We provide a covariant and gauge-invariant approach to the question of how pressure can be incorporated self-consistently in a cosmological scenario which is relevant to weakly self-interacting or warm dark matter models in the linear regime. New modes arise in the density perturbations because the number density fluctuations are no longer simply proportional to the density fluctuations.

I. INTRODUCTION

According to the current cosmological paradigm, the large-scale structure, as seen today, formed from the evolution of gravitational instabilities which originated from inflationary processes in the primordial fluid. In studying the evolution of such instabilities one usually assumes a dust or radiation energy momentum tensor. While these two extreme fluid states are reasonable for the most of the history of the universe, there are short, but at the same time important, specific transition periods in which matter is in the non-relativistic regime with a tiny although non-negligible pressure.

In this paper we shall be concerned with the evolution of matter perturbations in different cosmological settings. Historically most of the literature treats the cosmological non-relativistic matter as a pressureless fluid, i.e. dust. While this is a convenient approach for late stages of our evolving universe, some refinements must be introduced when we deal with earlier times during which collisions or slight random motions of collisionless matter give rise to a small but non-zero pressure. Even though in these cases the matter pressure in the background may be negligible, small disturbances (perturbations) of the spacetime induce a tiny pressure term which may modify the evolution of the inhomogeneities. The methods for carrying out gauge-independent descriptions of relativistic perturbations have been analyzed deeply in the literature [1], [2], [3]. Here we have chosen to use the Ellis-Bruni covariant and gauge-invariant approach [3], which over recent years has proved to be useful in different cosmological problems [4], [5], [6].

In addition to the pressure effects mentioned above, we analyze particle number density and energy density inhomogeneities. It must be born in mind that in structure formation one really needs to look at the clumping of a set of particles (baryons, stars or galaxies according to the scale and epoch being studied) rather than at the evolution of energy density instabilities. Although in taking the dust assumption to be valid such a distinction is not relevant, this is no longer true for those periods in which matter is undergoing a transition to the non-relativistic regime and, consequently, for which the dust approximation is not accurate enough, even in the linear regime. Our results may be useful for studying weakly self-interacting or warm dark matter models in the linear regime. Recent work using numerical integrations in the nonlinear regime, has investigated these models as possible resolutions of problems with the cold dark matter model on small scales (see [7]).

The approach we use here is to generalise the usual equations for the growth of cosmological perturbations for matter in a FRW background to include pressure effects due to a non-relativistic component, both in the radiation and matter dominated eras. Thus the hypothesis of dust, i.e. vanishing pressure, is relaxed to include a pressure due to the thermal motions of the particles which make up the fluid.

II. MATTER VERSUS ENERGY DENSITY PERTURBATIONS

We use the Ellis-Bruni [3] formalism which, because it is covariant and gauge-invariant, allows a clear and simple physical interpretation of the variables involved. The starting point is the choice of a time-like congruence $u^a$ along which typical (comoving) observers move. Once the four-velocity has been selected we split all the physical quantities into their spatial and time-like parts by using the spatial projector tensor $h^{ab}$. Physical quantities relevant for our purpose are defined as follows

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\[
\delta_a = a \frac{D_a \rho}{\rho}, \quad \theta_a = a D_a \theta, \quad p_a = a \frac{D_a p}{\rho}, \quad \nu_a = a \frac{D_a n}{n}
\]

(1)

where \(D_a \equiv h^b_a \nabla_b\) is the spatial projection of the covariant derivative, \(\theta \equiv u^a\) the expansion, \(p\) the pressure, \(\rho\) the energy density and \(a\) is a representative or average length scale defined by \(\theta = 3 \dot{a} / a\). In a Friedman-Robertson-Walker (FRW) metric \(a\) is the background scale factor. The \(\delta_a, p_a, \theta_a\) and \(\nu_a\) describe the spatial inhomogeneity of quantities which are homogeneous and isotropic in the background. Each of them is spatial (contraction with \(u^a\) vanishes), covariant and gauge-invariant to first order, as all of them vanish in the FRW background [1]. The scalar parts of perturbations described in (1) are obtained by taking the comoving spatial divergence, say, \(\delta = a D^b \delta_a\).

The general procedure to obtain evolution and constraint equations for the foregoing quantities starts from a suitable spatial projection of the Ricci and Bianchi identities as given in terms of the fluid variables once the field equations have been taken into account (see for example [8] for a pedagogical account). Here we shall follow the notation of [4].

Restricting ourselves to a spatially flat FRW background, the equations governing the evolution of the comoving fractional density gradient \(\delta_a\) and the comoving expansion gradient \(\theta_a\) are (see [4] equations 25,26)

\[
\dot{\delta}_a = 3wH\delta_a - (1 + w)\theta_a,
\]

(2)

\[
\dot{\theta}_a - \frac{9}{2} H^2 p_a + 2H\theta_a - aD_a b\dot{u}_b + \frac{3H^2}{2}(\rho_a + 3p_a) = 0,
\]

(3)

where

\[
w = \frac{p}{\rho}.
\]

(4)

A full understanding of large scale structure formation in the framework of gravitational instabilities requires an analysis of how particles, which comprise the cosmic fluid, clump together due to their mutual attraction. Although dealing with the energy density is the usual procedure, we believe that it is better to include the particle number density \(n\) because, ultimately, particles are the building blocks of all the cosmic structures we see today. For dust, it does not make any difference whether one chooses \(\rho\) or \(n\) because in such a case no random motions are present and the internal energy vanishes identically, yielding \(\rho = mn\), i.e,

\[
\delta_a = \nu_a.
\]

(5)

The situation is quite different when internal energy associated with thermal motions is introduced. The behaviour of \(\nu_a\) requires a separate investigation. In [4] it is shown that from the Gibbs equation

\[
TdS = d \left( \frac{\rho}{n} \right) + p d \left( \frac{1}{n} \right),
\]

(6)

we get

\[
e_a = \delta_a - (1 + w)\nu_a,
\]

(7)

where

\[
e_a = \frac{anTD_a s}{\rho},
\]

(8)

is a normalized entropy gradient. Perturbations which are non-dissipative (\(s = 0\)) and without spatial variation of the entropy \((e_a = 0)\) are called isentropic in [4] because they have the same entropy at all the points of the spacetime. When no spatial variation of the entropy is allowed (isentropic disturbances are a specific case) the number density perturbations are related algebraically to the energy density perturbations by

\[
\nu_a = \frac{\delta_a}{1 + w}.
\]

(9)

In fact, \(e_a = 0\) is a very strong assumption which in general does not have to be true. Its departure from zero could affect the growth of perturbations in an important way. For instance, from the particle conservation equation

\[
\dot{n} + n \theta = 0,
\]

(10)

we get an evolution equation for \(\nu_a\)
\[ \nu_a + \theta_a - \frac{\theta}{1+w}p_a = 0. \quad (11) \]

If we use the standard, but generally unphysical, equation of state \( w = \text{constant} \) then together with (2), the integral of (11) yields

\[ \nu_a = \frac{\delta_a}{1+w} + k_a, \quad \dot{k}_a = 0. \quad (12) \]

We see that in general, when this equation of state is used, a stationary mode appears in the evolution of \( \nu_a \) which could have an impact on the structure formation at certain stages of the evolution of inhomogeneities. The problem with (12) is that it makes sense only for the very special, but widely used, cases of dust and radiation. In more general circumstances; for instance those which occur in the radiation-matter transition, equations of state should involve two independent thermodynamical quantities, i.e. \( n \) and \( \rho \). To deal with the inhomogeneities that arise in non-relativistic matter with a non-vanishing pressure we may follow two different approaches.

(i) If we assume that the fluid has reached a collision-dominated equilibrium, we learn from kinetic theory that the equation of state in the non-relativistic regime is

\[ \rho = mn + \frac{3}{2}p, \quad (13) \]

which together the energy balance equation

\[ \dot{\rho} + 3H(\rho + p) = 0, \quad (14) \]

and the particle number balance equation (10) leads to

\[ \dot{p} = -5Hp, \quad (15) \]

with solution

\[ p = p_0 \left( \frac{a}{a_0} \right)^5. \quad (16) \]

Also from (13) we get the equation

\[ \delta_a = \frac{mn}{\rho} \nu_a + \frac{3}{2}p_a \quad (17) \]

which, together with (11) and the conservation equations, determines \( \nu_a \).

(ii) Without imposing the restriction of a collision-dominated equilibrium fluid, we could tackle the problem of a collisionless gas in which a small pressure arises because of random motions of the particles making up the system. In [5] it was shown that by neglecting higher powers of the velocity dispersion in the Boltzmann equation, the evolution equation for pressure is given by (15). We remark that in obtaining this equation no assumption about the equation of state was made. Only the absence of collisions and the smallness of the velocity dispersion was used. Despite the evident mathematical similarities to case (i), a crucial difference emerges here: for a collisionless gas the Gibbs equation (6) does not apply and, as we shall show below, this leads to some new results.

In the next section we discuss several physically motivated cosmological settings where the pressure enters in a natural way.

III. NON-RELATIVISTIC MATTER PERTURBATIONS WITH PRESSURE

We investigate the role played by the pressure in the non-relativistic regime on an FRW background filled by a dust gas. The point is that while the idealized background can be described by dust, as soon as it is disturbed a small nonzero pressure arises. This means that the pressure is a first order quantity, as it vanishes in the background. The usual approach takes an identically zero pressure in the background and in the real (perturbed) spacetime. We relax this assumption by allowing first order corrections to pressure due either to the fluid being in a collision-dominated regime (case (i) above) or through velocity dispersion without collisions (case (ii) above). One-component and two-component models will be considered in turn.
A. Dust background

For simplicity we use an Einstein-de Sitter background. As stated above we focus on perturbations for which pressure is a first order quantity. From the previous section we have that, both in case (i) and case (ii), \( p \) is given by the evolution equation (15) leading to (16)

\[
p = p_0 \left( \frac{a_0}{a} \right)^5. \tag{18}
\]

As \( p \) is now a first order quantity we get from (2) and (3),

\[
\ddot{\delta}_a - 2H \dot{\delta}_a + \frac{3}{2} H^2 \delta_a + \frac{a}{\rho} D_a D^2 p = 0, \tag{19}
\]

where the momentum balance equation

\[
(\rho + p)\dot{u}_a = -D_a p, \tag{20}
\]

has been used. Note that for large-scale perturbations the Laplacian term is negligible and we recover the standard evolution equation for dust. Hence, as expected, pressure plays a role only at small scales.

At this point we distinguish between the two cases discussed in section two. If we assume that the fluid is in collision-dominated equilibrium, then the Gibbs equation holds which, together the equation of state (13), leads to

\[
D_a p = 0, \tag{21}
\]

at first order reducing (19) to the standard equation valid for dust at any scale. Expression (21) implies that the spatial gradient of \( p \) is a gauge-invariant second order quantity and hence it allows us to go beyond first order up to second order corrections for pressure. A full covariant and gauge-invariant treatment for second order perturbations is not possible at the moment. The point is that in order to construct \( n \)-order gauge-invariant variables à la Ellis-Bruni, we have to ensure that they are constant (usually zero) at all orders below \( n \) [9]. It is not clear how to find (if at all possible) a second-order meaningful variable for the perturbed density in such a way that it vanishes at zero and first orders. Bruni and coworkers provide a systematic method to tackle relativistic perturbations beyond the linear order, but one has to pay the price of losing the gauge-invariant character [10].

From (15) we have

\[
\frac{p_a}{a} + \frac{2}{a} \dot{p}_a = \frac{5p}{3\rho} \delta_a, \tag{22}
\]

where a prime denotes a derivatives with respect to the scale factor \( a \). According to the argument above and the fact that \( p \) and \( \delta_a \) are first order quantities, equation (22) contains only second order quantities. On using (16), equation (28) below for the dust background and the standard first order solution for dust, i.e.,

\[
\delta_a = c_1 a + c_2 a^{-3/2}, \tag{23}
\]

in (22), we find

\[
p_a = \frac{A c_1}{a} + \frac{A c_2}{a^{7/2}} + \frac{c_2}{a^{2/2}} A \equiv \frac{5p_0 a_0^2}{3\rho_0}. \tag{24}
\]

This is a simple and exact example giving the evolution of a second order quantity in the Ellis-Bruni formalism. The point is that we managed to apply such a formalism because \( D_a p \) vanishes at the zero and at the first order approximation.

However, the more interesting and physically appealing situation is that of a collisionless gas for which the foregoing arguments are not applicable. Applying the operator \( aD_a D^2 \) to (15) and using the first order identity

\[
(aD_a f) = aD_a f, \tag{25}
\]

where \( f \) denotes any first order scalar or tensor quantity, we get
\[(aD_aD^2p)' = -\left(\frac{7}{a}\right) aD_aD^2p,\]  
(26)

and hence
\[aD_aD^2p = A^0_a \left(\frac{a_o}{a}\right)^7, \quad A^0_a = aD_aD^2p|_{a=a_o}, \quad \dot{A}^0_a = 0.\]  
(27)

For a dust background
\[H = \sqrt{\frac{\rho_0}{3}} \left(\frac{a_0}{a}\right)^{3/2}, \quad \rho = \rho_0 \left(\frac{a_0}{a}\right)^3,\]  
(28)

so equation (19) reduces to
\[\delta''_a + \delta'_a \left(\frac{H'}{H} + \frac{3}{a}\right) - \frac{3}{2} \frac{A^0_a}{\rho_0H^2a^2} \left(\frac{a_0}{a}\right)^4 = 0,\]  
(29)

which can be solved to give
\[\delta_a = K^1_a \left(\frac{a}{a_0}\right) + K^2_a \left(\frac{a_0}{a}\right)^{3/2} - \frac{3A^0_a}{\rho_0} \left(\frac{a_0}{a}\right),\]  
(30)

where \(\dot{K}^i_a = 0\). Thus a new mode is obtained which decays at slower rate than the standard decaying mode for dust.

**B. Dust-radiation background**

The novelty in this subsection is that the dynamics of the background is given by a decoupled mixture of dust and radiation. We will consider perturbations of the matter component which again will be taken as dust in the background but with a non-vanishing first order pressure in the real spacetime. This situation mimics that of the cosmological fluid prior to the transition from the radiation to the matter dominated era with a matter component which is acquiring a major role in the dynamics. In the sequel \(p_r, \rho_r\) and \(\rho_m\) denote the zero order pressure and energy density of radiation and matter respectively, whereas \(p_m\) is the first order pressure of matter as given by (16).

Assuming that the fluids are decoupled and share the same 4-velocity we get
\[(\rho_r + p_r)\dot{u}_a + D_a p_r = 0,\]  
(31)
\[\rho_m \dot{u}_a + D_a p_m = 0.\]  
(32)

From these equations and the equation of state for radiation we find that
\[\frac{D_a p_r}{\rho_r + p_r} = \frac{D_a p_m}{\rho_m} \Rightarrow \delta^r_a = 4p^m_a,\]  
(33)

where
\[\delta^r_a = a \frac{D_a \rho_r}{\rho_r}, \quad p^m_a = a \frac{D_a p_m}{\rho_m}.\]  
(34)

Note that keeping the matter component as dust both in the background and in the real spacetime we would have ended up with \(\dot{u}_a = 0 = D_a p_r\), i.e. radiation perturbations would not have been allowed.

In order to get an evolution equation for \(\theta_a\) we start with the Raychaudhry equation
\[\dot{\theta} + \frac{1}{3} \theta^2 - D^a \dot{u}_a + \frac{1}{2} (\rho + 3p) = 0,\]  
(35)

where \(\rho = \rho_r + \rho_m\) is the total energy density and \(p = p_r + p_m\) is the total pressure. We have to be very careful in dealing with \(p\) since in the background \(p = p_r\) but when we apply the spatial derivative operator we get \(D^a p = D^a p_r + D^a p_m\). Bearing all this in mind the evolution equation for \(\theta_a\) becomes
\[
\dot{\theta}_a + \frac{9}{2}H^2 a \dot{u}_a (1 + w) + 2 H \theta_a - a D_a D^b \dot{u}_b + \frac{1}{2}(8 \rho_r p_m^m + 3 \rho_m p_m^m + \rho_m \delta^m_a) = 0,
\]
where equation (33) has been used.

Before dealing with the acceleration term we recall the identity [4]

\[
D^2(D^a f) = D_a (D^2 f) + \frac{2}{3}(\rho - 3H^2) D_a f + 2 \dot{f} \text{curl} \omega_a.
\]

From this identity and bearing in mind that \(p_m\) is a first order quantity, we have

\[
a D_a D^b \dot{u}_b = -a \frac{D_a D^2 p_m}{\rho_m} = -a \frac{D^2 D_a p_m}{\rho_m} = -D^2 p_m^m.
\]

It is important to note that the result obtained from commuting the operators \(D_a\) and \(D^2\) would have been different if we had used \(p_r\) instead of \(p_m\). In the latter case a first order term \(\sim \dot{\rho}_r \text{curl} \omega_a\) would have arisen. This means that consistency requires

\[
\text{curl} \omega_a = 0,
\]

i.e., in a radiation-matter decoupled mixture for which (a) both fluids share the same 4-velocity and (b) the matter pressure arises from perturbations so \(p_m\) is a first order quantity, \text{curl} \omega_a vanishes at first order.

Changing the independent variables from \(t\) to \(a\), we get the evolution equation for \(\theta_a\),

\[
\theta_a' - \frac{9H}{2a} \left(4 + \frac{3\rho_m}{\rho_r}\right) p_m^m + \frac{2\theta_a}{a} + \frac{D^2 p_m^m}{aH} + \frac{1}{2aH} \left[p_m^m(8 \rho_r + 3 \rho_m) + \rho_m \delta^m_a\right] = 0.
\]

The background equations for dust plus radiation are

\[
\rho = \rho_m + \rho_r = \frac{3}{\beta a^4}(1 + a\alpha), \quad p = p_r = 1/\beta a^4, \quad H^2 = \frac{1}{\beta a^4}(1 + a\alpha),
\]

where

\[
\beta = \frac{3}{\rho_0^4 a_0^4}, \quad \alpha = \frac{\rho_m^4 a_0^4}{\rho_r^4 a_0^4}.
\]

From the evolution equation for \(p\) for a collisionless gas

\[
\dot{p}_m = -\frac{5}{3} \dot{\rho}_m,
\]

and using the equation (25) we have

\[
\dot{p}_a^m = -2H p_a^m \Rightarrow \dot{p}_a^m = -\frac{2}{a} p_a^m,
\]

which gives

\[
p_a^m = K_a^m \left(\frac{a_0}{a}\right)^2, \quad \dot{K}_a^m = 0.
\]

Using similar reasoning for \(D^2 p_a^m\) and with the help of the identity

\[
(D^2(f)) = D^2 \dot{f} - 2HD^2 f + f D^a \dot{u}_a,
\]

from [4] we get

\[
D^2 p_a^m = M_a^m \left(\frac{a_0}{a}\right)^4, \quad \dot{M}_a^m = 0.
\]

We now define the scalar parts of the perturbations by
\[ \delta \equiv aD^a \delta_a, \quad Z \equiv aD^a \theta_a. \]  
(48)

From (45) it follows that

\[ aD^a p_m^a = K \left( \frac{a_0}{a} \right)^2, \quad \dot{K} = 0, \]
(49)

where \( K = aD^a K^a_m \) and that

\[ aD^a [aD_a (D_b \dot{u}_b)] = -M \left( \frac{a_0}{a} \right)^4, \quad \dot{M} = 0 \]
(50)

where \( M \equiv aD^a M^a_m \). From equations (32) and (33) it follows that

\[ aD^a (a\dot{u}_a) = -K \left( \frac{a_0}{a} \right)^2. \]
(51)

Operating on equation (36) with \( aD^a \) gives

\[ \dot{Z} + 2HZ + \frac{1}{2} \rho_m \delta + M \left( \frac{a_0}{a} \right)^4 + K \left( \frac{a_0}{a} \right)^2 \left[ 4\rho_r + \frac{3}{2} \rho_m - \frac{9}{2} (1 + w)H^2 \right] = 0. \]
(52)

For the matter scalar perturbations, \( w = 0 \) and equation (2) becomes \( Z = -\delta \) and the evolution equation for \( \delta \) is

\[ \delta'' + \frac{3a + 2h}{2a(a + h)} \delta' - \frac{3}{2} \frac{1}{a(a + h)} \delta = F(a), \]
(53)

where \( h = \alpha^{-1} \), the independent variable has been changed to \( a \) and

\[ F(a) = \frac{1}{a^2 H^2} \left[ M \left( \frac{a_0}{a} \right)^4 + K \left( \frac{a_0}{a} \right)^2 \left[ 4\rho_r \left( \frac{a_0}{a} \right)^4 + \frac{3}{2} \rho_m \left( \frac{a_0}{a} \right)^3 - \frac{9}{2} H^2 \right] \right], \]
(54)

where \( H \) is given in terms of \( a \) in (41), is the inhomogeneous part of the ODE. It is immediately apparent that \( y_1 = 3a + 2h \) is a solution to the homogeneous part of equation (53). The method of reduction of order [11] then leads to a complete solution. The second solution of the homogeneous part is

\[ y_2 = y_1 \int \frac{da}{(3a + 2h)^2 a \sqrt{a + h}} \]
(55)

which can be integrated using partial fractions. The full solution is given by

\[ \delta = C_1 y_1 + C_2 y_2 + y_1 \int \frac{1}{Ey_1} \left( \int Ey_1 F da \right) da \]
(56)

where \( E = \exp \left( \int (3a + 2h) [2a(a + h)]^{-1} da \right) = a \sqrt{a + h} \).

The non-homogeneous nature of the evolution equation (53) leads to the appearance of a new mode: the \( C_1 = 0 = C_2 \) mode in equation (56). Two other features of the equation are worthy of note. First, for large \( a \) the asymptotic form of the homogeneous equation is the usual equation for first order perturbations in dust. The second feature that even for initial conditions in which \( \delta \) and its first derivative with respect to \( a \) are zero perturbations will arise from the influence of the function \( F(a) \) is possibly more significant for the formation of structure.

**IV. CONCLUSION**

For analytical as opposed to numerical modelling of the evolution of inhomogeneities in the universe, it is conventional to use a fluid approximation. At a detailed level this is at variance with reality because the matter is generally more particulate than in hydrodynamics. Also most of the analytical literature assumes that the cosmological non-relativistic matter is dust. If we take the particulate nature into account, then at early times weak self-interactions or small random motions of collisionless matter will give rise to a small but non-zero pressure. We have discussed some of the implications of treating the matter as particulate and of including a first order pressure in the matter distribution.
We have derived the relation between the energy perturbations and the number density perturbation for the isentropic case and we show that in the simple non-isentropic case with $p/\rho = \text{constant}$ a new stationary mode appears. This new mode will affect the formation of structure at certain stages in the evolution of the inhomogeneity. The general case does not yield a simple exact solution.

In the main body of the paper we derive an exact solution to the energy perturbation equation when the matter pressure is non-zero at first order. The implications of assuming a non-zero first order matter pressure are quite deep as is illustrated in equation (24), which is an equation for second order gauge invariant quantities, and in equation (32), which together with (31) shows that if $D_a p_m$ were equal to 0, then $\dot{u}_a = 0 = D_a p_r$, which would have reduced the problem to the standard case.

The complete solution (56) is different from the usual solutions used in CDM approximations. Having such a solution may be useful as an analytical tool for understanding some features of recent numerical work on weakly self-interacting or warm dark matter models in the non-linear regime [7]. This is a subject of further investigation.

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