In order to facilitate the application of standard renormalization techniques, gravitation should be described, if possible, in pure connection formalism, as a Yang–Mills theory of a certain spacetime group, say the Poincaré or the affine group. This embodies the translational as well as the linear connection. However, the coframe is not the standard Yang–Mills type gauge field of the translations, since it lacks the inhomogeneous gradient term in the gauge transformations. By explicitly restoring the “hidden” piece responsible for this behavior within the framework of nonlinear realizations, the usual geometrical interpretation of the dynamical theory becomes possible, and in addition one can avoid the metric or coframe degeneracy which would otherwise interfere with the integrations within the path integral. We claim that nonlinear realizations provide a general mathematical scheme clarifying the foundations of gauge theories of spacetime symmetries. When applied to construct the Yang-Mills theory of the affine group, tetrads become identified with nonlinear translational connections; the anholonomic metric does not constitute any more an independent gravitational potential, since its degrees of freedom reveal to correspond to eliminable Goldstone bosons. This may be an important advantage for quantization.

PACS numbers: 04.50.+h, 11.15.-q, 12.25.+e.

I. INTRODUCTION

On a macroscopic scale, gravity is empirically rather well described by Einstein’s general relativity theory (GR). However, quantum-field theoretically, Einstein’s theory is perturbatively nonrenormalizable [26,31] and plagued by anomalies if coupled to fundamental matter as the Dirac field of an electron, cf. [41,30]. As a matter of fact, in the standard GR theory, gravity is conceived as an interaction of a very different nature as the remaining forces, being supposed to be mediated by the spacetime metric rather than by Yang–Mills connections. Thus, it is reasonable to hope that some of the problems related to the quantization of this force might disappear if one were able to describe gravitation as an ordinary gauge theory. The search for such a formulation yielded in particular the different gauge theories of gravity proposed by Hehl and his Cologne group [24,23,25]. In all of them, the local treatment of translations reveals itself as a corner-stone of the Yang–Mills type interpretation of gravitation. As Feynman [9] has put it, “...gravity is that field which corresponds to a gauge invariance with respect to displacement transformations”, cf. [40]. Starting from the Metric-Affine Gravity (MAG) theory [25], our aim is to re-examine it from the point of view of nonlinear realizations (NLRs). We think that such approach enlightens several internal features of the theory. The interpretation of a few specific points becomes modified, but the main features of the formalism elaborated in the work by Hehl et al. [25], in particular the field equations, remain unchanged under our alternative derivation.

A. Tetrads as nonlinear connections

In a first order formalism, one introduces a local frame field (or vielbein), $e_\alpha = e^\iota_\alpha \partial_\iota$, together with the coframe field or one–form basis $\vartheta^\beta = e^\iota_\beta dx^\iota$, which is dual to the frame $e_\alpha$ with respect to the interior product: $e_\alpha [\vartheta^\beta = \delta^\alpha_\beta$. Quite often, $\vartheta^\alpha$ is advocated as the translational gauge potential, although it does not transform inhomogeneously under local frame rotations, as is characteristic for a connection. The rigouros explanation of this apparent paradox...
II. GAUGING SPACETIME GROUPS

As far as internal symmetries are concerned, the definition of gauge transformations as fibre-preserving bundle automorphisms [1] constitutes a satisfactory characterization of them. Accordingly, given a principal fibre bundle \( P( M, H) \) with the base space \( M \) representing spacetime, a gauge transformation is identical with the action of the structure group \( H \) along local fibres, being the spacetime base manifold not affected by such transformations.

Obviously, this scheme is not applicable to gauge theories of gravitation. In fact, they are theories of the Yang–Mills type based on the gauging of spacetime groups; and precisely these groups are symmetries which affect spacetime itself. Thus, one has to generalize the definition of gauge transformations in order to take account of such external symmetries as well. We follow Lord [32–34], who suppresses the restriction of no action on the base space. According to him, a gauge transformation is a general bundle automorphism, that is, a diffeomorphism that maps fibres to fibres.

The natural framework to define such an action is the following, based on the manifold character of Lie groups and on the simple properties of left and right multiplications of group elements. Let us choose a spacetime group \( G \), and a subgroup \( H \subset G \). We will construct the gauge theory of \( G \) on the principal fibre bundle \( G( G/H, H) \), where the group manifold \( G \) itself is the bundle manifold, and the subgroup \( H \) is taken to be the structure group — different \( H \)'s may be chosen to play this role —; the quotient space \( G/H \) will play the role of spacetime. Gauge transformations are defined on this bundle as follows. As mentioned above, the usual definition of (active) gauge transformations

B. Spacetime metric as a Goldstone boson

The recent paper by Gronwald et al. [15] adopts the “quartet” of scalar fields introduced by Guendelman and Kaganovich [16–22], originally used in the study of the cosmological constant problem. Gronwald et al. apply these fields in a different context, in order to remedy, as far as possible, the insatisfactory fact of the occurrence of the (anholonomic) metric tensor in MAG as a field different from the Yang-Mills ones. As declared in the introduction by the authors themselves, they look for an alternative way to define a volume element without reference to the metric.

In fact the introduction, besides the linear connection and the coframe, of the metric tensor as an independent gravitational potential [25] seems to be contrary to the spirit of a pure gauge approach, where the role of gauge potentials is played exclusively by connections. In standard Yang–Mills theories, no other quantity is required to carry interactions. Contrarily, in MAG, the metric tensor appears as a quantity with ten additional degrees of freedom, foreign to the otherwise standard Yang–Mills treatment.

In our opinion, nonlinear realizations [27] provide a different interpretation of the actually formulated theory in its present form [25]. In fact, most of the features of MAG — such as the vector character of the coframe despite its nature of (translational) connection, as already mentioned; the freedom to fix the metric tensor to be Minkowskian without loss of generality, the tensoriality of the nonmetricity, etc. — are consequences of the nonlinearity. For instance, as we will see, the Goldstone nature of the MAG metric highlights from a particular nonlinear realization of the affine group. According to this interpretation, the metric results to be a set of ten fields, rearrangeable in the connections. When rearranged in this way, the metric reduces to a constant one, without any dynamical degrees of freedom. Thus, the anholonomic MAG metric tensor is not a genuine gravitational potential. Accordingly, the nonmetricity is not to be interpreted as the corresponding field strength but simply as the connection component associated to the symmetric generators of the general linear group. Under gauge transformations, nonmetricity behaves as a tensor due to the (not immediately recognizable) underlying nonlinear realization of the affine group.
as vertical automorphisms along fibres, not affecting spacetime, must be modified to more general automorphisms affecting both, vertical fibres and the points of the quotient space $G/H$ the latter are attached to. Since the left and right multiplications of elements of $G$ commute, we have in particular $L_g \circ R_h = R_h \circ L_g$, with $g \in G$, $h \in H$. Thus $L_g$, acting on fibres defined as orbits of the right action $R_h$ (that is, as left cosets $gH$), constitutes an automorphism of the kind we are looking for, transforming in general fibres into fibres. To be explicit, we define the left action $L_g$ of $G$ on zero sections $\sigma : G/H \rightarrow G$ as follows:

$$L_g \circ \sigma(\xi) = R_h \circ \sigma(\xi') ,$$  \hspace{1cm} (2.1)$$

As observed by Lord [33], this equation coincides with the prescription for nonlinear transformations due to Coleman et al. [5]. Thus, the NLR-procedure rests on a particular definition (2.1) of the action of a given group $G$. As a prerequisite, a subgroup $H \subset G$ is necessary in order to perform a partition of the group manifold of $G$ into equivalence classes, namely the coset spaces $gH$, playing the role of fibres attached to each point of $G/H$. The group action (2.1) moves from a fibre to another. In accordance with what one expects for spacetime symmetries, a transformation is induced on the quotient space $G/H$, reflecting the mapping from a fibre to other. Indeed, taking into account that $\pi \circ R_h \circ \sigma = \pi \circ \sigma = id$, from (2.1) then follows

$$\xi' = \pi \circ L_g \circ \sigma(\xi) ,$$  \hspace{1cm} (2.2)$$

being the fields $\xi$ coset fields, characterized as continuous labels of the elements of the quotient space $G/H$. In particular, for $G/H \approx R^4$, they provide the translational fields destined to replace the ‘quartet’ of scalar fields of Refs. [15,14]. Actually, in the context of gauge theories of spacetime groups, the ‘Poincaré coordinates’ [13] or components [4] of “Cartan’s radius vector” $\xi^a$ are in fact translational coset fields. In the case of MAG we are interested in, they turn out to transform as affine covectors resembling coordinates, see (3.5) and (4.3) below.

### A. Nonlinear connections

For practical calculational purposes, the fundamental Eq. (2.1) defining the nonlinear group action may be rewritten in a more explicit form in terms of $g \in G$ and $h \in H$ as

$$g \sigma(\xi) = \sigma(\xi') h (\xi, g) ,$$  \hspace{1cm} (2.3)$$

or shortly as $\sigma' = g \sigma h^{-1}$. Due to this particular transformation law of $\sigma$, from the linear connection $\Gamma_{\approx}$ of $G$ it becomes possible to define the nonlinear connection $\Gamma$ with suitable transformation properties as follows

$$\Gamma := \sigma^{-1} \left( d + \Gamma_{\approx} \right) \sigma .$$  \hspace{1cm} (2.4)$$

Indeed, given the ordinary linear transformation law of the linear connection, $\Gamma_{\approx}$, namely

$$\Gamma_{\approx} = g \Gamma_{\approx} g^{-1} + g d g^{-1} ,$$  \hspace{1cm} (2.5)$$

and the transformation (2.3) of $\sigma$ in its shorted form $\sigma' = g \sigma h^{-1}$, it follows that the nonlinear connection (2.4) transforms as

$$\Gamma' = h \Gamma h^{-1} + h d h^{-1}$$  \hspace{1cm} (2.6)$$

under local transformations. Observe that, according to (2.6), only the components of $\Gamma$ defined on the Lie algebra of $H \subset G$ transform inhomogeneously as true connections; the remaining components of $\Gamma$ transform as tensors with respect to $H$.

The nonlinear connection allows to construct covariant derivatives (of nonlinear fields) as follows. Consider a field $\varphi$ transforming linearly under $G$ as $\varphi' = g \varphi$, and let us schematically define a correlated nonlinear field as

$$D\psi := (d + \Gamma) \psi = \sigma^{-1} \left( d + \Gamma_{\approx} \right) \varphi ,$$  \hspace{1cm} (2.7)$$

behaving as an $H$–covariant object, namely

$$(D\psi)' = h D\psi ,$$  \hspace{1cm} (2.8)$$

under the left action of the whole group $G$. 

3
By applying the previous results, taking $G$ to be the affine group $A(4, R) := R^4 \otimes GL(4, R)$, i.e. the semidirect product of translations and general linear transformations, we will show how the nonlinear approach leads to MAG. The commutation relations of the affine group are:

$$
\begin{align*}
[P_\alpha, P_\beta] &= 0, \\
[L^\alpha_\beta, P_\gamma] &= \delta^\alpha_\gamma P_\beta, \\
[L^\alpha_\beta, L^\gamma_\delta] &= \delta^\alpha_\delta L^\gamma_\beta - \delta^\gamma_\delta L^\alpha_\beta.
\end{align*}
$$

(3.1)

Observe that the physical dimensions of the generators of the linear group are $[L^\alpha_\beta] = \hbar$, whereas those of translations are $[P_\alpha] = \hbar/\text{length}$.

Let us construct in two steps the fibre bundle descriptions of the spacetime dynamics of $G = A(4, R)$, which we denote $G(G/H_1, H_1)$ and $G(G/H_2, H_2)$, corresponding to two consecutive smaller subgroups as structure group, namely $H_1 = GL(4, R)$ and $H_2 = SO(1, 3) \subset H_1$ respectively. Other choices of $H$ are possible, for instance $H_3 = SO(3)$, see Ref. [37], but this will not be considered here. The occurrence of a certain subgroup $H$ (or $H_1$, $H_2$ etc.) on which the action of the total group $G$ becomes projected, is a constitutive feature of NLRs; it should not be confused with symmetry breaking. Indeed, in the nonlinear approach the symmetry is not broken, so that alternative choices of the subgroup $H$ are mathematically equivalent. True symmetry breaking requires an additional mechanism involving the ground state of a dynamical theory of fundamental physics. Since we do not propose here such a symmetry breaking mechanism for the original group $G$, the gauge theories we are going to construct are to be seen as different realizations of the unique underlying gauge theory of the whole gauge group $G$ (in our case, the affine group). The subgroups $H \subset G$ (in particular, $H =$Lorentz) do not play a singularized dynamical role. In fact, different choices of $H_1, H_2 \subset G$ are interchangeable, in the sense that one can go from a realization $G(G/H_1, H_1)$ to another $G(G/H_2, H_2)$, being both, simply, different rearrangements of the degrees of freedom of the same gauge theory, namely MAG. The possibility of dynamically singling out one and only one of the subgroups $H \subset G$, say the Lorentz group, by means of a symmetry breaking mechanism, remains to be studied elsewhere.

First we consider the gauge theory of the affine group with the general linear group $H_1 = GL(4, R)$ as structure group. We will show that the coframe appears in a natural way as a nonlinear translative connection.

III. GAUGE THEORETICAL ORIGIN OF THE TETRADS

Using the Campbell–Hausdorff formula in (2.3) with (3.2–3.4), the variation of the coset parameters $\xi^\alpha$ from (3.5) that the coset parameters $\xi^\alpha$ transform as affine covectors, as postulated [4] for Cartan’s generalized radius vector. The nonlinear connection (2.4) will be constructed in terms of the linear one, namely

$$
\Gamma := -i \Gamma^{(T)}_{\alpha} P_\alpha - i \Gamma^{(GL)}_{\beta} L^\alpha_\beta,
$$

(3.6)

which includes the linear translational potential $\Gamma^{(T)}_{\alpha}$ and the $GL(4, R)$ connection $\Gamma^{(GL)}_{\alpha}$, whose infinitesimal transformations read

$$
\delta \Gamma^{(GL)}_{\alpha} = D^\alpha \omega_\alpha, \quad \delta \Gamma^{(GL)}_{\beta} = D^\alpha \omega_\beta \Gamma^{(T)}_{\beta}.
$$

(3.7)
Here \( D \) denotes the covariant differential constructed from the \( GL(4, R) \) connection. Making use of the definition (2.4), we get

\[
\tilde{\Gamma} := \tilde{\sigma}^{-1} \left( d + \tilde{\Gamma} \right) \tilde{\sigma} = -i \tilde{\vartheta}^\mu P_\mu - i \tilde{\Gamma}_\alpha^\beta L^\alpha_{\beta}. \tag{3.8}
\]

with

\[
\tilde{\Gamma}_\alpha^\beta = (GL) \Gamma_\alpha^\beta, \quad \tilde{\vartheta}^\mu := (T) D \xi^\mu. \tag{3.9}
\]

As in the case of (3.3), we denote these objects with a tilde for later convenience. Making use of (2.6), it is straightforward to prove that, whereas \( \tilde{\Gamma} \) transforms as a \( GL(4, R) \) connection, the coframe \( \tilde{\vartheta}^\mu \) defined as in (3.9) transforms as a \( GL(4, R) \) covector. Explicitly

\[
\delta \tilde{\Gamma}_\alpha^\beta = \tilde{D}\omega_\alpha^\beta, \quad \delta \tilde{\vartheta}^\mu = -\omega_\beta^\mu \tilde{\vartheta}^\beta,
\]

compare with (3.7). The nonlinear treatment of the affine group thus clarifies how the coframe can be constructed from gauge fields of the Yang–Mills type, in particular those of (3.6). The coset parameters \( \xi^\alpha \) play the role of Cartan’s generalized radius vector of Ref. [43], being not introduced \( ad \) hoc, since they are constitutive elements of the theory. They mainly contribute to the construction of the translational invariant \( \tilde{\vartheta}^\mu = (T) D \xi^\mu \); the variation of \( \xi \) under translations, see (3.5), is compensated by the variation of the translative connection, see (3.7), cf. [43]. Since \( \xi = \xi^\alpha P_\alpha \) acquires its values in the coset space \( A(n, R)/GL(n, R) \approx R^n \), it can be regarded as an affine vector field (or “generalized Higgs field” according to Trautman [53]) which “hides” [49] the action of the local translational symmetry \( T(n, R) \). Accordingly, conditions like \( (T) D \xi^\alpha = 0 \) break the translational symmetry. Only in the absence of gravitational interaction we recover the specially relativistic relation \( \vartheta^\alpha = d\xi^\alpha \) for the coframes (i.e., for the translational nonlinear connections), employed in Ref. [15] in order to derive a “metric–free” volume four-form. It is interesting to notice that, in this limit, the fields \( \xi^\alpha \) play the role of ordinary coordinates, see also (3.5). In other words, the spacetime manifold of special relativity is a residual structure of the nonlinear approach when gravitational forces are switched off.

### IV. THE METRIC IN MAG

In order to complete the MAG scheme, the next considerations are devoted to show how the metric tensor can be introduced, related to a particular choice of the subgroup \( H \subset G \).

Let us consider the second choice of structure subgroup in our bundle approach mentioned above, namely \( G(G/H_2, H_2) \) with \( G = A(4, R) \), as before, and \( H_2 = SO(1, 3) \). We split up the generators \( L^\alpha_{\beta} \) of the general linear transformations as \( L^\alpha_{\beta} = L^\alpha_{\beta} + S^\alpha_{\beta} \), being \( L^\alpha_{\beta} \) the Lorentz generators and \( S^\alpha_{\beta} \) those of the symmetric linear transformations. Now we apply the general formula (2.3) with the particular factorization

\[
g = e^{i e^\mu P_\mu} e^{i \alpha^{\mu\nu} S_{\mu\nu}} e^{i (\vartheta^\mu L^\mu)} \quad \sigma := e^{-i \xi^\alpha P_\alpha} e^{i h^\mu S_{\mu\nu}} \quad h := e^{i (w^\mu \tilde{L}^\mu)}. \tag{4.1}
\]

Being \( e^\mu, \alpha^{\mu\nu} \) and \( \beta^{\mu\nu} \) infinitesimal parameters of the affine group, the transformed coset parameters of \( \sigma \) reduce to \( \xi^\alpha = \xi^\alpha + \delta \xi^\alpha \) and \( h^\mu = h^\mu + \delta h^\mu \); the Lorentz parameters \( w^\mu \) (being Lorentz the structure group \( H_2 \)) are also infinitesimal. Let us define

\[
r_{\alpha}^\beta := (e^h)_\alpha^\beta := \delta_{\alpha}^\beta + h_{\alpha}^\beta + \frac{1}{2!} h_{\alpha}^\gamma h_{\gamma}^\beta + \cdots \tag{4.2}
\]

from the coset parameters \( h_{\alpha}^\beta \) associated to the generators of the symmetric part of \( GL(4, R) \), see (4.1). (In the following, \( r_{\alpha}^\beta \) rather than the coset parameters \( h_{\alpha}^\beta \) themselves, will play the fundamental role [3,35,].) We find the variations

\[
\delta \xi^\alpha = -\left( \alpha_{\beta}^{\alpha} + \beta_{\alpha}^{\gamma} \right) \xi^\beta - \epsilon^\alpha, \quad \delta r_{\alpha}^\beta = \left( \alpha_{\beta}^{\alpha} + \beta_{\alpha}^{\gamma} \right) r_{\beta}^\gamma + u_{\beta}^\gamma r_{\gamma}^\alpha.
\]

where \( \alpha_{\beta}^{\alpha} + \beta_{\alpha}^{\gamma} = \omega_{\beta}^\alpha \), compare with (3.5). Since \( r_{\alpha}^\beta \) is symmetric, the antisymmetric part of the second equation in (4.3) vanishes. From this condition we find the explicit form of the nonlinear Lorentz parameter
\[ u^{\alpha\beta} = \beta^{\alpha\beta} - \alpha^{\mu\nu} \tanh \left( \frac{1}{2} \log \left[ r^{\alpha}_{\mu} (r^{-1})^{\beta}_{\nu} \right] \right) , \]  

which obviously differs from the linear Lorentz parameter \( \beta^{\alpha\beta} \). It is precisely the nonlinear \( u^{\alpha\beta} \), and not the linear \( \beta^{\alpha\beta} \), which is relevant for nonlinear transformations, as becomes evident in (4.8), (4.9) and (4.10) below.

In order to define the nonlinear connection, let us first rewrite the linear affine connection (3.6) as

\[ \Gamma := -i \Gamma^\alpha P_\alpha - i \Gamma^{(GL)}_\alpha \left( S^\alpha_\beta + \varrho^\alpha_{\beta} \right) , \]  

The symmetric part of \( \Gamma^{\alpha\beta} \) will correspond to nonmetricity. Making use of the definition (2.4), we get

\[ \Gamma := \sigma^{-1} \left( d + \frac{\varpi}{\sigma} \right) \sigma = -i \varpi^\alpha P_\alpha - i \varpi^{(GL)}_\alpha \left( S^\alpha_\beta + \varrho^\alpha_{\beta} \right) , \]  

with the nonlinear \( GL(4,R) \) connection \( \Gamma^{\alpha\beta} \) and the nonlinear translational connection \( \varrho^{\alpha\beta} \) respectively defined as

\[ \varpi^{(GL)}_\alpha := \left( r^{-1} \right)_\alpha \gamma \Gamma^\lambda \gamma^\beta - \left( r^{-1} \right)_\alpha \gamma d \gamma^\beta , \quad \varpi^{(T)} := r^{\beta\alpha} \left( \Gamma^{\beta} + \varpi^{(GL)}_\xi \right) , \]  

We identify the components of \( \Gamma^{\alpha\beta} \) and \( \varrho^{\alpha\beta} \) of the (nonlinear) Yang–Mills connections of the affine group with the geometrical linear connection and with the coframe respectively. Thus Eqs. (4.7) establish the correspondence between the geometrical objects in the l.h.s. and the dynamical objects in the r.h.s. We find that the whole connection, consisting of the sum of the antisymmetric (Lorentz) part and the symmetric (nonmetricity) part, behaves as a Lorentz connection

\[ \delta \Gamma^{\alpha\beta} = Du^{\alpha\beta} , \]  

according to (2.6), with the nonlinear Lorentz parameter (4.4). The transformation of the symmetric part of \( \Gamma^{\alpha\beta} \) lacks an inhomogeneous contribution, see (4.10) below; it becomes a tensor as a result of the nonlinear realization. Let us mention that, in addition, the covariant differential \( D \) in (4.8) is constructed in terms of the Lorentz connection itself. On the other hand, the coframe transforms as a Lorentz covector

\[ \delta \varrho^{\alpha\beta} = -u^{\beta\alpha} \varrho^{\beta} , \]  

which constitutes a main result of the nonlinear approach.

Notice that, in view of the splitting of the general linear generators into a Lorentz plus a symmetric part as \( L^{\alpha_\beta} = \tilde{L}^{\alpha_\beta} + S^{\alpha_\beta} \), the connection is actually composed of two parts, defined on different elements of the Lie algebra.

In fact, only the antisymmetric part, defined on the Lorentz generators, behaves as a true connection of the Lorentz group playing the role of the structure group \( H_2 \). As already mentioned, the symmetric part \( \Gamma_{(\alpha\beta)} := \frac{1}{2} \varpi^{(GL)}_{\alpha\beta} \), i.e. the nonmetricity, is tensorial. Actually

\[ \delta \varpi^{(GL)}_{\alpha\beta} = 2 u_{(\alpha} \varpi^{(T)}_{\beta)\gamma} . \]  

On the other hand, being the structure group \( H_2 \) the Lorentz group, the Minkowski metric \( o_{\alpha\beta} \) is automatically present in the theory as a natural invariant: \( \delta o_{\alpha\beta} = 0 \). Thus, a metric occurs in the affine theory due to the particular choice of a (pseudo-)orthogonal group as the structure group, so that the corresponding Cartan-Killing metric becomes apparent. However, no degrees of freedom are related to the Minkowski metric. This seemingly makes a difference between the dynamical content of our theory and that of ordinary MAG, since in the latter the metric tensor involves ten degrees of freedom. Nevertheless, we will see immediately how these degrees of freedom, being of Goldstone nature, can be taken from the nonlinear connections where they are hidden. Actually, the Goldstone fields which will manifest themselves as the degrees of freedom of the MAG–metric are those of the matrix \( r^{\alpha\beta} \) defined in (4.2). They can be factorized into the nonlinear connections and the coframes, as shown in (4.7), in the presence of the Minkowskian metric we are discussing, or alternatively they can be explicitly displayed in the metric tensor, as we will show in (4.13) below. In this case, the metric becomes identical with the ordinary MAG metric.

In order to show how the transition between these alternative formulations takes place, we establish the correspondence between the objects of both choices \( H_1 \) and \( H_2 \) studied above. Formally, we find that this correspondence is
isomorphic to a finite gauge transformation, with the matrix $r^{\alpha\beta}$ of (4.2) standing for the symmetric affine transformations. But $r^{\alpha\beta}$ is not a transformation matrix; it is constructed in terms of coset fields. The relation between (3.9) and (4.7) reads

$$\tilde{\Gamma}_\alpha^\beta := \Gamma_\alpha^\beta = r_\alpha^\gamma \Gamma_\gamma^\lambda (r^{-1})^\lambda_\beta - r_\alpha^\gamma d (r^{-1})^\beta_\gamma, \quad (4.11)$$

and

$$\tilde{\vartheta}^\alpha := (T)^\alpha + (GL)^\alpha = (r^{-1})^\beta_\alpha \vartheta^\beta. \quad (4.12)$$

The standard metric–affine objects of ordinary MAG, such as connections and coframes (up to the metric), are identical to those with tilde in the l.h.s. of (4.11) and (4.12), studied in Sec. III, corresponding to a nonlinear realization of the affine group with $H_1 = GL(4, R)$ as the structure group. In the approach studied in Sec. III, the metric tensor was absent. However, in analogy to (4.11) and (4.12), it can be introduced as an object with tilde related to the Minkowski metric $o_{\alpha\beta}$ which appears in the case of $H_2 = SO(1, 3)$ studied in Sec. IV. Actually, we define $\tilde{g}_{\alpha\beta}$ from $o_{\alpha\beta}$ as

$$\tilde{g}_{\alpha\beta} := r_\alpha^\mu r_\beta^\nu o_{\mu\nu}. \quad (4.13)$$

The resulting MAG-metric tensor plays the role of a Goldstone field, cf. [5], that drops out after applying the inverse of the "gauge transformation" (4.13). By also inverting (4.11) and (4.12), one reaches the nonlinear realization studied in Sec. IV, with the Lorentz group as the structure group. This completes the correspondence between the nonlinear objects and those of the framework of ordinary metric–affine theory. The virtue of eqs. (4.11–4.13) consists in that they show explicitly the mechanism by means of which the rearrangement of gauge-theoretical degrees of freedom (namely those related to the symmetric general linear transformations) takes place by "choosing tetrads to be orthonormal" in the context of MAG. As a consequence, observe that invariants like the line element may be alternatively expressed in terms of the Lorentz-nonlinear or metric–affine objects respectively, namely as

$$ds^2 = o_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta = \tilde{g}_{\alpha\beta} \tilde{\vartheta}^\alpha \otimes \tilde{\vartheta}^\beta, \quad (4.14)$$

where the transition from $o_{\alpha\beta}$ to $\tilde{g}_{\alpha\beta}$ or vice versa takes place by means of the suitable factorization of the coset parameters associated to the symmetric affine transformations. Thus, given the standard MAG theory, if one fixes the metric to be globally Minkowskian, the degrees of freedom of the theory automatically rearrange themselves into the nonlinear theory developed in Sec. IV, with the Lorentz group as the structure group. However, the theory remains the same.

Due to the transformation law (4.3) of $r^{\alpha\beta}$, which involves both, general linear and Lorentz parameters, the indices of objects with tilde behave as general linear indices; those of objects without tilde are Lorentz indices. In the second case, the ten degrees of freedom corresponding to $r^{\alpha\beta}$ are rearranged into the coframe and connections, so that none of them remains in the metric tensor, which becomes Minkowskian. An action which is invariant under affine transformations can be alternatively expressed in terms of $GL(4, R)$ or $SO(1, 3)$ tensors, respectively. This corresponds to the choice of variables with or without tilde, as discussed above. The field equations derived, by means of a variational principle, from such an affine invariant action, are the same as those derived in [25]. As already known by the authors of the latter reference, the field equation obtained by varying with respect to the MAG metric tensor is redundant, in accordance with the fact that the Goldstone-like degrees of freedom of such quantity can be rearranged into the remaining fields of the theory. Being the metric tensor eliminable as a fundamental gravitational potential, the interpretation of nonmetricity as the corresponding field strength can be put aside. In the nonlinear approach, it manifests itself simply as the (tensorial) symmetric part of the linear connection, so that action terms quadratic in nonmetricity should be seen as mass terms rather than as kinetic terms. This alternative conceptual point of view leaves the theory formally unchanged.

**V. OUTLOOK: DYNAMICAL ORIGIN OF THE SIGNATURE?**

In the present work, we were not concerned with the coupling of the gravitational gauge theory to matter. We refer the interested reader to Ref. [36], where the relation between multispinors and ordinary spinors was studied. Such relation extends the correspondence between different realizations of the affine group, displayed in eqs. (4.11–4.13), to matter fields. Although the details of the coupling to dilatation and shear currents were not worked out, in principle they could be obtained from the correspondence established there.
Concerning the signature of the metric parametrized via \( \alpha_{\alpha\beta} := \text{diag}(e^{i\theta}, 1, 1, 1) \), cf. [47], the nonlinear approach is particularly adapted for dealing with spontaneous symmetry breaking. In fact, the Higgs mechanism can be understood as the way to select a particular structure group \( H \) by fixing the Goldstone fields in terms of suitable fields of the theory, see Ref. [52]. Thus, symmetry breaking could give a fundamental physical meaning to a particular structure subgroup \( H \), fixing it dynamically.

Previously to the symmetry breaking, the choices of different structure groups \( H \) are physically equivalent in the sense that they simply provide alternative ways to rearrange the degrees of freedom of the total gauge group \( G \). In particular, in the gauge theory of the affine group, in the absence of symmetry breaking one can freely choose the structure subgroup \( H \) to be the Lorentz group or \( SO(4) \) etc., so that the corresponding metric signatures become the Minkowskian or the Euclidean one, respectively. They constitute alternative realizations of the same theory, since it is the symmetry under the total group \( G \) the only relevant one.

In quantum field theory, Minkowskian or Euclidean signature are, however, quite different. Usually in the path integral approach, Euclidean signature is chosen in order to have a well defined measure. Moreover, tunneling between different topologies of instanton configurations may occur. (After applying a Wick rotation to \( e^{i\pi} = -1 \), the physical measurable quantities are regained.) In the path integral approach to quantum gravity, a summation over all inequivalent coframes and connections, and even topology [38] is understood. This summation will also involve degenerate \((\det e_{\beta}^\alpha = 0)\) or even vanishing coframes, cf. [28]. Macroscopically, this would imply the breakdown of any length measurement performed by means of the metric (4.14). Microscopically, then also signature changes of the metric are to be admitted, cf. Refs. [8,6]. These conceptual difficulties [26] are not encountered in the quantization of internal Yang–Mills theories on a fixed spacetime background.

Degenerate coframes, however, tend to jeopardize the coupling of gravity to matter fields, as exemplified by Dirac or Rarita–Schwinger fields, cf. [42]. The basic reason is that the local frame \( e_{\alpha} \), even if it still exists, is not invertible any more; i.e. the relation \( e_{\alpha}|_{\theta} \beta = \delta_{\alpha}^\beta \), which is needed in the formulation of matter Lagrangians, would then be lost.

These arguments seems to require to introduce a symmetry fixing mechanism which dynamically differentiates a particular structure group \( H \), and thus the signature. In other words, it remains to be seen if also the signature of the physical spacetime has a dynamical origin in such a framework, as is suggested by Sakharov [50] and Greensite [12], or arises naturally in string or M-theory [54,7].

ACKNOWLEDGMENTS

We would like to thank Friedrich W. Hehl, Alfredo Macías, Yuri Obukhov and Alfredo Tiemblo for useful hints and comments. This work was partially supported by CONACyT, grant No. 28339–E, and the joint German–Mexican project DLR–Conacyt E130–1148 and MXI 010/98 OTH. One of us (E.W.M.) thanks Noelia Méndez Córdova for encouragement.


