Goldstone bosons corresponding to broken \( O(2) \) symmetry, there appear the so-called quasi-Nambu-Goldstone bosons corresponding to broken \( O(2)^C \) symmetry [10]. With the fermions of their superpartner, they constitute massless chiral superfields. The vacuum manifold as the \( O(2)^C \)-orbit are parameterized by these massless bosons, and the quasi-Nambu-Goldstone bosons just parameterize the non-compact flat directions. It is known that, in the case of the F-term breaking, there must exist at least one quasi-Nambu-Goldstone boson, and then the vacuum manifold inevitably becomes non-compact [11].
$a$ is assumed to be a real positive constant for simplicity. This model has three disjoint vacua:

\begin{align*}
\text{Vac. I} & \quad \vec{\phi} = 0, \quad W = 0, \\
\text{Vac. II} & \quad \vec{\phi}^2 = \frac{a^2}{3}, \quad W = \frac{4}{81}a^6, \\
\text{Vac. III} & \quad \vec{\phi}^2 = a^2, \quad W = 0. 
\end{align*} \tag{3.6}

Setting $\vec{\phi} = \vec{x} + i\vec{y}$ as the previous model, Vac. II and Vac. III can be rewritten as two hyperboloid with different size and Vac. I as the origin in the space $(x^1, x^2, y^1)$ (see figure 1). We see that Vac. I is $O(2)^C$ symmetric but Vac. II and Vac. III break $O(2)^C$ symmetry spontaneously. We consider the two kinds of BPS walls, connecting Vac. I and Vac. II, and connecting Vac. II and Vac. III. Then we show that the BPS walls can connect the complex vacua of Vac. II and Vac. III, but cannot connect Vac. I and complex vacua of Vac. II.

### 3.2 BPS wall in the model I

We construct the BPS saturated wall in the model with one flat direction (Model I). The BPS equation (2.3) for this wall is

\[ \frac{\partial \phi^i}{\partial z} = \phi^*(\vec{\phi}^2 - a^2). \] \tag{3.7}

At first we show that there is no complex solution of this BPS equation. When we map the field space to the superpotential space, two disjoint vacua are mapped to two points. It is known that the configuration of the BPS wall can be mapped to a straight line segment connecting these two points in the superpotential space [9]. Now difference of the value of the superpotential for two vacua, $\Delta W = a^4/4$, is real. This means that the configuration of the BPS wall in the superpotential space is also real. If we set $\vec{\phi} = \vec{x} + i\vec{y}$, the imaginary part of the superpotential is $3W = 4(\vec{x} \cdot \vec{y})(\vec{x}^2 - \vec{y}^2 - a^2)$, so we can find that BPS solution must satisfy the constraint, $\vec{x} \cdot \vec{y} = 0$. Since the BPS equation of eq. (3.7) is covariant under the $O(2)$ transformation, we can set, by using this transformation,

\[ \phi^1 = v, \quad \phi^2 = iu, \] \tag{3.8}
where $u$ and $v$ are real scalar fields. In figure 2 (a), we show the moduli space of this model in the $(u, v)$-plane. Then the BPS equation (3.7) becomes

$$
\frac{dv}{dz} = v(v^2 - u^2 - a^2), \quad \frac{du}{dz} = - u(v^2 - u^2 - a^2).
$$

(3.9)

From these equations, we can immediately find

$$
\frac{d(uv)}{dz} = 0, \quad uv = \text{const.} \equiv \sqrt{c},
$$

(3.10)

where $c$ is a real integral constant. From figure 2 (a) we can find that there is no complex BPS solution connecting Vac. I and vacua along the flat direction of Vac. II: In order for the BPS wall to reach Vac. I, we need to set $uv = \sqrt{c} = 0$, and this is reduced to the real solution $(u(z) = 0)$ for the boundary condition of Vac. II on the other side.

Hence we consider the real solution of eq. (3.9). The real solution can be found as

$$
v = \phi_W \equiv a \sqrt{\frac{1}{1 + \exp[2a^2(z - z_0)]}}, \quad u = 0,
$$

(3.11)
where \( z_0 \) is an integral constant, representing the position of the center of the domain wall. We plot this real solution in figure 2 (b). Using the \( O(2) \) transformation, the general real solutions can be written as

\[
\phi^1 = \phi_W \cos \theta, \quad \phi^2 = \phi_W \sin \theta, \quad (3.12)
\]

where \( \theta \) is a real parameter. The wall separates the two vacua in the broken phase and the unbroken phase.

### 3.3 BPS wall in the model II

In this section, we construct the BPS wall in the model with two flat directions (Model II). This model has three disjoint vacua as Eq. (3.6). Difference of the values of the superpotential of each pairs of vacua out of three vacua is real as the previous model. There exists no BPS wall connecting Vac. I and Vac. III, because two superpotentials corresponding to these two vacua are the same and the BPS bound (2.2) becomes zero. So we consider two kinds of walls; the walls interpolating Vac. II and Vac. III (we call them “outer walls”), and the walls interpolating the Vac. I and Vac. II (we call them “inner walls”). The BPS equations (2.3) for these walls are

\[
\frac{\partial \tilde{\phi}^i}{\partial z} = \phi^* (\tilde{\phi}^2 - a^2)(\tilde{\phi}^2 - a^2), \quad (3.13)
\]

where boundary conditions are \( \tilde{\phi}(-\infty) = a^2 \) (\( \tilde{\phi}(-\infty) = 0 \)) and \( \tilde{\phi}(\infty) = a^2/3 \) for outer (inner) walls.

The map of the BPS wall into the superpotential space must be real as the previous model. If we put \( \tilde{\phi} = \vec{x} + i\vec{y} \), the imaginary part of the superpotential in this model becomes

\[
3W = \frac{1}{3}(\vec{x} \cdot \vec{y})[3(\vec{x}^2 - \vec{y}^2 - a^2)(\vec{x}^2 - \vec{y}^2 - a^2/3) - 4(\vec{x} \cdot \vec{y})^2]. \quad (3.14)
\]

So \( \vec{x} \cdot \vec{y} = 0 \) is a sufficient condition. As in the model I, we can put \( \tilde{\phi} = \begin{pmatrix} v \\ iu \end{pmatrix} \),

where \( v \) and \( u \) are real scalar fields, by using the \( O(2) \) transformation. In figure 3,

\[\text{We can show that this is also the necessary condition by the argument of the continuity of the solution.}\]
we illustrate the moduli space of this model in the \((u, v)\)-plane. Eq. (3.13) becomes
\[
\frac{dv}{dz} = -v(v^2 - u^2 - \frac{a^2}{3})(v^2 - u^2 - a^2),
\]
\[
\frac{du}{dz} = +u(v^2 - u^2 - \frac{a^2}{3})(v^2 - u^2 - a^2).
\] (3.15)

One can find
\[
\frac{d(uv)}{dz} = 0, \quad uv = \text{const.} \overset{\text{def}}{=} \sqrt{c}.
\] (3.16)

Hence, we can put \(u^2 = c/v^2\). We find, from the figure 3 (a), that there can exist complex BPS wall solution connecting Vac. II and Vac. III, but no complex BPS wall can connect Vac. I and Vac. II for the same reason as the model I. The first equation of Eq. (3.15) becomes
\[
\frac{dv^2}{dz} = -2\frac{1}{v^2}((v^2)^2 - \frac{a^2}{3}v^2 - c)((v^2)^2 - a^2v^2 - c).
\] (3.17)

This can be integrated to give
\[
e^{\frac{4\pi^2}{3}(z-z_0)} = \left| \frac{v^2 - \frac{1}{2}(\frac{a^2}{3} + \sqrt{\frac{a^2}{9} + 4c})}{v^2 - \frac{1}{2}(\frac{a^2}{3} - \sqrt{\frac{a^2}{9} + 4c})} \right| \left| \frac{1}{\sqrt{\frac{a^2}{9} + 4c}} \right| \left| \frac{v^2 - \frac{1}{2}(a^2 - \sqrt{a^4 + 4c})}{v^2 - \frac{1}{2}(a^2 + \sqrt{a^4 + 4c})} \right| \left| \frac{1}{\sqrt{a^4 + 4c}} \right|,
\] (3.18)
where $z_0$ is the center of the wall. For the complex solution interpolating Vac. II and Vac. III, this equation (3.18) can be rewritten as

$$e^{\frac{4a^2}{3}(z-z_0)} = \left[\frac{v^2 - \frac{1}{2}(a^2 + \sqrt{a^4 + 4c})}{v^2 - \frac{1}{2}(a^2 - \sqrt{a^4 + 4c})}\right]^{\frac{1}{\sqrt{a^4 + 4c}}} \left[\frac{v^2 - \frac{1}{2}(a^2 - \sqrt{a^4 + 4c})}{\frac{1}{2}(a^2 + \sqrt{a^4 + 4c}) - v^2}\right]^{\frac{1}{\sqrt{a^4 + 4c}}}.$$ (3.19)

Since we cannot obtain the explicit solution $v(z)$ of this equation, we plot $v(z)$ in the figure 3 (b) as the implicit solution of the complex BPS wall.

We can find the explicit solution for the real BPS walls. For the real solution, the integrated BPS equation can be obtained by putting $c = 0$ in eq. (3.18):

$$X \overset{\text{def}}{=} \exp \left[\frac{4a^4}{3} (z - z_0)\right] = \frac{|v^2 - a^2| v^4}{|v^2 - \frac{a^2}{3}|^3} = \frac{|\Phi - a^2| \Phi^2}{|\Phi - \frac{a^2}{3}|^3},$$ (3.20)

where we have defined $\Phi \overset{\text{def}}{=} v^2 = (\phi^1)^2$.

We solve this equation in the outer region, $\frac{a^2}{3} \leq (\phi^1)^2 \leq a^2$, and the inner region, $0 \leq (\phi^1)^2 \leq \frac{a^2}{3}$, respectively. In the case of the outer solutions, $\frac{a^2}{3} \leq (\phi^1)^2 \leq a^2$, eq. (3.20) can be rewritten as the third order equation:

$$(X + 1)\Phi^3 - a^2(X + 1)\Phi^2 + \frac{a^4}{3} X \Phi - \frac{a^6}{27} X = 0.$$ (3.21)

Thus third order equation can be solved to yield

$$(\phi^1)^2 = \frac{a^2}{3} \left[1 + \left(\frac{1}{1 + X} + \sqrt{\frac{X}{(X + 1)^3}}\right)^{\frac{1}{3}} + \left(\frac{1}{1 + X} - \sqrt{\frac{X}{(X + 1)^3}}\right)^{\frac{1}{3}}\right]$$ (3.22)

for a real solution. (Other two solutions are complex and then inappropriate.)

In the case of the inner solutions, $0 \leq (\phi^1)^2 \leq \frac{a^2}{3}$, eq. (3.20) can be rewritten as

$$(X - 1)\Phi^3 - a^2(X - 1)\Phi^2 + \frac{a^4}{3} X \Phi - \frac{a^6}{27} X = 0.$$ (3.23)

In this case, we must solve this equation in each case, $X = 1$ or $X \neq 1$, respectively. When $X = 1$, the solution of this equation is $(\phi^1)^2 = a^2/9$ and this corresponds to the expectation value at the center of the wall ($z = z_0$). When $X \neq 1 (z \neq z_0)$, there are three candidates for the solution of the outer wall:

$$\phi^2 = \frac{a^2}{3} \left[1 + \left(\frac{1}{1 - X} + \sqrt{\frac{X}{(X - 1)^3}}\right)^{\frac{1}{3}} \left(\frac{1}{e^{\frac{2\pi i}{3}}} e^{\frac{2\pi i}{3}}\right)\right].$$
Figure 4: The two kinds of real solutions, the outer solution, eq. (3.22), and the inner solution, eq. (3.25), are plotted in figures (a) and (b), respectively.

\[
\phi^2 = \begin{cases} 
\frac{a_2^2}{3} \left[ 1 + \left( \frac{1}{1-X} + \sqrt{\frac{X}{(X-1)^3}} \right)^\frac{2}{3} - \left( -\frac{1}{1-X} + \sqrt{\frac{X}{(X-1)^3}} \right)^\frac{2}{3} \right] & (z > z_0) \\
\frac{a_2^2}{3} \left[ 1 + \left( \frac{1}{1-X} + \sqrt{\frac{X}{(X-1)^3}} \right)^\frac{2}{3} e^{-\frac{2\pi i}{3}} + \left( \frac{1}{1-X} - \sqrt{\frac{X}{(X-1)^3}} \right)^\frac{2}{3} e^{\frac{2\pi i}{3}} \right] & (z < z_0)
\end{cases}
\]  \quad (3.25)

All of these solutions are not real positive, so we must choose correct one for the regions of \( z < z_0 \) \((X < 1)\) and \( z > z_0 \) \((X > 1)\). In the region of \( z > z_0 \), the first solution is appropriate for the real solution. In the region of \( z < z_0 \), the third solution is appropriate. (The first cannot satisfy the correct boundary conditions, \( \phi^2(-\infty) = 0 \). The second tends to infinity in the limit of \( z \to z_0 \).) In summary, we obtain the inner wall solution, by continuing the third one in the left \((z < z_0)\) and the first one in the right \((z > z_0)\):

The profiles of the outer and inner wall solutions are plotted in the figure 4.
4 Conclusions and discussions

We considered BPS walls in the models with continuously degenerated moduli spaces. We discussed only two $O(2)$ symmetric models concretely, but some results can be generalized to the models with other global symmetry $G$. When a model has continuous symmetry, $O(2)$ for example, the BPS equation of the wall becomes covariant under this symmetry, so the BPS wall also has this symmetry. If we can find a BPS solution, the configurations transformed from it by $G$ are also BPS solutions. Under this transformations, boundary conditions change, but the tensions of the walls never change.

In the supersymmetric field theories, the symmetry of superpotential is enlarged to the complexification of original symmetry because of holomorphy of the superpotential, and there exist the flat directions corresponding to the directions of imaginary parts of fields. As the BPS equation is not covariant under $O(2)^c$, it is nontrivial and depends on the detail of the models whether there exist complex BPS walls interpolating two disjoint flat directions. Actually we considered two models with flat directions, and showed that there is no complex BPS wall in the first model with one flat direction and there is the family of complex BPS walls in the second model with two flat directions. From the examinations of these two models, we learned that, in models with continuous symmetry, there can exist BPS domain walls interpolating not only two disjoint vacua in the real moduli spaces of the continuous symmetry, but also two disjoint vacua in the complex moduli spaces originated from the complexification of the original symmetry.

We have not found the criteria whether or not a complex BPS wall exists in general models yet. So let us now examine the general structure of the existence of complex BPS walls by using these two models. Since we considered the supersymmetric model with two chiral superfields, there are BPS equations according to four real degrees of freedom. From the condition on BPS states, we constrained the solutions by the reality of solutions in the superpotential space. We thus can expect that BPS solutions include maximally three free parameters as the integral constants, according to remaining three degrees of freedom, since the BPS equation
is the first order differential equation. In fact, three parameters, \( z_0, \theta \) and \( c \), occur as the integral constants in the BPS solution of the second model; on the other hand, the third parameter \( c \) is not contained in the BPS wall solution of the first model: it has been dropped by the boundary condition.

We can interpret parameters, \( z_0 \) and \( \theta \) (and \( c \)), as the “moduli” of the BPS wall solutions, since the tension of the wall does not change when we continuously vary the values of these parameters. These parameters, however, have slightly different meaning: since \( z_0 \) stands for the location of the center of the wall, we can vary this parameter without changing the boundary conditions; on the other hand, we cannot vary \( \theta \) and \( c \) without changes of the boundary condition.

Next we discuss the nature of these parameters in terms of symmetry. Two of the three parameters stand for the Nambu-Goldstone modes corresponding to the symmetries broken by the wall solution; the \( z_0 \) corresponds to the translation along to the \( z \)-axis, and the \( \theta \) to the continuous internal symmetry, \( O(2) \). Therefore the BPS wall solution contains these free parameters apparently. On the other hand, the superpotential is invariant under the complex symmetry and the parameter \( c \) can be considered to represent the deformation of the BPS wall along the flat direction. This complex symmetry, however, is not the symmetry of the whole model (the Kähler potential is invariant under \( G \) but not \( G^C \)), and therefore the \( c \) does not directly correspond to the complex symmetry. This is why the BPS wall does not always contain a parameter corresponding to the complex symmetry.

We give a comment on the similarity with the results in Ref. [8]. The parameter \( c \) in our model is the integral constant of the BPS equations and depends on the detail of the model. This quantity is similar to the additional integrals of motion in Ref. [8]. In both the models, the additional constants do not correspond directly to the symmetry of the theory. The distinct point is that additional integrals of motion in Ref. [8] represented the spatial distance (in the space-time) of two separated BPS walls; on the other hand, the quantity \( c \) in our model controls the shift of the BPS walls along the flat direction in the internal space.

Let us summarize the criteria for the existence of the complex solution learned from our models: we can expect the existence of the complex BPS wall interpolating
disjoint flat directions, when there remain some degrees of freedom, after taking account of the breaking of the translation and continuous internal symmetries. At the time, additional integrals of motion may play an important role. However there can be the case that the boundary condition does not allow the additional degrees of freedom, as in the $O(2)$ symmetric vacua in our two models.

Before closing the conclusion we point out an another interesting feature of our models: there exist BPS walls connecting $O(2)$ symmetric and $O(2)$ broken vacua. (In the case of standard BPS walls, broken symmetry is $Z_2$, and the two vacua separated by the wall are both the $Z_2$-broken phase.) Mass spectra are different in each side of the walls: in the broken phase of $O(2)$ (or $O(2)^C$), we can expect a massless Nambu-Goldstone boson (and a quasi-Nambu-Goldstone boson). It is a future problem to examine the wave functions of these massless modes in order to show this difference.

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