Gravitational Stability and Extra Timelike Dimensions

Satoshi Matsuda†

Department of Fundamental Sciences, FIHS
Kyoto University, Kyoto 606-8501, Japan

Shigenori Seki‡

Graduate School of Human and Environmental Studies
Kyoto University, Kyoto 606-8501, Japan

abstract

Tachyonic Kaluza-Klein modes normally appear from the compactification of extra timelike dimensions with scale $L$ and lead to some instabilities of physical systems. We calculate the contribution of tachyonic gravitons to the self energies of spherical massive bodies of radius $R$ and discuss their possible gravitational stability. We find that some spherical bodies can be stable at critical radii $R = 2\pi L p$ for some positive integers $p$. We also prove the generic property of massive bodies that for the range $0 < R \leq \pi L$ the gravitational force due to the ordinary massless graviton exchange is screened by the Kaluza-Klein mode exchange of tachyonic gravitons.

July 2000

† matsuda@phys.h.kyoto-u.ac.jp
‡ seki@phys.h.kyoto-u.ac.jp
1. Introduction

Recently large extra dimensions, which are normally spacelike ones, are discussed. In [1] it is argued that the large scale of extra spacelike dimensions has a size of, or smaller than, a millimeter range in order not to be in conflict with any experiments and observations. On the other hand, extra timelike dimensions can also be considered [2].

By compactifying extra dimensions, we find that Kaluza-Klein modes are produced. They are non-tachyonic when the extra dimensions are spacelike, but tachyonic Kaluza-Klein modes are obtained when the extra dimensions are timelike. Now for simplicity we consider one extra timelike dimension and compactify it on a circle of radius $L$ [3]. Let gravitons propagate in the extra dimension, then we obtain tachyonic gravitons of the Kaluza-Klein modes. Since their propagators are proportional to

$$-i \frac{1}{k_0^2 - k^2 + \frac{n^2}{L^2} + i\epsilon}, \quad n \in \mathbb{Z},$$

up to a spin tensor factor, we can calculate the gravitational potential between two unit mass points at distance $d$ as

$$V(d) = -G_N \frac{1}{d} - \sum_{n=-\infty, n\neq 0}^{\infty} G_N \frac{1}{d} \exp \left( i \frac{|n|}{L} d \right), \quad (1.1)$$

in the nonrelativistic tree-level approximation, where $G_N$ is the Newton constant. The first term of the potential (1.1) is the contribution of ordinary massless graviton and the second is the one of tachyonic gravitons.

Since the gravitational potential (1.1) has an imaginary part, which violates causality and conserved probability, it makes massive bodies unstable. But the effect is not unacceptable experimentally if the scale $L$ of the extra dimension is bounded below a sufficiently small size [2][3].

In this paper we calculate the gravitational self energies of spherical massive bodies exactly and discuss their stability on an analytic ground. In section 2 we assume their mass densities as some typical ones and calculate the corresponding self energies. In section 3 we devote ourselves to discussions on the gravitational screening effect of tachyonic gravitons and about the gravitational stability of spherical bodies. In the final section we present conclusions with some comments. We shall give some useful formulas in the appendix.
2. The self energy of spherical body

We consider the self energy of a spherical body of radius $R$ with mass density $\rho(r)$, which depends on the radial coordinate $r$. Using the form of the potential given in the second term of Eq.(1.1), and integrating out the relative polar angles between two mass points of densities $\rho(r)$ and $\rho(l)$ at distance $d = \sqrt{r^2 + l^2 - 2rl \cos\beta}$ with their opening angle $\beta$, we obtain the contribution of the $n$-th Kaluza-Klein mode, as a tachyonic graviton, to the self energy as follows:

$$E_n = 8i\pi^2 G_N L \int_0^R dr \int_0^r dl \rho(r)\rho(l) \frac{rl}{|n|} \left[ \exp\left(\frac{in}{L} (r + l)\right) - \exp\left(\frac{in}{L} (r - l)\right) \right], \quad (2.1)$$

whereas from the first term of Eq.(1.1) the contribution of an ordinary massless graviton to that is given by

$$E_0 = -16\pi^2 G_N \int_0^R dr \int_0^r dl \rho(r)\rho(l) rl^2. \quad (2.2)$$

Note that this can also be obtained simply as the limit of $|n| \to 0$ from Eq.(2.1). So we obtain the total gravitational self energy as

$$E(R) = E_0 + \sum_{n=-\infty, n\neq 0}^{\infty} E_n = E_0 + 2 \sum_{n=1}^{\infty} E_n. \quad (2.3)$$

In order to make definite and precise arguments on the implications of the tachyonic Kaluza-Klein modes to gravitational stability, we actually set $\rho(r)$ to a few typical forms of mass density and calculate the corresponding gravitational self energies as follows.

2.1. The $\rho(r) = \frac{D}{r}$ case

At first we set the density $\rho(r) = \frac{D}{r}$, where $D$ is constant. This provides a simple and interesting result. From Eq.(2.2) the contribution of massless graviton to the self energy becomes

$$E_0 = -16\pi^2 G_N D^2 \int_0^R dr \int_0^r dl \ l = -\frac{8\pi^2}{3} G_N D^2 R^3, \quad (2.4)$$

and from Eq.(2.1) the contribution of tachyonic gravitons becomes

$$2 \sum_{n=1}^{\infty} E_n = 16i\pi^2 G_N LD^2 \sum_{n=1}^{\infty} \frac{1}{n} \int_0^R dr \int_0^r dl \ f_n(r, l), \quad (2.5)$$

where

$$f_n(r, l) \equiv \exp\left(\frac{in}{L} (r + l)\right) - \exp\left(\frac{in}{L} (r - l)\right). \quad (2.6)$$
We set $R = 2\pi L + c$, where $k \in \{N, 0\}$ and $0 \leq c < 2\pi L$. From Eqs.(2.3), (2.4) and (2.5) the total self energy is given by

$$E(R) = -\frac{8\pi^2}{3} G_N D^2 (R^3 - 2\pi^3 L^3 k) + 16i\pi^2 G_N LD^2 \sum_{n=1}^{\infty} \frac{1}{n} \int_0^c dr \int_0^r dl \ f_n(r,l).$$

(2.7)

At $R = 2\pi L k (c = 0)$, we obtain

$$E(2\pi L) = -\frac{8\pi^2}{3} G_N D^2 2(\pi L)^3 k(4k^2 - 1),$$

(2.8)

and this is pure real. So we find that the spherical body with a critical radius $R = 2\pi L k$ is gravitationally stable.

From Eqs.(2.7) and (A.2) the imaginary part of the self energy can be expressed as

$$\Im E(2\pi L k + c) = 16\pi^2 G_N D^2 L \int_0^c dr \int_0^r dl \sum_{n=1}^{\infty} \frac{1}{n} \left( \cos \frac{n(r + l)}{L} - \cos \frac{n(r - l)}{L} \right)$$

$$= -16\pi^2 G_N D^2 L \int_0^c dr \int_0^r dl \ \log \left| \sin \frac{n(l)}{L} \right| \sin \frac{n(r)}{2L},$$

(2.9)

where $0 \leq c < 2\pi L$. Eq.(2.9) exhibits a periodicity of the imaginary part with respect to the size $R$ of spherical body and the period $2\pi L$ is determined by the scale of the extra timelike dimension. We can calculate Eq.(2.9) numerically and the result is presented in fig.1.

![fig. 1 The imaginary part of the self energy ($\rho = \frac{D}{r}$).](image-url)
From Eqs. (2.7) and (A.1) the real part of the self energy can be calculated as

\[ \Re E(2\pi Lk + c) = -\frac{8\pi^2}{3}G_N D^2 (R^3 - 2\pi^3 L^3 k) \]

\[ -16\pi^2 G_N D^2 L \int_0^c dr \int_0^r dl \sum_{n=1}^\infty \frac{1}{n} \left( \sin \frac{n(r+l)}{L} - \sin \frac{n(r-l)}{L} \right) \]

\[ = -32\pi^3 G_N D^2 L \left\{ \begin{array}{ll}
\frac{k}{2} [3c^2 + 6\pi Lkc + (4k^2 - 1)\pi^2 L^2] , & \text{if } 0 \leq c < \pi L , \\
(k + 1) [3c^2 + 6(k - 1)\pi Lc + (4k^2 - 4k + 3)\pi^2 L^2] , & \text{if } \pi L \leq c < 2\pi L .
\end{array} \right(2.10) \]

The behavior of Eq.(2.10) is shown in fig.2.

![fig. 2 The real part of the self energy (ρ = D_0).

Since Eq.(2.10) indicates that the real part of the self energy vanishes identically for the range \(0 < R \leq \pi L\) with \(k = 0\), we find that the gravitational force due to the ordinary massless graviton exchange is “screened” in this region by the exchange of the Kaluza-Klein mode tower of tachyonic gravitons.

2.2. The \(\rho(r) = D_0(\text{constant})\) case

We set \(\rho(r)\) to a constant value \(D_0\). This is also a physically normal and simple situation. Using Eq.(2.2) the contribution of an ordinary graviton to the self energy of a spherical body of radius \(R\) becomes

\[ E_0 = -16\pi^2 G_N D_0^2 \int_0^R dr \int_0^r dl \ r l^2 r = -\frac{16}{15} \pi^2 G_N D_0^2 R^5 , \quad (2.11) \]

and the one of tachyonic gravitons to that becomes

\[ 2 \sum_{n=1}^\infty E_n = 16i\pi^2 G_N LD_0^2 \sum_{n=1}^{\infty} \frac{1}{n} \int_0^R dr \int_0^r dl r l f_n(r,l) . \quad (2.12) \]
At \( R = 2\pi Lk \) with \( k \in \{N, 0\} \), we obtain

\[
E(2\pi Lk) = -\frac{16}{45} \pi^7 G_N D_0^2 L^5 k (4k^2 - 1)(24k^2 + 1) - 64iG_N D_0^2 \pi^4 L^5 k^2 \zeta(3),
\]  

(2.13)

where \( \zeta \) is a zeta-function and \( \zeta(3) = \sum_{1}^{\infty} \frac{1}{n^3} \sim \frac{\pi^2}{25.79436} \). Since the imaginary part of Eq.\( (2.13) \) does not vanish for any \( k > 0 \), at the points \( R = 2\pi Lk \) the spherical bodies are unstable and this result is quite different from the one in the case of \( \rho(r) = \frac{D_0}{r} \).

From Eqs.(2.3), (2.11) and (2.12) we calculate the imaginary part of the self energy as

\[
\Im E(R) = -16\pi^2 G_N D_0^2 L \int_0^R dr \int_0^r dl \ r l \log \left| \frac{\sin \frac{r+l}{2L}}{\sin \frac{r-l}{2L}} \right| ,
\]

and its numerical result is as shown in fig.3 .

![fig. 3 The imaginary part of the self energy (ρ = D0).](image)

From fig.3 we note that the imaginary part of self energy never vanishes for any \( R > 0 \). This implies that in the present case of mass density there is no critical value of radius \( R \) which stabilizes the corresponding spherical body.

Now we set \( R = 2\pi Lk + c \), where \( k \in \{N, 0\} \) and \( 0 \leq c < 2\pi L \). From Eqs.(2.3), (2.11) and (2.12) the real part of the self energy becomes at \( 0 \leq c < \pi L \)

\[
\Re E(2\pi Lk + c) = -\frac{16}{45} G_N D_0^2 L^2 \pi^4 (2k + 1)k \left[ 30c^3 + 15(8k - 1)\pi Lc^2 + 60k(2k - 1)(\pi L)^2 c + (48k^3 - 24k^2 + 2k - 1)(\pi L)^3 \right] ,
\]  

(2.14)


and at $\pi L \leq c < 2\pi L$

$$\Re E(2\pi Lk + c) = -\frac{16}{45} G_N D_0^2 L^2 \pi^4 (k+1)(2k+1) \left[ 30c^3 + 15(8k - 3)\pi Lc^2 + 60k(2k - 3)(\pi L)^2 c + (48k^3 - 72k^2 + 74k + 15)(\pi L)^3 \right]. \quad (2.15)$$

From Eqs.(2.14) and (2.15) the behavior of the real part of the self energy is shown in figs.4(a) and 4(b). The magnified plot of fig.4(a) for the range $0 < R < 2\pi L$ is given in fig.4(b).

![Graph showing the real part of the self energy.](image)

**fig. 4** The real part of the self energy ($\rho = D_0$). The magnified plot is shown in (b).

Since, substituting $k = 0$ in Eq.(2.14), we obtain $\Re E(R) = 0$, we again conclude that for $0 < R < \pi L$ the gravitational force of ordinary graviton exchange is “screened” by the coherent effect of tachyonic graviton exchange of the Kaluza-Klein mode tower.

### 3. Discussions

We first discuss the generic feature of the gravitational screening effect due to the tachyonic Kaluza-Klein modes. Let us note that the summation formulas, Eqs.(A.1) and (A.2), give the identity:

$$i \sum_{n=1}^{\infty} \frac{1}{n} f_n(r,l) = \frac{l}{L} - i \log \frac{\sin \frac{r+l}{2L}}{\sin \frac{r-l}{2L}}, \quad 0 < r - l \leq r + l < 2\pi L. \quad (3.1)$$

Actually the infinite sum on the left-hand side is a periodic function in $r + l$ and $r - l$ with period $2\pi L$. So, outside the range $0 < r - l \leq r + l < 2\pi L$ the identity should be modified properly to hold by taking into account the periodicity so that it matches the valid range.
in $x$ of Eqs.(A.1) and (A.2). This point is important in calculating the real part of the gravitational self energy for a larger value of radius $R$ in the range $R \geq \pi L$. Now, with the use of the formula (3.1) in Eqs.(2.1), (2.2) and (2.3) we find exactly

$$E(R) = -i16\pi^2 G_N L \int_0^R dr \int_0^r dl \rho(r)\rho(l)rl \log \frac{\sin \frac{r+l}{2L}}{\sin \frac{r-l}{2L}}, \quad 0 < R \leq \pi L, \quad (3.2)$$

which proves that the real part of the self energy vanishes identically for the range $0 < R \leq \pi L$ for any spherical mass density $\rho(r)$. We can therefore conclude generically that in the corresponding region the gravitational force due to the massless graviton exchange is “screened” by the effect of the tachyonic graviton exchange of the Kaluza-Klein modes.

Next we discuss a possible gravitational stability of a spherical body with critical radius $R$ being equal to some positive integer multiple of $2\pi L$. As presented in the previous section, the spherical body with the mass density $\rho(r) = D/r$ can be gravitationally stable at the critical radius of $R$ equal to every positive integer multiple of $2\pi L$. More generally, we can consider an onion-like hybrid model of spherical mass density with two kinds of the $D$ value. We assume $\rho(r)$ to be

$$\rho(r) = \begin{cases} \frac{D}{r} & \text{for } \max[0, (2km - 1)\pi L] < r < (2km + 1)\pi L \\ \frac{bD}{r} & \text{for } (2km + 1)\pi L < r < (2(k+1)m - 1)\pi L \end{cases}, \quad k = 0, 1, 2, \cdots \quad (3.3)$$

with a fixed positive integer $m$ and a positive constant $b \neq 1$. Then the imaginary part of the self energy $E(R)$ is found to vanish exactly at critical values of $R = k(2m\pi L)$ with $k = 1, 2, 3, \cdots$ and, in between any two consecutive critical values of $R$ at distance $2m\pi L$, it oscillates with a pitch of $2\pi L$ between the two values of weight proportional to $D^2$ and $(1 - b)^2D^2$. When $b = 1$, the onion-like hybrid model just reduces to the simple case of subsection 2.1, while by varying $b$ and $m$ we find to obtain a variety of the hybrid models with the common generic feature of the gravitational stability which shows up in each model at critical radius $R = 2\pi Lp$ with a corresponding positive integer $p$.

4. Conclusions

We considered one extra timelike dimension, where only gravitons propagate, and its compactification on a circle of a radius $L$, which leads to the tachyonic Kaluza-Klein modes of gravitons. And we calculated the gravitational self energies of spherical bodies of a radius $R$ which include the contributions of the tachyonic gravitons as well as of a massless graviton.
We discussed two typical models of mass density. At first we set the density as the spherically symmetric one, \( \rho(r) = \frac{D}{r} \). The imaginary part of the self energy has a periodicity of \( 2\pi L \) in \( R \). Since this imaginary part vanishes at \( R = 2\pi Lk \) for any \( k \in \mathbb{N} \), the spherical body of a size \( R = 2\pi Lk \) is gravitationally stable. On the other hand the real part of the self energy becomes zero for \( 0 < R \leq \pi L \). So the gravitational force is screened in the spherical body with a size \( 0 < R \leq \pi L \).

Next we set \( \rho(r) = D \) (constant). The imaginary part of the self energy is not zero for any \( R \), and there is no stable size of spherical bodies. This result is different from the one of \( \rho(r) = \frac{D}{r} \). On the other hand, since the real part of the self energy vanishes for \( 0 < R \leq \pi L \), the gravitational force is screened in the constant spherical body with a size \( 0 < R \leq \pi L \). This behavior is the same as the one of \( \rho(r) = \frac{D}{r} \). In fact we have shown that the gravitational screening due to tachyonic gravitons for the range \( 0 < R \leq \pi L \) is the generic feature of the timelike Kaluza-Klein modes for any spherical mass density.

Though we have considered one extra timelike dimension, there is no reason why no more extra timelike dimensions exist. Actually even in the case of any additional extra timelike dimensions we are able to calculate the gravitational self energies of spherical bodies in some approximation and to discuss their stability and screening effect in a convincing way based on the obtained analytic results [4].

As a final comment we raise the possibility that what we have presented in this paper, that is, the fact that the scale of critical radius \( R \) in the spacelike dimensions typically represented by a massive particle size is simply related to the scale \( 2\pi L \) of the timelike dimension through the gravitational stability condition given by the vanishing imaginary part of the self-energy might involve some implications in the construction of a final theory of particle physics like superstring theory or brane theory.

**Acknowledgments**

This work is supported in part by the Grant-in-Aid for Scientific Research on Priority Area 707 “Supersymmetry and Unified Theory of Elementary Particles”, Japan Ministry of Education. S. M. is also funded partially by the Grant-in-Aid for Scientific Research (C) (2)-10640260, while S. S. is supported in part by JSPS Research Fellowship for Young Scientists.
Appendix A.

The following infinite summations are known to hold:

\[ \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{1}{2} (\pi - x), \quad 0 < x < 2\pi, \quad (A.1) \]

\[ \sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\log \left( \frac{2 \sin \frac{x}{2} }{2} \right), \quad 0 < x < 2\pi, \quad (A.2) \]

References


