Minimal extended flavor groups, matter fields chiral representations, and the flavor question

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Abstract

We show the specific unusual features on chiral gauge anomalies cancellation in the minimal, necessarily 3-3-1, and the largest 3-4-1 weak isospin chiral gauge semisimple group leptoquark–bilepton extensions of the 3-2-1 conventional standard model of nuclear and electromagnetic interactions. In such models a natural explanation for the fundamental question of fermion generation replication arises from the self-consistency of a local gauge quantum field theory, which constrains the number of the QFD fermion families to the QCD color charges.

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Flavor question has been a long standing puzzle. It was addressed in the SU(11) and SU(14) generalizations of the SU(5) grand unified theory with three and twelve families, [1] in the minimal supersymmetric model of quarks and leptons preon compositeness, [2] in the 3-3-1 leptoquark-bilepton model, [3–6] [7] and in the Sp(6)×U(1)$_Y$ symplectic gauge group extension. [8] The generation patterns of the reduced four-dimensional spinor fields with repeated copies of the same gauge quantum numbers is also a prediction of the 5-dimensional space-time Kaluza–Klein theory. [9] The anomaly cancellation of a gauge theory in all orders of perturbative expansion, which derives from the renormalizability condition, constrains the fermion representation content. Three perturbative anomalies have been identified [10] for chiral gauge theories in four dimensional space-time: (i) The triangle chiral gauge anomaly [11] must be cancelled to avoid violations of gauge invariance and the renormalizability of the theory; (ii) The global non-perturbative SU(2) chiral gauge anomaly, [12] which must be absent in order for the fermion integral to be defined in a gauge invariant way; (iii) The mixed perturbative chiral gauge gravitational anomaly [13,14] which must be cancelled in order to ensure general covariance. In the standard model, the anomalies were cancelled into each generation of quarks and leptons. This is a very important issue, because in order to cancel anomalies we must sum over all hypercharges of quarks and leptons in each representation. In the standard model the anomaly-free conditions and flavor question do not seem to have any connection. This leads us to a question: Are the anomalies always cancelled automatically only by generation of quarks or leptons? And do the anomaly cancellation conditions have any connection with flavor puzzle? These questions of course cannot be answered within the standard model. It is of interest to find out some models in which these questions can be answered. In this note, we would like to provide some models in which the above questions can be answered. Particularly, we consider the anomaly cancellation and flavor question in the 3-3-1 leptoquark-bilepton model and the 3-4-1 semisimple group generalization where there is a non-trivial very specific anomaly-free representation content. [3,4,15] A bilepton is a boson which couples minimally with dimension four interactions to standard model leptons and can carry lepton number zero or 2. [16] The semisimple chiral gauge groups are

$$G_{3n1} \equiv \text{SU}(3)_c \otimes \text{SU}(n)_L \otimes \text{U}(1)_N$$

(1)

with $n = 3, 4$, from now on 3-$n$-1 models and $N$ is the new U(1)$_N$ charge of the Abelian semisimple group factor. For the standard model weak isospin factor, $n = 2$. The general anomaly-free condition is

$$\sum_{\text{representations}} \text{Tr}[\{T^a_L, T^b_L\}T^c_L - \{T^a_R, T^b_R\}T^c_R] = 0$$

(2)

where $T^a$ are the gauge group generators and the $a$ index runs over the dimension of the simple SU($n$) group, $a = 1, 2, \ldots, n^2 - 1$, with a rank $n - 1$, and $a = 0$ for the Abelian factor. The standard model independent families have the following structure under the chiral 3-2-1 gauge group,

$$f_{iL} \sim (1_c, 2_L, Y = -1), \quad \ell_{iR} \sim (1_c, 1_R, -2)$$

(3a)

for the leptonic sector with the flavor labels $\ell_i = e, \mu, \tau$ ($i = 1, 2, 3$) and
\[ Q_{iL} \sim (3c, 2_L, 1/3), \quad u_{iR} \sim (3c, 1_R, 4/3), \quad d_{iR} \sim (3c, 1_R, -2/3) \] (3b)

for the quark flavors. These symmetry eigenstates are related to the mass eigenstates by three Cabibbo–Kobayashi–Maskawa rotation angles. The electric charge operator is defined as

\[ Q = T_3 + T_0 = T_3 + \frac{Y}{2}, \tag{4} \]

with \( e = g \sin \theta_W = g' \cos \theta_W = 1.602 177 33(49) \times 10^{-19} \) C = 4.803 206 8(15) \times 10^{-8} \) esu. [17] The matrices \( T^a \) in Eq. (2) will be either a half of the Pauli matrices, \( \tau^a, \quad a = 1, 2, 2^2 - 1, \) or the weak hypercharge generator \( T^0 \equiv Y/2 \) for \( a = 0 \) proportional to the \( n \times n \) identity operator. Since the weak isospin group SU(2) is a safe group, [18] then

\[ \text{Tr} (\{\tau^a, \tau^b\} \tau^c) = 2 \delta^{ab} \text{Tr}(\tau^c) = 0, \tag{5} \]

but in the case where at least one of the \( T \) generators is the hypercharge \( Y \), we have

\[ \text{Tr}(\{\tau^a, \tau^b\} Y) = 2 \delta^{ab} \text{Tr}(Y) \tag{6} \]

and

\[ \text{Tr}(\tau^a Y Y) \propto \text{Tr}(\tau^a). \tag{7} \]

The anomaly contribution in Eq. (6) is proportional to the sum of all fermionic discrete hypercharge values

\[ \text{Tr}(Y) = \sum_{\text{lepton}} (Y_L + Y_R) + \sum_{\text{quark}} (Y_L + Y_R). \tag{8} \]

The \( \text{Tr}(Y) \) in Eq. (8) vanishes for the fermion content in the equations (3a) and (3b),

\[ \sum_{\text{lepton}} (Y_L + Y_R) = Y(\nu_{iL}) + Y(\ell_L) + Y(\ell_R) = 2Y(f_{iL}) + Y(\ell_R) = -4 \tag{9a} \]

\[ \sum_{\text{quark}} (Y_L + Y_R) = 3[2Y(Q_{iL}) + Y(u_{iR}) + Y(d_{iR})] = +4 \tag{9b} \]

where the global factor in the last equation takes into account the number of SU(3)\(_c\) color charges. The summations indicated on the left-handed side involve the color, flavor, and the weak hypercharge degrees of freedom. For the case in which all \( T \)'s are the hypercharge \( U(1)_Y \) generator, we have from Eq. (4)

\[ \text{Tr}(Y^3) \propto \text{Tr}(Q^2 T_3 - QT_3^2) \tag{10} \]

where we used the fact that the electromagnetic vector neutral current vertices do not have anomalies. Thus the right-handed side of Eq. (10) yields to

\[ \sum_{\text{lepton}} (Q^2 T_3 - QT_3^2) = [(0)^2(1/2) - (0)(1/2)^2] + [(-1)^2(-1/2) - (-1)(-1/2)^2] = \frac{1}{4} \tag{11a} \]
\[ \sum_{\text{quark}} (Q^2 T_3 - Q T_3^2) = 3[(2/3)^2(1/2) - (2/3)(1/2)^2] + 3[(-1/3)^2(-1/2) - (-1/3)(-1/2)^2] = +\frac{1}{4} \]  

(11b)

which guarantees that the standard fermion assignments turn the 3-2-1 standard theory chiral anomaly free. Taking into account the electric charge operator, the trace of an odd number of \( Y \) generator in Eq. (6) and Eq. (10) can be re-written as follows:

\[ \text{Tr}(Y) \text{ or } \text{Tr}(Y^3) \propto \text{Tr}(Q) = \sum_j Q_j. \]  

(12)

Two remarks can be considered here. The first one concerns the fact that in the standard model we have anomaly cancellation within each independent family of chiral fermions and not by generations. The other one is that the anomaly cancellation occurs in each chiral sector.

Now, let us consider the following 3-3-1 non-universal fermion representation content

\[
\begin{align*}
    f_{1L} &\sim (1_c, 3_L, 0), \\
    Q_{1L} &\sim (3_c, 3_L, 2/3), \\
    u_{1R} &\sim (3_c, 1_R, 2/3), \\
    d_{1R} &\sim (3_c, 1_R, -1/3), \\
    J_{1R} &\sim (3_c, 1_R, 5/3), \\
\end{align*}
\]

(13a)

for the first generation and

\[
\begin{align*}
    f_{iL} &\sim (1_c, 3_L, 0), \\
    Q_{iL} &\sim (3_c, 3_L, -1/3), \\
    J_{iR} &\sim (3_c, 1_R, -4/3), \\
    u_{iR} &\sim (3_c, 1_R, 2/3), \\
    d_{iR} &\sim (3_c, 1_R, -1/3), \\
\end{align*}
\]

(13b)

when \( i = 2, 3 \) for the second and third generations. The \( N \times N \) flavor mixing matrix is parametrized by \( N(N - 1)/2 \) rotation angles and \( (N - 1)(N - 2)/2 \) phases. Three generations of leptons, \( f_{iL}, i = 1, 2, 3 \) transform in the same way, and right-handed flavor singlet fields are not introduced. In the electroweak semisimple group factor of the 3-3-1 gauge symmetry the electric charge operator is defined as

\[ \frac{Q}{e} = \frac{1}{2}(\lambda_3 - \sqrt{3}\lambda_8) + N \]  

(14)

where \( \lambda_{3,8} \) are diagonal Gell-Mann matrices associated to the rank two of the SU(3)\(_L\) flavor simple group and \( N \) is the new U(1)\(_N\) hypercharge,

\[ N = \frac{\sqrt{3}}{2}\lambda_8 + \frac{Y}{2} \]  

(15)

equivalent to the mean electric charge of the fields contained in the multiplet. The electron electric charge in terms of the \( \theta_{3-1} \) mixing angle of the SU(3)\(_L\) and U(1)\(_N\) simple and semisimple groups with the associated \( g \) and \( g' \) gauge coupling constants is
\[ e^2 = \frac{g^2 \sin^2 \theta_{3-1}}{1 + 3 \sin^2 \theta_{3-1}} = \frac{g'^2 \cos^2 \theta_{3-1}}{1 + 3 \sin^2 \theta_{3-1}} \]  

(16)

and with \( e^2 = g^2 \sin^2 \theta_W = g'^2 (1 - 4 \sin^2 \theta_W) \) the Eq. (16) becomes

\[ \tan^2 \theta_{3-1} = \frac{\sin^2 \theta_W}{1 - 4 \sin^2 \theta_W} \]  

(17)

with the bound on the Weinberg electroweak angle \( \sin^2 \theta_W < 1/4 \). The SU\( (n) \) groups for \( n > 2 \) are not safe in the sense of Eq. (5). For \( n = 3 \) the generators are proportional to the Gell-Mann matrices, closing among them the Lie algebra structure,

\[ [\lambda_a, \lambda_b] = 2i f_{abc} \lambda_c \]  

(18a)

\[ \{\lambda_a, \lambda_b\} = \frac{4}{3} \delta_{ab} + 2d_{abc} \lambda_c, \]  

(18b)

where the structure constants \( f_{abc} \) are totally antisymmetric and \( d_{abc} \) are totally symmetric under the exchange of indices. For any \( n \times n \) \( \lambda \)-matrices with arbitrary \( n \),

\[ f_{abc} = \frac{1}{4i} \text{Tr}([\lambda_a, \lambda_b] \lambda_c) \]  

(19a)

\[ d_{abc} = \frac{1}{4} \text{Tr}\{\lambda_a, \lambda_b\} \lambda_c \],  

(19b)

in general the anomaly is proportional to \( d_{abc} \). The \( d_{abc} \) constants vanish for \( n = 2 \) case. When \( n > 2 \) the generalization of the anti-commutation relation in Eq. (5) is

\[ \{\lambda_a, \lambda_b\} = \frac{4}{n} \delta_{ab} + 2d_{abc} \lambda_c. \]  

(20)

The anomaly-free conditions are

\[ \text{Tr}([\text{SU}(3)_c]^2[U(1)_N]) = 0 \]  

(21a)

\[ \text{Tr}([\text{SU}(3)_L]^2[U(1)_N]) = 0 \]  

(21b)

\[ \text{Tr}([U(1)_N]^3) = 0 \]  

(21c)

\[ \text{Tr}([\text{graviton}]^2[U(1)_N]) = 0. \]  

(21d)

Each generation is anomalous. Let us take the condition in Eq. (21c) containing only the Abelian factor. The general cancellation scheme can be found in Ref. [6]. We can see that the \( N^3 \) anomaly of the first generation is

\[ \sum_{\text{lepton}} N_L + \sum_{\text{quark}} (N_L + N_R) = \\ 0 + 3 [(2/3) \times 3 + 5/3 + 2/3 + (-1/3)] = +12, \]  

(22)

and the anomalies of the second or the third generations are

\[ \sum_{\text{lepton}} N_L + \sum_{\text{quark}} (N_L + N_R) = \\ 0 + 3 [(-1/3) + (-4/3) + 2/3 + (-1/3)] = -6. \]  

(23)
It is clear that anomalies are not cancelled within each generation as in the standard model. The global factor 3 takes into account the SU(3)\(_c\) three color charges. The general condition that, for a given fermionic representation \(\mathbf{D}\) it holds [19]

\[
\mathcal{A}(\mathbf{D}) = -\mathcal{A}(\bar{\mathbf{D}})
\]  

(24)

where \(\mathcal{A}(\mathbf{D})\) is the anomaly of the conjugate representation of \(\mathbf{D}\), and the fact that, as can be seen from Eqs. (13), there are the same number of multiplets in \(\mathbf{3}\) and \(\mathbf{3}^\ast\) representation turn the theory anomaly-free. The anomaly cancellation in this case does not occur by chiral sector.

Now let us display a second representation content which contains heavy leptons. [20]

For the first generation

\[
f_{1L} \sim (1_c, 3_L, 0), \quad E_{1R} \sim (1_c, 1_R, 1), \quad \ell_{1R} \sim (1_c, 1_R, -1),
\]

\[
Q_{1L} \sim (3_c, 3_L, 2/3), \quad u_{1R} \sim (3_c, 1_R, 2/3), \quad d_{1R} \sim (3_c, 1_R, -1/3), \quad J_{1R} \sim (3_c, 1_R, 5/3),
\]

(25a)

and

\[
f_{iL} \sim (1_c, 3_L, 0), \quad E_{iR} \sim (1_c, 1_R, 1), \quad \ell_{iR} \sim (1_c, 1_R, -1),
\]

\[
Q_{iL} \sim (3_c, 3_L, -1/3), \quad u_{iR} \sim (3_c, 1_R, 2/3), \quad d_{iR} \sim (3_c, 1_R, -1/3), \quad J_{iR} \sim (3_c, 1_R, -4/3),
\]

(25b)

for the second and third generations, \(i = 2, 3\). The \(N^3\) anomaly of the first generation is

\[
\sum_{\text{lepton}} (N_L + N_R) + \sum_{\text{quark}} (N_L + N_R) =
\]

\[
0 \times 3 + (-1) + 1 + 3 [(2/3) \times 3 + (5/3) + (2/3) + (-1/3)] = +12
\]

(26a)

and

\[
\sum_{\text{lepton}} (N_L + N_R) + \sum_{\text{quark}} (N_L + N_R) =
\]

\[
0 \times 3 + (-1) + 1 + 3 [(-1/3) \times 3 + (-4/3) + (2/3) + (-1/3)] = -6
\]

(26b)

are the anomalies of the second and the third generations and the complete generations \(N^3\) anomaly vanishes again.

Let us now consider the following 3-3-1 fermionic assignments for the first and second generations [4]

\[
f_{iL} \sim (1_c, 3_L, -1/3), \quad \ell_{iR} \sim (1_c, 1_R, -1),
\]

\[
Q_{iL} \sim (3_c, 3_L, 0), \quad u_{iR} \sim (3_c, 1_R, 2/3), \quad d_{iR} \sim (3_c, 1_R, -1/3), \quad J_{iR} \sim (3_c, 1_R, 5/3),
\]

(27a)

\(i = 1, 2\) and

\[
f_{3L} \sim (1_c, 3_L, -1/3), \quad \ell_{3R} \sim (1_c, 1_R, -1),
\]

\[
Q_{3L} \sim (3_c, 3_L, 1/3), \quad u_{3R} \sim (3_c, 1_R, 2/3), \quad d_{3R} \sim (3_c, 1_R, -1/3), \quad J_{3R} \sim (3_c, 1_R, 5/3),
\]

(27b)
for the third generation. Such peculiar representation structure contains the same number of flavor triplets and anti-triplets. The anomaly of the first or the second generation is
\[
\sum_{\text{lepton}} (N_L + N_R) + \sum_{\text{quark}} (N_L + N_R) =
\]
\[
(-1/3) \times 3 + (-1) + 3 [(0) \times 3 + (2/3) + (-1/3) + (-1/3)] = -2
\]
and for the third generation
\[
\sum_{\text{lepton}} (N_L + N_R) + \sum_{\text{quark}} (N_L + N_R) =
\]
\[
(-1/3) \times 3 + (-1) + 3 [(1/3) \times 3 + (2/3) + (2/3) + (-1/3)] = +4,
\]
so the \(N^3\) anomalies cancel. Note that we do not have anomaly cancellation by chiral sector.

Finally, we consider the following 3-4-1 fermionic representation content. [15] In the first generation we have a \(4\)-plet of leptons
\[
f_{1L} \sim (1_c, 4_L, 0)
\]
which contains all light leptonic degrees of freedom so that we do not need to introduce right-handed singlet fields. The assignments of the quarks are
\[
Q_{1L} \sim (3_c, 4_L, 2/3), \quad u_{1R} \sim (3_c, 1_R, 2/3), \quad d_{1R} \sim (3_c, 1_R, -1/3),
\]
\[
u'_{1R} \sim (3_c, 1_R, 2/3), \quad J_{1R} \sim (3_c, 1_R, 5/3).
\]
and the last two families have the following fundamental representation transformation properties under the 3-4-1 gauge group
\[
f_{iL} \sim (1_c, 4_L, 0),
\]
\[
Q_{iL} \sim (3_c, 4_L, -1/3), \quad J_{iR} \sim (3_c, 1_R, -4/3),
\]
\[
d'_{iR} \sim (3_c, 1_R, -1/3), \quad u_{iR} \sim (3_c, 1_R, 2/3), \quad d_{iR} \sim (3_c, 1_R, 1/3).
\]
In this extension we enlarge the electric charge operator to
\[
\frac{Q}{e} = \frac{1}{2} (\lambda_3 + \frac{1}{\sqrt{3}} \lambda_8 - \frac{2\sqrt{6}}{3} \lambda_{15}) + N'
\]
where \(\lambda_{3,8,15}\) are diagonal \(4 \times 4\) matrices whose number is equivalent to the rank of the SU(4) simple group. In the same way as the 3-3-1 fermion representations we can verify that the anomaly cancellation occurs between generations. The anomaly of the first generation is
\[
\sum_{\text{lepton}} N'_L + \sum_{\text{quark}} (N'_L + N'_R) =
\]
\[
0 + 3 [(2/3) \times 3 + (2/3) + (-1/3) + (2/3) + (5/3)] = +14
\]
and for each of the last two generations
\[
\sum_{\text{lepton}} N'_L + \sum_{\text{quark}} (N'_L + N'_R) =
\]
\[
0 + 3 [(-1/3) \times 3 + (-4/3) + (-1/3) + (2/3) + (-1/3)] = -7
\]
so that all $N^3$ Abelian contributions cancel.

It should be emphasized that there is the same number of triplets (4-plets) and antitriplets (4-plets) if each quark flavor appears with three color charges, so that the non-Abelian anomalies cancel in all these cases. Contrary to the massive neutrino extensions [21] of the conventional standard model in the 3-3-1 and 3-4-1 leptoquark-bilepton models the electric charge quantization is unavoidable and does not depend on the Dirac, Majorana or Dirac–Majorana [22] character of the neutral fermions. [23] There is always a set of leptonic generations transforming as $(1_c, 3_L, 0)$ or as $(1_c, 4_L, 0)$ with $N = N' = 0$ and the electric charge operator is a linear combination of the weak isospin simple group generators whose number is the rank of the group. In the leptonic sector there is the fundamental representation content structure of grand unified theories.

This sort of minimal gauge semisimple group models has became an interesting possibility not only for an extension of the standard model but alternative theoretical possibilities at near energy scales below a few TeV answering several open fundamental questions such as the flavor question and addressing new physics close to the Fermi scale. [24–26] The flavor question has no explanation within the 3-2-1 standard model. The anomaly cancellation in the 3-$n$-1 leptoquark-bilepton models with $n = 3, 4$ is possible only if the number of generations of quarks and leptons is equal to the number of color charges. In general, these theories will be anomaly-free if the number of generations is a multiple of the number of colors with the lepton families having identical transformation properties with $N = 0$ and without right-handed singlets even for massive neutrinos. In fact, the lightest known particles could be the sector in which a symmetry is uncovered, then the lepton sector is the part of the model determining new approximate symmetries. There is a close connection of the fundamental fermions generations, the three color charges, and the self-consistency of a local gauge quantum field theory. It is possible to answer the flavor question offering a connection between the QFD and QCD which is lost in the standard model. In these models the Weinberg angle has a Landau pole and is bounded from above giving an intrinsic limitation on the range of validity of the theory. This is another issue which does not have an equivalent one in the standard model.

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