Vacuum Structure and the Axion Walls in Gluodynamics and QCD with Light Quarks

Gregory Gabadadze\(^1\) and M. Shifman\(^2\)

\(^1\)Department of Physics, New York University, New York, NY 10003
\(^2\)Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455

Abstract

Large $N_c$ pure gluodynamics was shown to have a set of metastable vacua with the gluonic domain walls interpolating between them. The walls may separate the genuine vacuum from an excited one, or two excited vacua which are unstable at finite $N_c$. One may attempt to stabilize them by switching on the axion field. We study how the light quarks and the axion affect the structure of the domain walls. In pure gluodynamics (with the axion field) the axion walls acquire a very hard gluonic core. Thus, we deal with a wall “sandwich” which is stable at finite $N_c$. In the case of the minimal axion, the wall “sandwich” is in fact a “$2\pi$” wall, i.e., the corresponding field configuration interpolates between identical hadronic vacua. The same properties hold in QCD with three light quarks and very large $N_c$. However, in the realistic case of three-color QCD the phase corresponding to the axion field profile in the axion wall is screened by a dynamical phase associated with the $\eta'$, so that the gluon component of the wall is not excited. We propose a toy Lagrangian which models these properties and allows one to get exact solutions for the domain walls.

\(^*\)Address after September 1, 2000: Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455
1 Introduction

The early studies [1] of the chiral Ward identities in QCD revealed that the vacuum energy density depends on the vacuum angle $\theta$ through the ratio $\theta/N_f$, where $N_f$ is the number of quarks with mass $m_q \ll \Lambda$. Shortly after, Witten [2] and Di Vecchia and Veneziano [3] showed that this structure occurs naturally, provided that there exist $N_f$ states in the theory such that one of them is the true vacuum, while others are local extrema; all are intertwined in the process of “the $\theta$ evolution.” Namely, in passage from $\theta = 0$ to $\theta = 2\pi$, from $\theta = 2\pi$ to $\theta = 4\pi$, and so on, the roles of the above states interchange: one of the local extrema becomes the global minimum and vice versa. This would imply, with necessity, that at $\theta = k\pi$ (where $k$ is an odd integer) there are two degenerate vacuum states. Such a group of intertwined states will be referred to as the “vacuum family.” The crossover at $\theta = \pi, 3\pi, \text{etc.}$ is called the Dashen phenomenon [4].

This picture was confirmed by a detailed examination of effective chiral Lagrangians [2, 3, 5, 6] (for a recent update see [7]). For two and three light quarks with equal masses it was found that the vacuum family consists of two or three states respectively; one of them is a global minimum of the potential, while others are local extrema. At $\theta = \pi$ the levels intersect. Thus, Crewther’s dependence [1] on $\theta/N_f$ emerges.

On the other hand, the examination of the effective chiral Lagrangian with the realistic values of the quark masses, $m_d/m_u \sim 1.8$, $m_s/m_d \sim 20$, yields [2, 3, 7] a drastically different picture – the vacuum family disappears (shrinks to one state); the crossover phenomenon at $\theta = \pi$ is gone as well.

This issue remained in a dormant state for some time. Recently arguments were given that the “quasivacua” (i.e. local minima of the energy functional), which together with the true vacuum form a vacuum family, is an indispensable feature of gluodynamics. The first argument in favor of this picture derives [8] from supersymmetric gluodynamics, with supersymmetry softly broken by a gluino mass term. The same conclusion was reached in Ref. [9] based on a D-brane construction in the limit of large $N_c$. In fact, one can see that in both approaches the number of states in the vacuum family scales as $N_c$. Finally, an additional argument may be found in a cusp structure which develops once one sums up [10] subleading in $1/N_c$ terms in the effective $\eta'$ Lagrangian. At $N_c = \infty$ the states from the vacuum family are stable, and so are the domain walls interpolating between them [9, 11].

When $N_c < \infty$ the degeneracy and the vacuum stability is gone, strictly speaking. It is natural to ask what happens if one switches on the axion field. This generically leads to the formation of the axion domain walls. The axion domain wall [12] presents an excellent set-up for studying the properties of the QCD vacuum under the $\theta$ evolution. Indeed, inside the axion wall, the axion field (which, in

\footnote{We stress that the states from the vacuum family need not necessarily lie at the minima of the energy functional. As was shown by Smilga [7], at certain values of $\theta$ some may be maxima. Those which intersect at $\theta = k\pi$ ($k$ odd) are certainly the minima at least in the vicinity of $\theta = k\pi$.}
fact, coincides with an effective $\theta$) changes slowly from zero to $2\pi$. The characteristic length scale, determined by the inverse axion mass $m_a^{-1}$, is huge in the units of QCD, $\Lambda^{-1}$. Therefore, by visualizing a set of spatial slices parallel to the axion wall, separated by distances $\gg \Lambda^{-1}$, one obtains a chain of QCD laboratories with distinct values of $\theta_{\text{eff}}$ slowly varying from one slice to another. In the middle of the wall $\theta_{\text{eff}} = \pi$.

Intuitively, it seems clear that in the middle of the axion wall, the effective value of $\theta_{\text{eff}} = \pi$. Thus, in the central part of the wall the hadronic sector is effectively in the regime with two degenerate vacua, which entails a stable gluonic wall as a core of the axion wall. In fact, we deal here with an axion wall “sandwich.” Its core is the so-called D wall, see [13].

Below we will investigate this idea more thoroughly. We also address the question whether this phenomenon persists in the theory with light quarks, i.e., in real QCD. Certainly, in the limit $N_c = \infty$ the presence of quarks is unimportant, and the axion wall will continue to contain the D-wall core. As we lower the number of colors, however, below some critical number it is inevitable that the regime must change, the gluonic core must disappear as a result of the absence of the crossover. The parameter governing the change of the regimes is $\Lambda/N_c$ as compared to the quark mass $m_q$. At $m_q \ll \Lambda/N_c$, even if one forces the axion field to form a wall, effectively it is screened by a dynamical phase whose origin can be traced to the $\eta'$, so that in the central part of the axion wall the hadronic sector does not develop two degenerate vacua. The D walls cannot be accessed in this case via the axion wall.

A part of this paper is of a review character. We collect relevant assertions scattered in the literature. The main original results – the occurrence of the D-wall core inside the axion wall in pure gluodynamics and in QCD with $\Lambda \gg m_q \gg \Lambda/N_c$ – are presented in Secs. 4 – 7.

Recently, the issue of hadronic components of the axion wall in the context of a potential with cusps [10] was addressed in [14, 15, 16]. However, the gluonic component of the axion walls was not studied. The $\eta'$ component in the axion walls was considered in [17, 14]. As far as we understand, in actuality the $\eta'$ component is totally unstable, and cannot be discussed in the static regime.

## 2 Invisible Axion and Axion Walls

In this section we briefly review the axion set-up, mainly with the purpose of setting the relevant notation.

The axion was originally introduced by Weinberg [18] and Wilczek [19] to solve the strong $CP$ problem which arises in QCD if physics at very short distances (say, of order the Planck scale) generates a non-vanishing $\theta$ term. In the original version the axion was coupled directly to the light $u, d$ and $s$ quarks.

Shortly after, it became clear that the original construction of Weinberg and
Wilczek is not viable from the phenomenological standpoint and the axion mechanism was further developed: “invisible axions” were introduced. The concrete version we will keep in mind is the KSVZ axion \([20, 21]\) (although other versions can be considered too \([22]\)). One introduces a complex scalar field \(\phi\) coupled to a hypothetical quark field \(Q\) in the fundamental representation of the color SU(3), with no weak interactions,

\[
\Delta \mathcal{L} = \phi \bar{Q}_R Q_L + \text{H.c.}.
\]  

(1)

The modulus of \(\phi\) is assumed to develop a large vacuum expectation value \(f/\sqrt{2}\), while the argument of \(\phi\) becomes the axion field \(a\), modulo normalization,

\[
a(x) = f \alpha(x), \quad \alpha(x) \equiv \text{Arg}\phi(x), \quad f \gg \Lambda.
\]  

(2)

Then the low-energy coupling of the axion to the gluon field is

\[
\Delta \mathcal{L} = \frac{1}{f} a \frac{g^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu},
\]  

(3)

so that the QCD Lagrangian depends on the combination \(\theta + \alpha(x)\).

In general, one could introduce more than one fundamental field \(Q\), or introduce them in a higher representation of the color group. Then, the axion-gluon coupling (3) acquires an integer multiplier \(N\),

\[
\Delta \mathcal{L}' = \frac{1}{f} a N \frac{g^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}.
\]  

(4)

This \(N\) is sometimes referred to as the axion index, not to be confused with \(N\) of extended supersymmetry, nor with \(N_c\), the number of colors. The minimal axion corresponds to \(N = 1\). In the general case the QCD Lagrangian depends on the combination \(\theta + N\alpha(x)\). The phenomenon of formation of the axion domain walls is being discussed in the literature for a long time \([12]\). The character of the axion walls depends on \(N\). For \(N = 1\) there is no physical vacuum degeneracy (except at \(\theta = \pi\)). Since the wall interpolates between the vacuum and its “\(2\pi\) copy” it can be bounded by a closed axion string (see Ref. \([23]\) for a review). Thus, such a wall can have a finite longitudinal extent. This wall is classically unstable as it shrinks its size down by emitting axions. Moreover, the \(N = 1\) axion walls are, strictly speaking, unstable even if they have an infinite extent. They can decay quantum mechanically. The decay process is due to tunneling between the identical vacua separated by a barrier. In fact, a hole can be created in the wall – a domain where the modulus of the field \(\phi \equiv f/\sqrt{2}\) vanishes, and its phase can be “unwind.” This hole then expands to infinity removing the wall completely. Numerically this process is extremely suppressed due to the fact that \(f\) is very large in the vacuum, and suppressing \(|\phi|\) to make a hole in the wall costs a lot of energy. The suppression factor for tunneling was estimated \([12]\) to be \(\sim \exp\left\{(-\text{const} \ f^2 m_\pi^{-2}) \ln(f^2 m_\pi^{-2})\right\}\). Thus, the infinite-extent wall can be considered stable for all practical purposes.
If \( N \geq 2 \), there is a residual vacuum degeneracy of the \( Z_N \) type; the walls connecting distinct vacua must have infinite area and must be perfectly topologically stable (they are cosmologically unacceptable, since they would over-close the Universe [24]).

Since we have little to add on the process of the wall formation in the early universe, for our purposes – consideration of the walls in the static environment – the distinction between \( N = 1 \) and \( N \geq 2 \) is unimportant. For simplicity we will deal with the \( N = 1 \) axions. All formulae are readily adjustable for \( N = 2 \) and higher.

## 3 Two Scenarios (A Signature of the Hadronic Core)

The invisible axion is very light. Integrating out all other degrees of freedom and studying the low-energy axion effective Lagrangian must be a good approximation. The axion effective potential in QCD can be of two distinct types.

Assuming that for all values of \( \theta \) the QCD vacuum is unique one arrives at the axion effective Lagrangian of the form

\[
\mathcal{L}_a = f^2 \left[ \frac{1}{2} (\partial_\mu \alpha)^2 + m_a^2 (\cos(\alpha + \theta) - 1) \right].
\]  

(5)

The axion potential does not have to be (and generically is not) a pure cosine; it may have higher harmonics. In the general case it is a smooth periodic function of \( \alpha + \theta \), with the period \( 2\pi \). For illustration we presented the potential as a pure cosine. This does not change the overall picture in the qualitative aspect.

As we will see below, a smooth effective potential of the type (5) emerges even if the (hadronic) vacuum family is non-trivial, but the transition between the distinct hadronic vacua does not occur inside the axion wall. This is the case with very light quarks, \( m_q \ll \Lambda/N_c \). In the opposite limit, one arrives at the axion potential with cusps, considered below.

In the theory (5) one finds the axion walls interpolating between the vacuum state at \( \alpha = -\theta \) and the same vacuum state at \( \alpha = -\theta + 2\pi \),

\[
\alpha(z) + \theta = 4 \arctan(e^{m_a z}),
\]  

(6)

where the wall is assumed to lie in the \( xy \) plane, so that the wall profile depends only on \( z \). This is the most primitive “\( 2\pi \) wall.”

The tension of this wall is obviously of the order of

\[
T_1 \sim f^2 m_a.
\]  

(7)

Taking into account that \( f^2 m_a^2 \sim \chi \) where \( \chi \) is the topological susceptibility of the QCD vacuum, we get

\[
T_1 \sim \chi/m_a.
\]  

(8)
The inverse proportionality to $m_a$ is due to the fact that the transverse size of the axion wall is very large.

Let us now discuss the axion effective potential of the second type. In this case the potential has cusps, as is the case in pure gluodynamics, where the axion effective Lagrangian is of the form

$$\mathcal{L}_a = \frac{f^2}{2} (\partial \phi)^2 + \min_{\ell} \left\{ N_c^2 \Lambda^4 \cos \frac{\alpha + 2\pi \ell}{N_c} \right\},$$

(for a more detailed discussion see below). Here the $\theta$ angle was absorbed in the definition of the axion field. The axion wall interpolates between $\alpha = 0$ and $\alpha = 2\pi$.

What is the origin of this cusp? The cusps reflect a restructuring in the hadronic sector. When one (adiabatically) interpolates in $\alpha$ from 0 to $2\pi$ a gluonic order parameter, for instance $\langle \tilde{G}G \rangle$, necessarily experiences a restructuring in the middle of the wall corresponding to the restructuring of heavy gluonic degrees of freedom. In other words, one jumps from the hadronic vacuum which initially (at $\alpha = 0$) had $\langle \tilde{G}G \rangle = 0$ into the vacuum in which initially $\langle \tilde{G}G \rangle \neq 0$. Upon arrival to $\alpha = 2\pi$, we find $\langle \tilde{G}G \rangle = 0$ again. This implies that the central part of such an axion wall is dominated by a gluonic wall. Thus, the cusp at $\alpha = \pi$ generically indicates the formation of a hadronic core, the D wall [13] in the case at hand.

Returning to the question of the tension we note that

$$\chi \sim \Lambda^4 N_c^0, \quad m_a \sim \Lambda^2 N_c^0 f^{-1} \text{ in pure gluodynamics},$$
$$\chi \sim \Lambda^3 N_c m_q, \quad m_a \sim \Lambda^{3/2} m_q^{1/2} N_c^{1/2} f^{-1} \text{ in QCD with light quarks},$$

which implies, in turn,

$$T_1 \sim \begin{cases} f \Lambda^2 N_c^0 & \text{in pure gluodynamics} \\ f \Lambda^{3/2} m_q^{1/2} N_c^{1/2} & \text{in QCD with light quarks} \end{cases}$$

Here $m_q$ is the light quark mass.

The presence of the large parameter $f$ in $T_1$ makes the axion halo the dominant contributor to the wall tension. The contribution of the hadronic component contains only hadronic parameters, although it may have a stronger dependence on $N_c$. Examining the cusp with an appropriately high resolution one would observe that it is smoothed on the hadronic scale, where the hadronic component of the axion wall “sandwich” would become visible. The cusp carries a finite contribution to the wall tension which cannot be calculated in the low-energy approximation [25]. To this end one needs to consider the hadronic core explicitly. The tension of the core $T_{\text{core}} \sim \Lambda^3 N_c$, while the tension of the axion halo $T_{\text{halo}} \sim f \Lambda^2$ (in pure gluodynamics).

We pause here to make a comment on the literature. The consideration of the axion walls in conjunction with hadrons dates back to the work of Huang and Sikivie, see Ref. [12]. This work treats the Weinberg-Wilczek $N = 2$ axion in QCD with
two light flavors, which is replaced by a chiral Lagrangian for the pions, to the leading order (quadratic in derivatives and linear in the light quark masses). It is well-known [2, 3, 7] that in this theory the crossover phenomenon takes place at \( m_u = m_d \). In the realistic situation, \( (m_d - m_u)/(m_d + m_u) \sim 0.3 \) considered in Ref. [12], there is no crossover. The pions can be integrated over, leaving one with an effective Lagrangian for the axion of the type (5) (with \( \alpha \to 2\alpha \)). The potential is not pure cosine, higher harmonics occur too. The axion halo exhausts the wall, there is no hadronic core in this case.

At the same time, Huang and Sikivie (see Ref. [12]) found an explicit solution for the “\( \pi^0 \)” component of the wall. In fact, this is an illusion. The Huang-Sikivie (HS) solution refers to the bare \( \pi^0 \) field. To find the physical \( \pi^0 \) field one must diagonalize the mass matrix at every given value of \( \alpha \) (the bare \( f\alpha \) is the physical axion field up to small corrections \( \sim f_\pi^2/f^2 \) where \( f_\pi \) stands for the pion decay constant). Once this is done, one observes that the physical pion field, which is a combination of the bare pion and \( f\alpha \), is not excited in the HS solution. The equation (2.16) in the HS paper is exactly the condition of vanishing of the physical pion in the wall profile. This explains why the wall thickness in the HS work is of order \( m_\alpha^{-1} \), with no traces of the \( m_\pi^{-1} \) component. The crossover of the hadronic vacua at \( \alpha = \pi/2 \) (remember, this is \( N = 2 \) model) could be recovered in the Huang-Sikivie analysis at \( m_u = m_d \).

However, the chiral pion Lagrangian predicts in the two-quark case the vanishing of the pion mass in the middle of the wall, for accidental reasons. This is explained in detail by A. Smilga, Ref. [7].

### 4 Vacuum Structure in Gluodynamics with Invisible Axion

First we will summarize arguments in favor of the existence of a nontrivial vacuum family in pure gluodynamics.

The first indication that the crossover phenomenon may exist in gluodynamics comes [8] from supersymmetric Yang-Mills theory, with supersymmetry being broken by a gluino mass term. The same conclusion was reached in Ref. [9] based on a D-brane construction in the limit of large \( N_c \). In both approaches the number of states in the vacuum family is \( N_c \).

The Lagrangian of softly broken supersymmetric gluodynamics is

\[
L = \frac{1}{g^2} \left\{ -\frac{1}{4} G_{\mu \nu}^a G_{\mu \nu}^a + i \bar{\lambda}_a^\alpha D^\alpha \chi^a_\alpha - (m\lambda_a^\alpha \lambda_a^{\alpha} + \text{H.c.}) \right\} \\
+ \theta \frac{1}{32\pi^2} G_{\mu \nu}^a \tilde{G}_{\mu \nu}^a ,
\]

where \( m \) is the gluino mass which is assumed to be small, \( m \ll \Lambda \).

There are \( N_c \) distinct chirally asymmetric vacua, which (in the \( m = 0 \) limit) are
labeled by
\[
\langle \lambda^2 \rangle_\ell = N_c \Lambda^3 \exp \left( i \frac{\theta + 2\pi \ell}{N_c} \right), \quad \ell = 0, 1, \ldots, N_c - 1.
\] (13)

At \( m = 0 \) there are stable domain walls interpolating between them [26]. Setting \( m \neq 0 \) we eliminate the vacuum degeneracy. To first order in \( m \) the vacuum energy density in this theory is
\[
\mathcal{E} = \frac{m}{g^2} \langle \lambda^2 \rangle + \text{H.c.} = -m N_c^2 \Lambda^3 \cos \frac{\theta + 2\pi \ell}{N_c}.
\] (14)

Degeneracy of the vacua is gone. As a result, all the metastable vacua will decay very quickly. Domain walls between them, will be moving toward infinity because of the finite energy gradient between two adjacent vacua. Eventually one ends up with a single true vacuum state in the whole space.

For each given value of \( \theta \) the ground state energy is given by
\[
\mathcal{E}(\theta) = \min_\ell \left\{ -m N_c^2 \Lambda^3 \cos \frac{\theta + 2\pi \ell}{N_c} \right\}.
\] (15)

At \( \theta = \pi, 3\pi, \ldots \), we observe the vacuum degeneracy and the crossover phenomenon. If there is no phase transition in \( m \), this structure will survive, qualitatively, even at large \( m \) when the gluinos disappear from the spectrum, and we recover pure gluodynamics.

Based on a D-brane construction Witten showed [9] that in pure SU(\( N_c \)) (non-supersymmetric) gluodynamics in the limit \( N_c \to \infty \) a vacuum family does exist:\(^2\) the theory has an infinite group of states (one is the true vacuum, others are non-degenerate metastable “vacua”) which are intertwined as \( \theta \) changes by \( 2\pi \times \) (integer), with a crossover at \( \theta = \pi \times \) (odd integer). The energy density of the \( k \)-th state from the family is
\[
\mathcal{E}_k(\theta) = N_c^2 \Lambda^4 F \left( \frac{\theta + 2\pi k}{N_c} \right),
\] (16)
where \( F \) is some \( 2\pi \)-periodic function, and the truly stable vacuum for each \( \theta \) is obtained by minimizing \( \mathcal{E}_k \) with respect to \( k \),
\[
\mathcal{E}(\theta) = N_c^2 \Lambda^4 \min_k F \left( \frac{\theta + 2\pi k}{N_c} \right),
\] (17)

\(^2\)This was shown in Ref. [9] assuming that there is no phase transition in a certain parameter of the corresponding D-brane construction. In terms of gauge theory, this assumption amounts of saying that there is no phase transition as one interpolates to the strong coupling constant regime. Thus, the arguments of [9] have the same disadvantage as those of SUSY gluodynamics where one had to assume the absence of the phase transition in the gluino mass.
much in the same way as in Eq. (15).

At very large $N_c$ Eq. (17) takes the form

$$\mathcal{E}(\theta) = \Lambda^4 \min_k (\theta + 2\pi k)^2 + \mathcal{O}\left(\frac{1}{N_c}\right).$$

(18)

The energy density $\mathcal{E}(\theta)$ has its absolute minimum at $\theta = 0$. At $N_c = \infty$ the “vacua” belonging to the vacuum family are stable but non-degenerate. To see that the lifetime of the metastable “vacuum” goes to infinity in the large $N_c$ limit one can consider the domain walls which separate these vacua [11, 27]. These walls are seen as wrapped D branes in the construction of [9], and they indeed resemble many properties of the QCD D branes on which a QCD string could end. We refer to them as D walls because of their striking similarity to D2 branes. The consideration of the D walls has been carried out [11] and leads to the conclusion that the lifetimes of the quasivacua go to infinity as $\exp(\text{const} N_c^4)$.

Moreover, it was argued [28, 13] that the width of these wall scales as $1/N_c$ both, in SUSY and pure gluodynamics. To reconcile this observation with the fact that masses of the glueball mesons scale as $N_c^0$, we argued [13] that there should exist heavy (glue) states with masses $\propto N_c$ out of which the walls are built. The D-brane analysis [29], effective Lagrangian arguments and analysis of the wall junctions [30], support this interpretation. These heavy states resemble properties of the D0 branes. The analogy is striking, as the D0 branes make the D2 branes from the standpoint of the M(atrix) theory [31], so these QCD “zero-branes” make the QCD D2 branes (i.e. domain walls).\(^3\) The distinct vacua from the vacuum family differ from each other by a restructuring of these heavy degrees of freedom. They are essentially decoupled from the glueballs in the large $N_c$ limit.

Now we switch on the axion

$$\Delta \mathcal{L} = \frac{1}{2} f^2 (\partial_\mu \alpha) (\partial^\mu \alpha) + \frac{1}{32\pi^2} C^a_{\mu\nu} \tilde{G}^a_{\mu\nu},$$

(19)

with the purpose of studying the axion walls. The potential energy $\mathcal{E}(\theta)$ in Eq. (17) or (18) is replaced by $\mathcal{E}(\theta + \alpha)$.

Since the hadronic sector exhibits a nontrivial vacuum family and the crossover\(^4\) at $\theta = \pi, 3\pi$, etc., strictly speaking, it is impossible to integrate out completely the hadronic degrees of freedom in studying the axion walls. If we want to resolve the cusp, near the cusp we have to deal with the axion field plus those hadronic degrees of freedom which restructure. In the middle of the wall, at $\alpha = \pi$, it is mandatory to jump from one hadronic vacuum to another – only then the energy of the overall field configuration will be minimized and the wall be stable. Thus, in gluodynamics the axion wall acquires a D-wall core by necessity.

One can still integrate out the heavy degrees of freedom everywhere except a narrow strip (of a hadronic size) near the middle of the wall. Assume for simplicity

\(^3\)See also closely related discussions in Ref. [32].

\(^4\)For nonminimal axions, with $N \geq 2$, the crossover occurs at $\alpha = k\pi/N$. 

9
that there are two states in the hadronic family. Then the low-energy effective
Lagrangian for the axion field takes the form (9). The domain wall profile will also
exhibit a cusp in the second derivative. The wall solution takes the form
\[
\alpha(z) = \begin{cases} 
8 \arctan \left( e^{m_a z} \tan \frac{\pi}{8} \right), & \text{at } z < 0 \\
-2\pi + 8 \arctan \left( e^{m_a z} \tan \frac{3\pi}{8} \right), & \text{at } z > 0,
\end{cases}
\] (20)
where the wall center is at \( z = 0 \).

Examining this cusp with an appropriately high resolution one would observe
that it is smoothed on the hadronic scale, where the hadronic component of the
axion wall “sandwich” would become visible. The cusp carries a finite contribution
to the wall tension which cannot be calculated in the low-energy approximation but
can be readily estimated, \( T_{\text{core}} \sim \Lambda^3 N_c \).

Below we will examine this core manifestly in a toy solvable model. Before doing
so, however, we want to elucidate the issue of the peculiar \( N_c \) dependence (or, better
to say, its absence), in Eq. (18).

5 Description in Terms of a Three-Index Field

The expression for the vacuum energy density (18) seems somewhat puzzling from
the point of view of the gluon Lagrangian. Indeed, there are \( N_c^2 - 1 \) degrees of
freedom in gluodynamics. Therefore, naively, one expects that the vacuum energy
density in the large \( N_c \) limit scales as \( \sim N_c^2 \). However, the leading term in Eq. (18)
scales as \( N_c^0 \). As a possible explanation, one could think of a \textit{colorless massless}
excitation which would give rise to the energy density (18). However, there are no
\textit{physical massless} states in gluodynamics.

The explanation to this apparent puzzle might come if one introduces a colorless
composite three-index field which does not propagate any \textit{physical} degrees of freedom
[33, 34]. On the other hand, this field gives rise [27] to precisely the vacuum energy
(18). In a sense, this field is similar to the photon in (1+1)-dimensional QED, where
a vector particle has no physical degrees of freedom, but it can create a constant
electric field background which produces a nonzero energy.

The three-index field in gluodynamics is defined as follows:
\[
\frac{g^2}{32\pi^2} G^a_{\mu\nu} \hat{G}^a_{\mu\nu} = \frac{\varepsilon^{\mu\nu\alpha\beta} H_{\mu\nu\alpha\beta}}{4!} = \frac{\varepsilon^{\mu\nu\alpha\beta} \partial_{\mu} C_{\nu\alpha\beta}}{4!},
\] (21)
where \( H_{\mu\nu\alpha\beta} \) is the field strength for the potential \( C_{\mu\nu\alpha} \), and the square brackets
denote antisymmetrisation over all indices. Hence, the \( C_{\mu\nu\alpha} \) field can be expressed
through the gluon fields \( A^a_{\mu} \) as follows:
\[
C_{\mu\nu\alpha} = \frac{1}{16\pi^2} (A^a_{\mu} \tilde{\nabla}_\nu A^a_{\alpha} - A^a_{\nu} \tilde{\nabla}_\mu A^a_{\alpha} - A^a_{\alpha} \tilde{\nabla}_\mu A^a_{\nu} + 2 f_{abc} A^a_{\mu} A^b_{\nu} A^c_{\alpha}).
\] (22)
Here $f_{abc}$ denote the structure constants of the corresponding gauge group. The derivative in this expression is defined as $A \partial B \equiv A(\partial B) - (\partial A)B$. Note that the $C_{\nu\alpha\beta}$ field is not a gauge invariant quantity. If the gauge transformation parameter is denoted as $\Lambda^\alpha$, the three-index field transforms as

$$C_{\nu\alpha\beta} \rightarrow C_{\nu\alpha\beta} + \partial_\nu \Lambda_{\alpha\beta} - \partial_\alpha \Lambda_{\nu\beta} - \partial_\beta \Lambda_{\alpha\nu},$$

(23)

where $\Lambda_{\alpha\beta} \propto A^a_\alpha \partial_\beta \Lambda^a - A^a_\beta \partial_\alpha \Lambda^a$. However, the expression for the field strength $H_{\mu\nu\alpha\beta}$ is gauge invariant.

At energies below $\Lambda$, all massive glueballs decouple from the effective Lagrangian of gluodynamics. Thus, no physical excitations are left. However, there should exist a kinetic term for the $C$ field in the low-energy Lagrangian [27]. This is related to the fact that the correlator of the vacuum topological susceptibility $\chi$ at zero momentum is non-zero in gluodynamics. Neglecting all higher derivative terms and also terms suppressed in the large $N_c$ limit one arrives at the effective Lagrangian for the $C$ field of the form

$$-\frac{1}{2 \times 4!} \frac{H^2_{\mu\nu\alpha\beta}}{\chi} + \theta \frac{\varepsilon^{\mu\nu\alpha\beta} H_{\mu\nu\alpha\beta}}{4!} + O\left(\frac{\partial^2}{\Lambda^2}, \frac{1}{N_c^2}\right).$$

(24)

The first term in this expression reproduces the proper correlation function for the topological susceptibility. The second contribution is just the $\theta$ term. Once this Lagrangian is set, it is easy to show that the classical equations of motion have a constant solution

$$H_{\mu\nu\alpha\beta} = -\chi \left(\theta + 2\pi k\right) \varepsilon_{\mu\nu\alpha\beta},$$

(25)

which reproduces the correct large $N_c$ expression for the energy density (18). Note that this solution persists even if higher derivatives are included in (24). Moreover, since the $H$ field does not propagate the dynamical degrees of freedom, the large $N_c$ classical solution (25) is also exact quantum-mechanically.

In this approach, the multiple structure in (18) is related to the quantization of the topological charge [27]. This provides an explanation for the expression (18) from the point of view of gluodynamics.

The three-index field $C$ can naturally couple to a D wall. The corresponding charge of the D wall is related to the instanton number in gluodynamics [27]. Thus, the D walls are the sources of a constant “electric” field (25) which produces the vacuum energy density (18).

Let us now discuss the mixing of the three-index field with the axion, after the latter is switched on. At low energies, when all glueballs are decoupled, two new terms emerge in the effective Lagrangian,

$$\frac{1}{2} \left(\partial_\mu a\right)^2 + \frac{a}{f} \frac{\varepsilon^{\mu\nu\alpha\beta} \partial_\mu C_{\nu\alpha\beta}}{3!}.$$  

(26)

11
It is known that the pseudoscalar field in four-dimensions is dual to a two-index antisymmetric gauge field, $B_{\mu\nu}$ [35, 36]. That is to say, the axion Lagrangian (26) can be rewritten in terms of a two-index field. The topological charge density, to which the axion is coupled in (26), is rewritten in terms of a three-index field $C_{\mu\nu\alpha}$ (21). It is intriguing to understand what happens with these three- and two-index fields after they are coupled to each other (see also a related discussions in [37]).

We can rewrite (26) in the following equivalent form:

$$-\frac{1}{2}\rho_{\mu}^2 + \rho_{\mu}\partial^\mu a + \frac{a}{f} \frac{\varepsilon^{\mu\nu\alpha\beta} \partial_\mu C_{\nu\alpha\beta}}{3!}.$$  \hspace{1cm} (27)

Here we have introduced an auxiliary field $\rho_{\mu}$. Equations (26) and (27) are equivalent – to see this one integrates out $\rho_{\mu}$ and substitutes the result $\rho_{\mu} = \partial_\mu a$ into (27).

On the other hand, we could first integrate over the axion field in (27). This gives rise to the following relation:

$$\rho_{\mu} = \frac{1}{f} \frac{\varepsilon_{\mu\nu\alpha\beta} C^{\nu\alpha\beta}}{3!} + \partial^{[\nu} B^{\alpha\beta]}.$$ \hspace{1cm} (28)

where we have introduced an antisymmetric two-index field $B_{\alpha\beta}$. Using the relation (28), we find that (27) (or equivalently (26)) is proportional to

$$\frac{1}{f^2} \left( C_{\nu\alpha\beta} + \partial^{[\nu} B_{\alpha\beta]} \right)^2.$$ \hspace{1cm} (29)

The sum in the parenthesis is gauge invariant. In fact, it is invariant under both, the Abelian transformations on the $B$ field, and the non-Abelian transformations of gluons. As we mentioned earlier, this latter transformation does not leave $C$ invariant. The invariance in (29) is restored, however, due to compensating transformations of the $B$ field [38].

At low energies, when all glueballs are decoupled, the expression (29) should be combined with the $\theta$ term and the gauge invariant kinetic term for $C$ given in (24). As a result, the expression (29) is nothing but the gauge invariant mass term for the three-index field which is a superposition of $C$ and $B$ fields. In other words, a mixed state of the $C$ field and the $B$ field produces a state with the mass

$$m_a^2 = \frac{\chi}{f^2}.$$ \hspace{1cm} (30)

This is the physical axion (similar results were first obtained in a different context in Ref. [37] by studying correlation functions of the three-index field. This is equivalent to the effective Lagrangian approach adopted here).

Summarizing, we started from gludynamics where the $C$ field had no physical components. The D walls were the sources of the $C$ field. After the axion (represented by $B$) is switched on, the $C$ field and the bare axion mix. The mixed
three-index field becomes massive and propagates one massive physical degree of freedom, the physical axion.

The direct physical consequence of this phenomenon is that the three-form charge of a D wall in a theory with the axion is screened. As a result, there will be a stationary and stable wall in the theory – a superposition of the axion and D wall in its core. In the next section we will explicitly find this domain wall “sandwich” in a toy model.

6 An Illustrative Model

To quantitatively describe the axion walls with the D wall core one has to solve QCD, which is way beyond our possibilities. Our task is more modest. We would like to obtain a qualitative description of the axion wall sandwich which, with luck, can become semi-quantitative. To this end we want to develop toy models. An obvious requirement to any toy model is that it must qualitatively reproduce the basic features of the vacuum structure which we expect in QCD. In SUSY gluodynamics it was possible to write down a toy model with a $\mathbb{Z}_{N_c}$ symmetry [39] which “integrates in” the heavy degrees of freedom and allows one to investigate the BPS domain walls in the large $N_c$ limit [28] (see also [40]). We will suggest a similar model in (non-supersymmetric) QCD, then switch on axions, and study the axion domain walls in a semi-realistic setting. In this model we will be able to find exact solutions for the D walls and the axion walls.

Here is our a simple toy model which has a proper vacuum structure. If an appropriate (complex) glue order parameter is denoted by $\Phi$, the modulus and phase of this field describe the $0^{++}$ and $0^{-+}$ channels of the theory, respectively. The toy model Lagrangian is

$$\mathcal{L} = \frac{N_c^2}{2} (\partial_{\mu} \Phi)^* (\partial_{\mu} \Phi) - V(\Phi, \Phi^*) , \quad V = V_0 + V_1 ,$$

$$V_0 = \frac{N_c^2 A^2}{2} |1 - \Phi^{N_c e^{-i\theta}}|^2 ,$$

$$V_1 = \left\{ -\frac{\chi N_c^2}{2} \Phi \left[1 + \frac{1}{N_c} (1 - \Phi^{N_c e^{-i\theta}})\right] + \frac{\chi N_c^2}{2} \right\} + \text{H.c.}$$

Here $A$ is a numerical constant of order one, and $\chi$ is the vacuum topological susceptibility in pure gluodynamics (note that $\chi$ is independent of $N_c$). The scale parameter $\Lambda$ is set to unity.

This model has the vacuum family composed of $N_c$ states. Indeed, the minima of the energy are determined from the equations

$$\left. \frac{\partial V}{\partial \Phi} \right|_{\text{vac}} = \left. \frac{\partial V}{\partial \Phi^*} \right|_{\text{vac}} = 0 ,$$

$$13$$
which have the following solutions:

$$\Phi_{\text{vac}} = \exp \left( i \frac{\theta + 2\pi \ell}{N_c} \right), \quad \ell = 0, 1, \ldots, N_c - 1. \quad (33)$$

In the $\ell$-th minimum $V_0$ vanishes, while $V_1$ produces a non-vanishing vacuum energy density,

$$\mathcal{E}_\ell = \chi N_c^2 \left\{ 1 - \cos \left( \frac{\theta + 2\pi \ell}{N_c} \right) \right\}. \quad (34)$$

For each given $\theta$ the genuine vacuum is found by minimization,

$$\mathcal{E}(\theta) = N_c^2 \chi \min_{\ell} \left\{ 1 - \cos \left( \frac{\theta + 2\pi \ell}{N_c} \right) \right\}. \quad (35)$$

The remaining $N_c - 1$ minima are quasivacua. Once the heavy field $\Phi$ is integrated out, the vacuum energy is given by the expression (35); it has cusps at $\theta = \pi, 3\pi$ and so on. Needless to say that the potential (31) has no cusps.

We will first consider the model (31) without the axion field, at $\theta = 0$, in the limit $N_c = \infty$. In this limit the false vacua from the vacuum family are stable.

The classical equation of motion defining the wall is

$$N_c^2 \Phi^{\prime\prime} = \frac{\partial V}{\partial \Phi}, \quad (36)$$

where primes denote differentiation with respect to $z$ (we look for a solution which depends on the $z$ coordinate only).

This is a differential equation of the second order. It is possible, however, to reduce it to a first order equation. Indeed, Eq. (36) has an obvious "integral of motion" ("energy"),

$$N_c^2 \Phi^{\prime} \Phi'' - V = \text{Const} = 0, \quad (37)$$

where the second equality follows from the boundary conditions. In the large $N_c$ limit one can parametrize the field $\Phi$ as follows ($\rho \sim 1$):

$$\Phi \equiv 1 + \frac{\rho}{N_c}. \quad (38)$$

Taking the square root of Eq. (37), substituting Eq. (38) and neglecting the terms of the subleading order in $1/N_c$ we arrive at

$$\rho' = iAN_c \left( 1 - \exp \rho \right). \quad (39)$$

The phase on the right-hand side can be chosen arbitrarily. The choice in Eq. (39) is made in such a way as to make it compatible with the boundary conditions for
the wall interpolating between $\Phi_{\text{vac}} = 1$ and $\Phi_{\text{vac}} = \exp(2\pi i/N_c)$. This is precisely the expression that defines the domain walls in SUSY gluodynamics [28, 40]. It is not surprising that the same equation determines the D walls in non-SUSY gluodynamics – the fermion-induced effects are not important for the D walls in the large $N_c$ limit.

The solution of this equation was obtained in [40]. In the parametrization $\rho = \sigma + i\tau$ the solution takes the form:

$$\cos \tau = (\sigma + 1) \exp(-\sigma),$$

$$\int_{\sigma(0)}^{\sigma(z)} \left[ \exp(2t) - (1 + t)^2 \right]^{-1/2} dt = -A N_c |z|. \tag{40}$$

The real part of $\rho$ is a bell-shaped function with an extremum at zero; it vanishes at $\pm \infty$. The imaginary part of $\rho$, on the other hand, changes its value from 0 to $2\pi$. This determines a D wall in the large $N_c$ gluodynamics. The width of the wall scales as $1/N_c$.

The solution presented above is exactly the same as in SUSY gluodynamics. This is not surprising since the ansatz (38) implies that $V_1$ does not affect the solution – its impact is subleading in $1/N_c$, while $V_0$ is exactly the same as in the SUSY gluodynamics-inspired model of Ref. [28]. Moreover, for the same reason the domain wall junctions emerging in this model will be exactly the same as in the SUSY gluodynamics-inspired model [13]. Inclusion of $V_1$ in the subleading order makes the wall to decay.

Inclusion of the $N = 1$ axion field amounts to the replacement

$$\theta \rightarrow \theta + \alpha$$

in Eq. (31), plus the axion kinetic term

$$L_{\text{kin}} = \left( \frac{f^2 + 2\Phi^* \Phi}{2} \right) (\partial_\mu \alpha)^2 + i N_c (\partial_\mu \alpha)(\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*). \tag{41}$$

The occurrence of the mixing between $\alpha$ and the phase of $\Phi$ is necessary, as is readily seen from the softly broken SUSY gluodynamics. (To get the potential of the type (31) in this model, one must eliminate the $G\tilde{G}$ term by a chiral rotation. Then $m \rightarrow m \exp((\theta + \alpha)/N_c)$ and, additionally one gets $\partial_\mu \alpha \times [\text{the gluino axial current}]$.) The term $2\Phi^* \Phi$ in the brackets has to be included to reproduce the correct mass for the axion after the physical heavy state is integrated out. The presence of this term signals that QCD dynamics generates not only the potential for the axion but also modifies its kinetic term. On the other hand, since $\Phi^* \Phi \leq \Lambda^2$ and, moreover, $\Lambda \ll f$, this term can be neglected for all practical purposes.

We are interested in the configuration with $\alpha$ interpolating between 0 and $2\pi$. The phase of $\Phi$ will first adiabatically follow $\alpha/N_c$, then at $\alpha \approx \pi$, when the phase of $\Phi$ is close to $\pi/N_c$, it will very quickly jump by $-2\pi/N_c$, and then it will continue
to grow as $\alpha/N_c$, so that when $\alpha$ reaches $2\pi$ the phase of $\Phi$ returns to zero. This jump is continuous, although it occurs at a scale much shorter than $m_a^{-1}$. This imitates the D-wall core of the axion wall. One cannot avoid forming this core, since otherwise the interpolation would not connect degenerate states – on one side of the wall we would have (hadronic) vacuum, on the other side an excited state.

In the large $N_c$ limit one can be somewhat more quantitative. Indeed, in this approximation the model admits the exact solutions. The gluonic core of the wall has the same form as before, Eq. (40), but the phase $\tau$ is now substituted by the superposition $\tau - (\alpha + \theta)$ since the axion field is mixed with the phase of the $\Phi$ field.

This very narrow core is surrounded by a diffused axion halo. The axion field is described in this halo by the solution to the Lagrangian (9). This takes the form:

$$\theta + \alpha(z) = -2\pi + 4N_c \arctan \left( e^{ma z} \tan \frac{\pi}{4N_c} \right), \quad z > 0,$$

$$\theta + \alpha(z) = -4N_c \arctan \left( e^{-ma z} \tan \frac{\pi}{4N_c} \right), \quad z < 0. \quad (42)$$

Thus, we find explicitly the stable axion wall with a D-wall core. Note that this is a usual “$2\pi$” wall as it separates two identical hadronic vacua. As we discussed in the introduction, this wall is harmless cosmologically. It will be produced bounded by global axion strings in the early universe. Bounded walls shrink very quickly by decaying into axions and hadrons.

### 7 QCD with Three Light Quarks and Axion

So far we discussed pure gluodynamics with the axion. Our final goal is to study QCD with $N_f = 3$. There are two, physically distinct regimes to be considered in this case. In real QCD

$$m_u, m_d \ll m_s \sim \frac{\Lambda}{N_c}, \quad m_u, m_d, m_s \ll \Lambda. \quad (43)$$

In this regime the consideration of the chiral Lagrangians [2, 3, 7], does not exhibit the vacuum family. We will comment on why the light quarks screen the vacuum family of the glue sector, so that the axion domain wall provides no access to it. In the limit (43) the effects due to the D walls will be marginal.

On the other hand, in the genuinely large $N_c$ limit

$$\frac{\Lambda}{N_c} \ll m_u, m_d \ll m_s \ll \Lambda, \quad (44)$$

physics is rather similar to that of pure gluodynamics. The light quarks are too heavy to screen the vacuum family of the glue sector.

In what follows we study the axion walls and their hadronic components in the limits (43) and (44), separately.
7.1 One Light Quark

To warm up, let us start from the theory with one light quark. In the limit of large $N_c$ this introduces a light meson, “$\eta$”. An appropriate effective Lagrangian can be obtained by combining the vacuum energy density of gluodynamics with what remains from the Witten-Di Vecchia-Veneziano Lagrangian at $N_f = 1$,

$$\mathcal{L} = \frac{F^2}{2} (\partial_\mu \beta)^2 - V(\beta),$$

$$V = -m_q \Lambda^3 N_c \cos \beta + \min_{\ell} \left\{ -N_c^2 \Lambda^4 \cos \frac{\beta + \theta + 2\pi \ell}{N_c} \right\}. \tag{45}$$

Here $\beta$ is the phase of $U \sim \bar{q}_L q_R$, while $F^2 \sim \Lambda^2 N_c$ is the “$\eta$” coupling constant squared. The product $F\beta$ is the “$\eta$” field. The first term in $V$ corresponds to the quark mass term, $\mathcal{M}U + \text{h.c.}$ (see Eq. (7) in Witten’s paper [2]). At $N_c = \infty$ the second term in $V$ becomes $(\beta + \theta)^2$. It corresponds to $(\text{Im} \det U + \theta)^2$ in Eq. (11) in [2]. The subleading in $1/N_c$ terms sum up into a $2\pi$ periodic function of the cosine type, with the cusps. It is unimportant that we used cosine in Eq. (45). Any $2\pi$ periodic function of this type would lead to the same conclusions. The second term in Eq. (45) differs from the vacuum energy density in gluodynamics by the replacement $\theta \to \beta + \theta$.

If $m_q \ll \Lambda/N_c$, the first term in $V$ is a small perturbation; therefore, in the vacuum, $\beta + \theta = 2\pi k$, and, hence, the $\theta$ dependence of the vacuum energy is

$$\mathcal{E}_{\text{vac}}(\theta) = -m_q \Lambda^3 N_c \cos \theta. \tag{46}$$

It is smooth, $2\pi$ periodic and proportional to $m_q$ as it should be on general grounds in the theory with one light quark.

The condition $m_q \ll \Lambda/N_c$ precludes us from sending $N_c \to \infty$. The would be “$2\pi$” wall in the variable $\beta$ is expected to be unstable. This is due to the fact that at $N_c \sim 3$ the absolute value of the quark condensate $\bar{\psi}\psi$ is not “harder” than the phase of the condensate $\beta$, and the barrier preventing the creation of holes in the “$2\pi$” wall is practically absent.

If one closes one’s eyes on this instability one can estimate that the tension of the “$\eta$” wall is proportional to $\Lambda^3 N_c^{1/2}$, with a small correction $m_q \Lambda^2 N_c^{3/2}$ from the quark mass term. The tension of the D-wall core is, as previously, $\Lambda^3 N_c$.

In the opposite limit

$$m_q \gg \frac{\Lambda}{N_c}, \quad \text{but } m_q \ll \Lambda, \tag{47}$$

the situation is trickier. Now the first term in $V$ is dominant, while the second is a small perturbation. There are $N_c$ distinct vacua in the theory,

$$\beta_\ell = - \frac{2\Lambda}{m_q N_c} (\theta + 2\pi \ell). \tag{48}$$
Then the $\theta$ dependence of the vacuum energy density is

$$E_{\text{vac}}(\theta) = \Lambda^4 \min(\theta + 2\pi\ell)^2,$$

(49)

this is similar to that in the theory without light quarks (i.e., the same as in gluodynamics). The “$\eta'$” wall is stable at $N_c \to \infty$, with a D-wall core in its center. The $\eta'$ wall is a “2$\pi$” wall.

From this standpoint, the quark with the mass (47) is already heavy, although the “$\eta'$” is still light on the scale of $\Lambda$,

$$M_{\eta'} \sim m_q^{1/2} \Lambda^{1/2} \ll \Lambda.$$

So far the axion was switched off. What changes if one includes it in the theory?

The Lagrangian now becomes

$$\mathcal{L} = \frac{F^2}{2} (\partial_\mu \beta)^2 + \frac{f^2}{2} (\partial_\mu \alpha)^2 - V(\beta, \alpha),$$

$$V = -m_q \Lambda^3 N_c \cos \beta + \min_{\ell} \left\{ -N_c^2 \Lambda^4 \cos \frac{\beta + \alpha + 2\pi\ell}{N_c} \right\},$$

(50)

where the $\theta$ angle is absorbed in the definition of the axion field.

The bare “$\eta'$” mixes with the bare axion. It is easy to see that in the limit $m_q \ll \Lambda / N_c$ the physical “$\eta'$” is proportional to $\beta + \alpha$, rather than to $\beta$. Therefore, even if we force the axion wall to develop, (i.e. $\alpha$ to evolve from 0 to $2\pi$) the “$\eta'$” wall need not develop. It is energetically expedient to have $\beta + \alpha = 0$. Thus, the effect of the axion field on the hadronic sector is totally screened by a dynamical phase $\beta$ coming from the quark condensate. In other words, the axion wall with the lowest tension corresponds to the frozen physical “$\eta'$”,

$$\beta + \alpha = 0.$$

There is no hadronic core. The tension of this wall is determined from the term $\propto m_q \Lambda^3 N_c$.

[If one wishes, one could add an (unstable) “$\eta'$” wall to the axion wall. Then the “$\eta'$” wall, with the D-wall core will appear in the middle of the axion wall, but they are basically unrelated. This will be a secondary phenomenon, and the D wall core will be, in fact, the core of the “$\eta'$” wall rather than the axion wall.]

If the quark mass is such that (47) applies, then the axion field $\alpha$ cannot be screened, since we cannot freeze $\beta + \alpha$ everywhere in the axion wall profile at zero – at $m_q \gg \Lambda / N_c$, $\beta$ is proportional to the physical “$\eta'$” and is much heavier than the axion field. Thus, in this case the axion wall will be described by the Lagrangian (9) and will have a D-wall core. One may also add, on top of it, the “$\eta'$” wall. This will cost $m_q^{1/2} \Lambda^{5/2} N_c$ in the wall tension – still much less than $\Lambda^3 N_c$ of the D-wall core of the axion wall.
The limit (47) is unrealistic. Moreover, in this limit the D walls taken in isolation, without the axion walls, are stable by themselves, although they interpolate between nondegenerate states [11].

### 7.2 Three Light Quarks

Let us turn to the case of three light flavors. The physical picture is quite similar to that of the one-flavor case, see Sec. 7.1.

We assume the mass matrix $\mathcal{M}$ in the meson Lagrangian to be diagonal. Therefore, we will look for a diagonal $U(3)$ meson matrix which minimizes the potential,

$$U = \text{diag} \left( e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3} \right). \quad (51)$$

The potential takes the form

$$V = -\sum_i m_i \Lambda^3 N_c \cos \phi_i + \min_{\ell} \left\{ -N_c^2 \Lambda^4 \cos \frac{\sum_i \phi_i + \theta + 2\pi \ell}{N_c} \right\}. \quad (52)$$

As before, we will consider two limiting cases, (43) and (44).

Let us switch off the axion field first. In the limit of genuinely light quarks, Eq. (43), when the second term in the potential (52) is dominant, the solutions for $\phi$'s were found in [2, 3]. They satisfy the relation $\phi_3 \simeq 0$ and $\phi_1 + \phi_2 = -\theta$. The corresponding expression for the vacuum energy density is

$$E_{\text{vac}}(\theta) = -N_c \Lambda^3 \sqrt{m_u^2 + m_d^2 + 2m_u m_d \cos \theta}. \quad (53)$$

As in Sec. 7.1, we deal here with a smooth single-valued function of $\theta$. The inclusion of the axion replaces $\theta \to \theta + \alpha \to \alpha$. The physical $\eta'$ field is given by the sum $\sum_i \phi_i + \alpha$. It is energetically favorable to freeze this state. Thus, the situation is identical to that in the one-flavor case: even if the axion wall is forced to develop, the physical $\eta'$ wall (which is now the $\sum_i \phi_i + \alpha$ wall) does not have to occur. The $\eta'$ wall is a gateway to the D wall. In the theory at hand the vacuum angle is screened in the axion wall, and there is no D-wall core.

If, nonetheless, the $\eta'$ wall is formed due to some cosmological initial conditions, it will have a D-wall core (albeit the $\eta'$ wall is unstable in the limit at hand and cannot be considered in the static approximation). The would-be $\eta'$ wall is independent of the axion wall; its effect on the axion wall formation is rather irrelevant.

In addition to this, a “$2\pi$” wall could develop built of nonsinglet mesons, at certain values of the quark masses. There is nothing new we could add to this issue which is decoupled from the issue of the vacuum family in the glue sector and D walls.

We now pass to the opposite limit (44), when the first term in the potential (52) is dominant. As in Sec. 7.1, there are $N_c$ distinct vacua with the energy given by (49). It is straightforward to show that the potential for the axion in this case is
of the form (9), with the cusps which signal the presence of the D-wall core. This is similar to what happens in gluodynamics. One cannot avoid having an $\eta'$ wall in the middle of the axion wall, which entails a D wall too. The D walls separate the degenerate vacua. Since they “live” in the middle of the axion wall, they are perfectly stable.

(In addition, there can be “$2\pi$” walls in either of $\phi$’s or their linear combinations. However, these latter are unstable and do not appear in the physical spectrum of the theory.)

8 Conclusions

Summarizing, we have found that the presence of the axion field and the axion wall makes the D wall perfectly stable in gluodynamics at finite $N_c$. The D wall develops as a core of the axion wall. It is unavoidable.

In QCD with light quarks the axion wall may or may not generate the D-wall core. Everything depends on the interplay between the quark masses, $\Lambda$ and $N_c$. In the realistic case of genuinely light quarks, see Eq. (43), the phase associated with the axion field in the wall profile is screened by a dynamical phase (which can be traced back to the presence of $\eta'$). The $\eta'$ is not excited, and neither is the D wall. There is no D-wall core in the axion wall.

If $N_c$ is increased so that Eq. (44) holds the picture changes essentially to that one deals with in gluodynamics: the $\eta'$ wall is excited, opening the access to the D wall. The D-wall core develops in the central part of the axion wall. Unfortunately, this limit is unrealistic.

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