Methods for the Nonperturbative Approximation of Form Factors and Scattering Amplitudes

John R. Hiller

Department of Physics
University of Minnesota Duluth
Duluth Minnesota 55812

Abstract. Methods are described for the nonperturbative calculation of wave functions and scattering amplitudes in light-cone quantization. Form factors are computed from the boost-invariant wave functions, which appear as coefficients in a Fock-state expansion of the field-theoretic eigenstate. A technique is proposed for calculating scattering amplitudes from matrix elements of a $T$ operator between such composite-particle eigenstates.

INTRODUCTION

To benefit from the recent progress on the calculation of field-theoretic bound states in light-cone quantization [1,2], we explore methods by which form factors and scattering amplitudes can be extracted nonperturbatively. In the case of form factors, this is relatively straightforward; well-known formulas [3] yield the form factors as overlap integrals of Fock-state wave functions. For scattering amplitudes, the way is less certain. One possible method [4] is discussed briefly here. Others have been considered by Kröger [5], Ji and Surya [6], and Fuda [7].

The formulations given are in terms of light-cone coordinates [8,1], where $x^+ \equiv t + z$ plays the role of time and the conjugate variable $p^- \equiv E - p_z$ is the light-cone energy. The light-cone three-momentum is $p_\mu = (p^+ \equiv E + p_z, p_\perp)$. An eigenstate $|P, \sigma\rangle$ of the light-cone Hamiltonian operators $\mathcal{P}^\pm$, $\mathcal{P}_\perp$ and helicity $\sigma$ is written as a Fock-state expansion

$$|P, \sigma\rangle = \sum_n \int [dx][d^2k_\perp]|\psi^{(n)}_{P,\sigma}(x, k_\perp)|n : p_\perp\rangle,$$

2) Work supported in part by the Department of Energy, contract DE-FG02-98ER41087.
with
\[
\int [dx][d^2k_\perp] = \int \delta(1 - \sum_i x_i) \prod_i \frac{dx_i}{\sqrt{x_i}} \frac{16\pi^3}{3}\delta\left(\sum_i k_{\perp i}\right) \prod_i \frac{d^2k_{\perp i}}{16\pi^3} \tag{2}
\]
and where the \(\psi^{(n)}\) are wave functions for \(n\) particles, \(x_i \equiv p_i^+/P^+\) are longitudinal momentum fractions, and \(k_{\perp i} = p_{\perp i} - x_iP_\perp\) are relative transverse momenta. Use of light-cone coordinates brings several advantages, including boost invariance of the wave functions.

The eigenvalue problem \(\mathcal{P}|P,\sigma\rangle = P|P,\sigma\rangle\) for fixed \(\sigma\) determines the wave functions as solutions of a coupled set of integral equations. A method frequently applied to these equations is discrete light-cone quantization (DLCQ) [9,1], which approximates the integrals by the trapezoidal rule and computes the wave functions on an equally spaced momentum grid. Any bound-state property can then, in principle, be calculated from these wave functions. The grid is parameterized by a longitudinal resolution \(K\) and transverse resolution \(N_\perp\), such that longitudinal momentum fractions are multiples of \(1/K\) and transverse momenta have as many as \(2N_\perp+1\) values in each direction.

The value of \(N_\perp\) is associated with a cutoff \(\Lambda^2\) on the invariant mass of each constituent and with the choice of transverse momentum scale \(\pi/L_\perp\).

**FORM FACTORS**

For a spin-1/2 fermion, the two form factors can be obtained from matrix elements of the plus component of the electromagnetic current \(J\)

\[
F_1(Q^2) = \frac{1}{2}\langle P + Q, \sigma|J^+(0)/P^+|P, \sigma\rangle, \tag{3}
\]

\[
-\left(\frac{Q_x - iQ_y}{2M}\right) F_2(Q^2) = \frac{1}{4\sigma}\langle P + Q, \sigma|J^+(0)/P^+|P, -\sigma\rangle. \tag{4}
\]

These can be reduced to overlap integrals [3]

\[
F_1(Q^2) = \sum \sum e_j \int [dx][d^2k_{\perp}][\psi^{(n)}_{P+Q,1/2}(x,k_{\perp}')\psi^{(n)}_{P,1/2}(x,k_{\perp})], \tag{5}
\]

\[
-\left(\frac{Q_x - iQ_y}{2M}\right) F_2(Q^2) = \sum \sum e_j \int [dx][d^2k_{\perp}][\psi^{(n)*}_{P+Q,1/2}(x,k_{\perp}')\psi^{(n)}_{P,-1/2}(x,k_{\perp})], \tag{6}
\]

in the frame where the photon momentum \(Q\) is written \((0,2Q \cdot P/P^+, Q_\perp)\) and

\[
k_{\perp i}' = \begin{cases} k_{\perp i} - x_iQ_\perp, & i \neq j \\ k_{\perp i} + (1 - x_j)Q_\perp, & i = j. \end{cases} \tag{7}
\]
For the model studied by Brodsky, Hiller, and McCartor [2], an explicit calculation of $F_1$ has been done [10]. In this model, a bare fermion acts as a source and sink for bosons of mass $\mu$. The lowest massive eigenstate is a fermion dressed by a boson cloud. The theory is regulated by a Pauli–Villars boson [11] with an imaginary coupling, and renormalized by fits of physical quantities to “data.” Because no spin-flip interactions are included, $F_2$ is zero. Results for $F_1$ are shown in Fig. 1. The large-momentum-transfer value of $F_1$ is the bare fermion probability and therefore is not zero.

**FIGURE 1.** The form factor $F_1$ for fixed longitudinal resolution $K = 9$ and transverse scale $L_\perp = 2\pi/\mu$, and for a particular set of model parameters. Various cutoffs $\Lambda^2$ are considered, with the transverse resolution $N_\perp$ ranging from 5 to 9.

### SCATTERING AMPLITUDES

The center-of-mass cross section for two-body scattering ($A+B \rightarrow C+D$) is [12]

$$
\frac{d\sigma}{d\Omega_{\text{cm}}} = \frac{1}{2E_A 2E_B v_{\text{rel}}} \frac{|\vec{p}_C| |M_{fi}|^2}{16\pi^2 E_{\text{cm}}},
$$

where $M_{fi}$ is the invariant amplitude obtained from the $S$ matrix

$$
S_{fi} = \langle f | i \rangle + (2\pi)^4 \delta^{(4)}(p_f - p_i) iM_{fi} = \delta_{CD,AB} - 2\pi i\delta(s_{AB} - s_{CD}) T_{LCfi},
$$

with $s_{AB} = \frac{m_A^2 + p_{A\perp}^2}{p_A^2/P_+} + \frac{m_B^2 + p_{B\perp}^2}{p_B^2/P_+}$. The $T$ matrix for scattering of composites is given by [4,13]

$$
T_{LCfi} = P^+ T_{fi} = \langle C | V_D^\dagger | s_{AB} + i\epsilon - H_{LC} | V_B | A \rangle + \langle C | D V_B | A \rangle.
$$
Here $|A\rangle$ and $|C\rangle$ are composite-particle eigenstates of the light-cone Hamiltonian $H_{LC}$, and the operator $V_B$ is defined by

$$V_B = [H_{LC}, B^\dagger] - \frac{m_B^2 + p_{B\perp}^2}{p_B^+/P^+} B^\dagger,$$

(11)

with $B^\dagger$ the creation operator for the $B$ particle, i.e. $|B\rangle = B^\dagger|0\rangle$. This construction generalizes one presented some time ago by Wick [13]. Details can be found in Ref. [4]. Given numerical solutions for the composite-particle eigenstates, obtained with DLCQ, the most difficult remaining task is the estimation of the matrix element of $(s+i\epsilon - H_{LC})^{-1}$. For this type of matrix element, the recursion method of Haydock [14] has worked well. A nonrelativistic application is described in [4]; an application to the field-theoretic model studied in [2] is in progress.

ACKNOWLEDGMENTS

This work was supported in part by the Minnesota Supercomputing Institute through grants of computing time and by the Department of Energy, contract DE-FG02-98ER41087.

REFERENCES