Fermion Mass Hierarchies and Small Mixing Angles from Extra Dimensions

Masako Bando*, Tatsuo Kobayashi†, Tatsuya Noguchi‡ and Koichi Yoshioka§

*Aichi University, Aichi 470-0296, Japan
††Department of Physics, Kyoto University, Kyoto 606-8502, Japan
§Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

Abstract

In this paper we study renormalization-group evolutions of Yukawa matrices enhanced by Kaluza-Klein excited modes and analyze their infrared fixed-point structure. We derive necessary conditions to obtain hierarchies between generations on the fixed point. These conditions restrict how the fields in the models can extend to higher dimension. Several specific mechanisms to realize the conditions are presented. We also take account of generation mixing effects and find a scenario where the mixing angles become small at low energy even with large initial values at high-energy scale. A toy model is shown to lead realistic quark mass matrices.

*E-mail: bando@aichi-u.ac.jp
†E-mail: kobayash@gauge.scphys.kyoto-u.ac.jp
‡E-mail: noguchi@gauge.scphys.kyoto-u.ac.jp
§E-mail: yoshioka@yukawa.kyoto-u.ac.jp
1 Introduction

The fermion mass hierarchy problem has been one of the most challenging problems in particle physics: the observed quarks and leptons show hierarchical mass difference between three generations. In the Standard Model, their particle masses are only input parameters. However, from the viewpoint of the unification scenario, this fermion mass hierarchy should also be explained. A number of ideas have been proposed so far. Among them the Froggatt-Nielsen mechanism [1] is one of possibilities which can explain the power hierarchy of Yukawa couplings via higher-dimensional operators constrained by some extra symmetries. Unfortunately, however, there is no principle to determine the coefficients of these operators and we can at most predict the order of Yukawa couplings. The coefficients could be determined by some more fundamental principle.

Recently, models with extra spatial dimensions have been investigated [2], and they have provided novel approaches to gauge coupling unification [3], fermion masses [4], possibilities of the experimental observations [5] and so forth. In these models, the existence of Kaluza-Klein (KK) excited modes plays an important role: because of the contribution from the towers of KK modes (of the Standard Model fields), the renormalization-group equations (RGE) of gauge and Yukawa couplings have “power-law” dependence on energy scale [6, 3]. The emergence of the power-law running behavior is very attractive when it comes to discussing the fermion mass hierarchy.

In our previous paper [7], we have shown that Yukawa couplings from higher-dimensional models can be highly suppressed under RG evolution with the value of coupling being stabilized as an infrared stable fixed point. Furthermore, extending the analysis to the case with two Yukawa couplings, we have shown that hierarchically different values can be realized as infrared fixed point values.

The aim of this paper is to study further the power-law running effect on Yukawa couplings and its infrared fixed-point structure in more detail. We give the necessary conditions for field configurations to yield mass hierarchies, and propose some mechanism realizing the conditions. That gives the hint how to arrange the matter, Higgs and gauge fields in the extra dimensions. Although in this paper, we do not perform explicit model-buildings, the obtained conditions are useful in constructing realistic higher-dimensional models.

The article is organized as follows. In section 2, we discuss the power-law running behavior of gauge and Yukawa couplings and their infrared fixed-point structure in the case
with no generation mixing. In section 3, we study under what conditions the Yukawa beta-
functions are made generation-dependent and hierarchical Yukawa couplings are realized. We investigate RG flows of Yukawa couplings in section 4, taking account of generation-
mixing and give a toy model which reproduces realistic quark matrices. Section 5 is devoted
to the summary and discussion.

2 RGE running of Yukawa couplings

In this section, we study power-law running behavior of gauge and Yukawa couplings and
their fixed-point structure in supersymmetric (SUSY) models with extra spatial dimensions. We assume that matter, Higgs and gauge fields can extend into the extra dimensions other than our four-dimensional spacetime. Models with extra dimensions generally involve different compactification scales depending on their directions, but for simplicity, we here take all of these scales as the universal compactification scale $\mu_0$, above which KK modes appear. Below $\mu_0$, the KK modes decouple and the models become four-dimensional $N = 1$ SUSY theories with only light modes. Thus, the infrared fixed point values at $\mu_0$ are regarded as the boundary values at $\mu_0$ of ordinary $N = 1$ SUSY theory. Our aim is to realize hierarchical values of Yukawa couplings at the scale $\mu_0$.

Let us consider the case with one gauge coupling constant $g$, which may be identified
with the $SU(3)$ gauge coupling in the SUSY standard models. We consider an $N = 1$ SUSY
gauge-Yukawa system with Yukawa couplings $y_i$,

$$ W = \sum_i y_i \Psi_{L_i} \Psi_{R_i} H, \quad (2.1) $$

where $\Psi_{L_i}$ and $\Psi_{R_i}$ are the $i$-th generation of matter superfields. Here we have neglected
generation mixing.

In this setup, the beta-functions of gauge and Yukawa couplings above $\mu_0$ generally
contain power terms of energy scale $[6, 3]$ while the beta-functions below $\mu_0$ are controlled
by usual logarithmic terms. Above $\mu_0$, the RGE of the gauge coupling $\alpha \equiv g^2/4\pi$ is written as

$$ \frac{\partial \alpha}{\partial t} = -\frac{b}{2\pi} \left( \frac{\Lambda}{\mu} \right)^{\delta_g} \alpha^2 + \cdots, \quad t = \ln \left( \frac{\Lambda}{\mu} \right), \quad (2.2) $$

where $\Lambda$ is the cutoff energy scale ($\mu_0 < \mu < \Lambda$) and $b$ is determined by group-theoretical and
phase-space integration factors. The ellipsis denotes sub-leading terms. The power of $\Lambda/\mu$,
\( \delta_g \), is generated by the KK-mode contribution of gauge fields (Fig. 1a), matter and Higgs fields (Fig. 1b), which propagate in the loops. Note that all of the external lines in Figs. 1, 2 and 3 are KK zero-modes. The case with no extra dimension \( (\delta_g = 0) \) corresponds to usual logarithmic behavior in four dimensions. Similarly the RGEs of the Yukawa couplings \( \alpha_{yi} \equiv y_i^2/4\pi \) are expressed as

\[
\frac{\partial \alpha_{yi}}{\partial t} = \frac{\alpha_{yi}}{2\pi} (\gamma_{Li} + \gamma_{Ri} + \gamma_H),
\]

where \( \gamma_{Li} \) (\( \chi = L \) or \( R \)) and \( \gamma_H \) are the anomalous dimensions of \( \Psi_{\chi_i} \) and \( H \), respectively:

\[
\gamma_{\chi_i} = -a_{\chi_i} \left( \frac{\Lambda}{\mu} \right) \delta_{\chi_i} + c_{\chi_i} \left( \frac{\Lambda}{\mu} \right) \delta_{\chi_i} + \cdots,
\]

\[
\gamma_H = -\sum_i a_{H_i} \left( \frac{\Lambda}{\mu} \right) \delta_{H_i} + c_{H_i} \left( \frac{\Lambda}{\mu} \right) \delta_{H_i} + \cdots
\]

with \( a \) and \( c \) being written in terms of group-theoretical and phase factors. In the following, we drop the beta-function coefficients, \( a, b \) and \( c \) since only exponents of \( \Lambda/\mu \) are important for our discussion.* The exponents \( \delta_{\chi_i}'s \) in the \( i \)-th generation anomalous dimension \( \gamma_{\chi_i} \) reflect the properties of the intermediating gauge fields (Fig. 2a) and Higgs fields (Fig. 2b). As for the Higgs anomalous dimension \( \gamma_H \), the exponents are determined by the contributions from the gauge fields (Fig. 3a) and matter fields (Fig. 3b). Note that \( \gamma_H \) includes the contributions from all generations.

Let us consider the two-generation case for concreteness. The RGEs of the Yukawa couplings \( \alpha_{y1} \) and \( \alpha_{y2} \) are given by

\[
\frac{d}{dt} \left( \frac{\alpha_{y1}}{\alpha} \right) = \frac{1}{2\pi} \left[ -\left( \frac{\Lambda}{\mu} \right) \delta_{y1} \alpha_{y1} - \left( \frac{\Lambda}{\mu} \right) \delta_{y2} \alpha_{y2} + \left( \frac{\Lambda}{\mu} \right) \delta_{y1} \right] \alpha_{y1},
\]

\[
\frac{d}{dt} \left( \frac{\alpha_{y2}}{\alpha} \right) = \frac{1}{2\pi} \left[ -\left( \frac{\Lambda}{\mu} \right) \delta_{y1} \alpha_{y1} - \left( \frac{\Lambda}{\mu} \right) \delta_{y2} \alpha_{y2} + \left( \frac{\Lambda}{\mu} \right) \delta_{y2} \right] \alpha_{y2},
\]

where \( \delta_{y1} \equiv \text{Max} (\delta_{\chi1}, \delta_{H1}) \) and \( \delta_{y2} \equiv \text{Max} (\delta_{\chi2}, \delta_{H2}) \). The fixed-point solutions at \( \mu = \mu_0 \) for \( \alpha_{y1}/\alpha \) and \( \alpha_{y2}/\alpha \) satisfy the equations \( \frac{d(\alpha_{y1}/\alpha)}{dt}|_{\mu=\mu_0} = 0 \), i.e.,

\[
\left( \frac{\Lambda/\mu_0}{\Lambda/\mu_0} \right)^{\delta_{y1}} \left( \frac{\Lambda/\mu_0}{\Lambda/\mu_0} \right)^{\delta_{y2}} \left( \frac{\alpha_{y1}/\alpha}{\alpha_{y2}/\alpha} \right) = \left( \frac{\Lambda/\mu_0}{\Lambda/\mu_0} \right)^{\delta_{y1}} \left( \frac{\Lambda/\mu_0}{\Lambda/\mu_0} \right)^{\delta_{y2}}.
\]

*One can easily restore the omitted coefficients, associating them with the exponents \( \delta \)'s.
Figure 1: The Feynman diagrams contributing to $\gamma_g$. $\chi$ represents $L$ and $R$ in the diagrams (b).

Figure 2: The Feynman diagrams contributing to $\gamma_{L/Ri}$.

Figure 3: The Feynman diagrams contributing to $\gamma_H$. The index $k$ runs over all generations in the diagrams (b).
Then the solutions are found as,

\[
\left( \frac{\alpha_{y1}}{\alpha} \right)^* = D^{-1} \left[ \left( \frac{\Lambda}{\mu_0} \right)^{\delta_{1}^y + \delta_{2}^y} - \left( \frac{\Lambda}{\mu_0} \right)^{\delta_{2}^y + \delta_{y_2}^H} \right],
\]

(2.9)

\[
\left( \frac{\alpha_{y2}}{\alpha} \right)^* = D^{-1} \left[ \left( \frac{\Lambda}{\mu_0} \right)^{\delta_{2}^y + \delta_{1}^y} - \left( \frac{\Lambda}{\mu_0} \right)^{\delta_{1}^y + \delta_{y_1}^H} \right],
\]

(2.10)

with the determinant of the matrix on the left hand side of Eq. (2.8):

\[
D = \left( \frac{\Lambda}{\mu_0} \right)^{\delta_{1}^y + \delta_{2}^y} - \left( \frac{\Lambda}{\mu_0} \right)^{\delta_{2}^y + \delta_{y_2}^H}.
\]

(2.11)

Since the determinant \( D \), which is non-negative by definition, must be “positive” in order to have definite solutions, the following condition must be satisfied (Condition I)

\[
\delta_{1}^y + \delta_{2}^y > \delta_{y_1}^H + \delta_{y_2}^H.
\]

(2.12)

Condition I implies that at least one of \( \delta^y \) must be larger than \( \delta^H \). Moreover, the requirement that the solutions (2.9) and (2.10) should be positive imposes additional constraints for \( \delta \)'s,

(Condition II)

\[
\begin{align*}
\delta_{1}^y + \delta_{2}^y &> \delta_{2}^y + \delta_{y_2}^H, \\
\delta_{2}^y + \delta_{1}^y &> \delta_{1}^y + \delta_{y_1}^H.
\end{align*}
\]

(2.13) (2.14)

To see the fixed-point solutions more explicitly, let us define \( R_i \):

\[
R_1 = D \left[ \left( \frac{\Lambda}{\mu} \right)^{\delta_{1}^y + \delta_{2}^y} - \left( \frac{\Lambda}{\mu} \right)^{\delta_{2}^y + \delta_{y_2}^H} \right]^{-1} \frac{\alpha_{y1}}{\alpha},
\]

(2.15)

\[
R_2 = D \left[ \left( \frac{\Lambda}{\mu} \right)^{\delta_{2}^y + \delta_{1}^y} - \left( \frac{\Lambda}{\mu} \right)^{\delta_{1}^y + \delta_{y_1}^H} \right]^{-1} \frac{\alpha_{y2}}{\alpha}.
\]

(2.16)

Under Condition I, we have \( D \simeq (\Lambda/\mu)^{\delta_{1}^y + \delta_{2}^y} \) and their RGEs are written

\[
\begin{align*}
\frac{dR_1}{dt} &= -\frac{\alpha}{2\pi} \left( \frac{\Lambda}{\mu} \right)^{\delta_{1}^y} R_1 \left[ R_1 + R_2 \left( \frac{\Lambda}{\mu} \right)^{\delta_{2}^y + \delta_{y_2}^H - \delta_{1}^y} - 1 \right],
\quad (2.17) \\
\frac{dR_2}{dt} &= -\frac{\alpha}{2\pi} \left( \frac{\Lambda}{\mu} \right)^{\delta_{2}^y} R_2 \left[ R_1 \left( \frac{\Lambda}{\mu} \right)^{\delta_{2}^y + \delta_{y_1}^H - \delta_{1}^y - \delta_{2}^y} + R_2 - 1 \right].
\quad (2.18)
\end{align*}
\]

Condition II implies that the \( R_2 \) (\( R_1 \)) term in the first (second) equation is suppressed enough than the other terms. We then find from Eqs. (2.17) and (2.18) that \( R_i \) converge
into $R_i = 1$. The convergence behavior onto their fixed points is important to regard the fixed-point values as boundary conditions for the realistic quark Yukawa couplings. If the above two conditions are satisfied, the RGEs of $R_i$ are solved as

$$\frac{R_i(t) - 1}{R_i(t)} = \xi_i \frac{R_i(0) - 1}{R_i(0)},$$

with $R_i(0)$ being initial values at the cutoff scale $\Lambda$. The quantity $\xi_i$ is obtained

$$\xi_i = \frac{1}{E_i(t)} \left( \frac{\alpha(t)}{\alpha(0)} \right),$$

where

$$E_i(t) = \exp \left[ \int_0^t dt' \frac{1}{2\pi} \left( \frac{\Lambda}{\mu'} \right)^{\delta_i} \right].$$

The quantity $\xi_i$ measures the rapidity of convergence onto the fixed points $R_i = 1$. In extra-dimensional models, the Yukawa couplings converge rapidly even in asymptotically-free gauge theories, if the relation $\delta_i^y > \delta_y$ is satisfied. That is in contrast to the slow convergence in four-dimensional asymptotically-free gauge theories. This property is interesting for constructing asymptotically-free models with strong infrared convergence [7]. Note that even in the case of $\delta_i^y < \delta_y$, strong convergence of Yukawa couplings is utilized in infrared-free models ($b < 0$) [8].

The dashed lines in Figure 4 show typical running behavior of $\alpha_{y_1}$ and $\alpha_{y_2}$ calculated from RGEs (2.6) and (2.7). In this figure, we have used the RGEs (2.6) and (2.7). The solid lines which correspond to the fixed-point solutions (2.9) and (2.10). From the figure, we see that the good convergence to the fixed points and a large hierarchy between two couplings are indeed viable.

It is straightforward to extend the above discussions to the cases with more than two generations. First, to obtain definite fixed-point values, at least one of the Yukawa terms in matter anomalous dimensions should be larger than that in the Higgs anomalous dimension, i.e. $\delta_i^y > \delta_y^H$ (the non-vanishing determinant condition). Furthermore, the positiveness of fixed-point values must be satisfied. That requires the condition similar to (2.13), etc. If these conditions are satisfied, Yukawa couplings $\alpha_{y_i}$ are fixed by the infrared fixed-point values, $\alpha_{y_i} \sim (\mu_0/\Lambda)^{\delta_i^y - \delta_y} \alpha$ at the KK threshold scale $\mu_0$. 

6
Figure 4: Typical running behaviors of Yukawa couplings. The exponents in the RGEs are assumed as follows: $\delta_y = 2$, $\delta_y^1 = 5$, $\delta_y^2 = 4$, $\delta^1_h = 3$, $\delta^2_h = 2$, $\delta^1_g = 2$ and $\delta^2_g = 4$. The solid lines are fixed-point solutions (that are energy-dependent). The dashed and dot-dashed lines denote the RGE running of $\alpha_{y_1}$ and $\alpha_{y_2}$, respectively. In the figure, we take two initial values, $\alpha_{y_1}(0) = \alpha_{y_2}(0) = 1$ and 0.1.
3 Realization for hierarchies

We have derived Conditions I and II for realizing hierarchically suppressed couplings on infrared fixed points of RG evolution. In this section, we propose several mechanisms to realize the above conditions.

First, we consider the condition to have generation-dependent fixed-point values of Yukawa couplings. The beta-function of Yukawa coupling is given by \( \beta \propto y(\gamma_L + \gamma_R + \gamma_H) \). If the Higgs contribution, \( \gamma_H \), is dominant, generation-dependent terms become sub-dominant since the \( \gamma_H \) contribution commonly appears in all Yukawa beta-functions. A key ingredient for Yukawa hierarchy is therefore to suppress the Higgs anomalous dimension: the matter contributions, \( \gamma_L \) and/or \( \gamma_R \), should be larger than \( \gamma_H \), i.e. \( \delta_y > \delta^H_y \). Note that this condition corresponds to Condition I. There are two possibilities for suppressing the Higgs effects: (i) the couplings between Higgs zero-modes and KK modes of \( \Psi_L \) and \( \Psi_R \) are forbidden and (ii) the loop diagrams contributing to \( \gamma_H \) cancel out. We describe the models satisfying the condition (i) in sections 3.1 and 3.2, and the condition (ii) in section 3.3.

3.1 Charge conservation rule

Generally, each bulk field is allowed to have various values of KK momentum under the compactification. At interaction vertices, however, the momentum conservation or some charge-conservation rules associated with extra dimensions restricts possible combinations of KK modes. To put it more precisely, the loop diagrams where two KK modes extend in common directions of extra dimensions can contribute to wave-function renormalization.

Now let us consider the Yukawa term \( \Psi_L \Psi_R H \). The KK modes contributing to the anomalous dimension \( \gamma_{L(R)} \) (Fig. 2b) are those along the directions to which \( \Psi_{R(L)} \) and \( H \) extend in common. The KK modes with nonzero charges for other directions cannot propagate. Likewise, in Fig. 3b, only the KK modes of \( \Psi_L \) and \( \Psi_R \) extending into same extra dimensions contribute to the Higgs anomalous dimension. It is hence required for suppressing \( \gamma_H \) to reduce the number of dimensions that \( \Psi_L \) and \( \Psi_R \) expand in common. For example, in the case that there is no common extra dimensions between \( \Psi_L \) and \( \Psi_R \), we obtain \( \delta^H_y = 0 \) and the anomalous dimension \( \gamma_H \) reduces to logarithmic behavior. Then, assuming the Higgs field extends to some numbers of extra dimensions, we can obtain a large \( \delta^L_y \) and/or \( \delta^R_y \) whose values are generation-dependent and thereby induce hierarchies.

To see how this mechanism works, consider a two-generation model with the configuration
shown in Table 1. In the table, each circle denotes that the corresponding field propagates through that direction. With this setup, we have $\delta^H = 0$ because $\Psi_L$ and $\Psi_R$ have no overlapping dimensions (except for our four dimensions). We also find $\delta_1^y = 1$ and $\delta_2^y = 2$. It is interesting to note that these non-zero powers come from the anomalous dimensions of the right-handed field, $\gamma_{R_i}$, while those of the left-handed fields, which spread into extra dimensions, do not contribute to $\delta_i^y$. The above model thus realizes a hierarchy under the RG evolutions down to the infrared.

In this scenario, the number of extra dimensions is responsible for hierarchically different mass parameters. For example, since the perturbative string theories demand six extra spatial dimensions, it can generate at most six mass gaps. Another interesting point is that the ‘left-right asymmetry’ is essential for generating family-dependences in the beta functions, $\beta_i \propto y_i(\gamma_{L_i} + \gamma_{R_i} + \gamma_H)$. We will apply this left-right asymmetric scenario to find a realistic form of quark mass matrices in later section, including generation-mixing effects.

### 3.2 $N = 2$ hypermultiplet

Another mechanism satisfying the condition (i) is to utilize $N = 2$ (or larger) supersymmetry. In generic $N = 2$ SUSY models, Yukawa couplings among $N = 2$ hypermultiplets are forbidden by supersymmetry.

We suppose that the models consist of two parts. One is a four-dimensional theory derived from higher dimension by orbifold ($Z_2$) projections. It is noted that the $Z_2$ symmetry is only responsible for having chiral nature of matter fields and not for suppressing anomalous dimensions. The other sector is a genuine four-dimensional theory (defined on the $Z_2$ fixed point).
point). In addition, we assume that the former part respects $N = 2$ SUSY before the projection and forbids Yukawa interactions among hypermultiplets in the model. On the other hand, there could exist Yukawa couplings between the hypermultiplets and $N = 1$ superfields on the four-dimensional theory even with this assumption.

For simplicity, let us again consider a two-generation model with the Higgs $H$ and the second-generation fields $\Psi_{L,R_2}$ coming from $N = 2$ multiplets while $\Psi_{L,R_1}$ belong to four-dimensional $N = 1$ multiplets. In this case, all the Yukawa couplings of the KK modes of $H, \Psi_{L,2}$ and $\Psi_{R,2}$ are not allowed and consequently we have $\delta^H = \delta^\nu = 0$. Since the Yukawa coupling of the first-generation fields is allowed, we obtain $\delta^\nu \neq 0$ from the Higgs KK-modes contribution. We thus obtain a Yukawa hierarchy between the generations. It turns out that the Yukawa coupling originated from $N = 1$ multiplets has power-law RG running while that from $N = 2$ theory runs logarithmically (see Table 2). The order of hierarchy is determined by the number of extra dimensions in which the Higgs multiplet lives. Note that with this mechanism, we can accomplish only one difference between the couplings: the power-law and logarithmic running behaviors.

<table>
<thead>
<tr>
<th>$H$</th>
<th>$N = 2$ hyper</th>
<th>$N = 1$</th>
<th>$\rightarrow$ power running</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi_{L_1, R_1}$</td>
<td>$N = 1$</td>
<td>$N = 2$ hyper</td>
<td>$\rightarrow$ logarithmic running</td>
</tr>
</tbody>
</table>

Table 2: The suppression of anomalous dimensions due to the absence of Yukawa couplings among hypermultiplets.

### 3.3 $N = 2$ vector multiplet

In this subsection, we discuss the possibility that the KK contributions to the anomalous dimension $\gamma_H$ (Fig. 3) cancel out, i.e. the condition (ii) stated in the beginning of this section. Recall that in models with $N = 2$ SUSY, hypermultiplets receive no wave function renormalization. This is because Yukawa couplings are forbidden among hypermultiplets and in addition, the radiative corrections from an $N = 2$ vector multiplet are cancelled.

---

\[ \text{If one assumes the KK-momentum conservation, either of } \Psi_{L,R_1} \text{ should be assigned to an } N = 2 \text{ hypermultiplet. Otherwise, we have no enhanced RG-running effect (} \delta^\nu = 0). \text{ It may be, however, natural not to assume such a conservation rule as long as one exploits orbifold symmetries.} \]
between $N = 1$ vector and chiral multiplets. We use this non-renormalization property to suppress the Higgs anomalous dimension.

Let us consider a two-generation model. We assume that $\Psi_{L1}$ is originated from an $N = 2$ vector multiplet. That is, in higher dimension, the standard gauge symmetry is enhanced to a larger gauge group $G$ and broken by some symmetry (projection), leaving $\Psi_{L1}$ as a part of broken gauge multiplet. In this case, we should suppose that the Higgs field $H$ and $\Psi_{R1}$, which are combined with $\Psi_{L1}$ into a Yukawa coupling, are originated from an $N = 2$ hypermultiplet. Note that above the KK threshold, since the $\Psi_{L1}$ field belongs to the $N = 2$ vector multiplet of the gauge group $G$, $H$ and $\Psi_{R1}$ are in a single representation of $G$. With this assumption, there is no radiative corrections in the Higgs anomalous dimension. The anomalous dimension $\gamma_{R1}$ is also suppressed while $\gamma_{L1}$ receives large contributions from KK modes.

Now there are two options for the assignment of the second-generation fields. One is to consider $\Psi_{L,R2}$ have no KK modes (live only in four-dimensional spacetime). Then no KK contribution to $\gamma_H$ appears, and moreover the anomalous dimensions $\gamma_{L,R2}$ are also zero due to the charge conservation law discussed before. The other choice is that we assume the $\Psi_{L,R2}$ fields belong to $N = 2$ hypermultiplets and have no Yukawa coupling. After all, in both cases, we find no enhanced behaviors of the RG evolution for the second-generation Yukawa coupling. We thus have a hierarchy between two Yukawa couplings.

The appearance of suppressed Yukawa coupling depends on whether it is originated from $N = 2$ vector multiplet. The mechanism can hence generate a single mass gap between generations.

<table>
<thead>
<tr>
<th>$H$</th>
<th>$N = 2$ hyper</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi_{L1}$</td>
<td>$N = 2$ vector</td>
</tr>
<tr>
<td>$\Psi_{R1}$</td>
<td>$N = 2$ hyper $\rightarrow$ power running</td>
</tr>
<tr>
<td>$\Psi_{L2}, \Psi_{R2}$</td>
<td>$N = 2$ hyper or $N = 1$ $\rightarrow$ logarithmic running</td>
</tr>
</tbody>
</table>

Table 3: The suppression of anomalous dimensions due to the vanishing anomalous dimensions of hypermultiplets.
3.4 Constraint on gauge contribution

So far, we have considered generic fixed-point values of Yukawa couplings and their suppression mechanisms. If one applies these mechanisms to the observed fermion mass hierarchy, one should impose an additional phenomenological constraint on the beta-function coefficients (the exponents of $\Lambda/\mu$). That is, the fixed-point values must be of order one or less because the $SU(3)$ gauge and top Yukawa couplings are of order one and the other Yukawa couplings are smaller than them. It is found from the fixed-point solutions discussed in section 2 that this phenomenological requirement results in the suppression of KK gauge contributions as seen below.

Since all the KK modes contribute to the gauge anomalous dimension, we naively have $\delta_g \geq \delta_y^i$. If this is the case, it follows from the fixed-point solutions that the leading terms turn out to be $\alpha_y \sim (\mu_0/\Lambda)^{\delta_y - \delta_g} \alpha$. However, with the requirement of gauge invariance, vector multiplets must spread into the extra dimensions in which charged fields live. That may imply $\delta_g \geq \delta_y^i$ and the available values of Yukawa couplings become larger than order one: $\alpha_y \sim (\mu_0/\Lambda)^{c_i} \alpha$ with $c_i < 0$. If one identifies $\alpha$ with the $SU(3)$ gauge coupling, one can only have Yukawa couplings larger than the top Yukawa coupling on the fixed point. We are thus led to a phenomenological requirement that the exponent $\delta_g$ should be smaller than those in the matter anomalous dimensions. It is interesting that this condition is favorable for Yukawa couplings to realize a rapid convergence into the infrared fixed points.

One way to satisfy the above condition is to require the coefficient of $(\Lambda/\mu)^{\delta_g}$ term vanishes. The exponent of the gauge anomalous dimension is then reduced to a smaller one. Generally, the gauge beta function is determined by the gauge and matter loop diagrams (Fig. 1). In the matter loops, all gauge non-singlet fields can propagate and contribute to the anomalous dimensions. In this case, the beta function is expressed in terms of $N = 1$ basis,

$$\beta(\alpha) \propto -3C_2(G) \left( \frac{\Lambda}{\mu} \right)^{\delta_V} + \sum_M T(R_M) \left( \frac{\Lambda}{\mu} \right)^{\delta^M} + \sum_{M'} T(R_{M'}) \left( \frac{\Lambda}{\mu} \right)^{\delta^{M'}} + \cdots, \quad (3.1)$$

where $C_2(G)$ is the quadratic Casimir and $T(R)$ denotes the Dynkin index of the representation $R$. In the above, the first term is the contribution of $N = 1$ vector (and ghost) fields and the following part comes from that of the KK matter fields ($\delta^M > \delta^{M'} > \cdots$). The exponent $\delta_g$ in the beta function is fixed by $\delta_V$, $\delta^M$, and so on. Suppose that the first two terms in Eq. (3.1) cancel, we have a possibility of $\delta_g < \delta_y^i$. An interesting by-product of this cancel-
lation is that it is preferable to infrared fixed-point behavior. That is, in the beta-function, 
the negative contribution from gauge fields is cancelled and only the positive matter contributions remain. The resultant infrared-free behavior of gauge coupling enhances the rate at which Yukawa couplings approach to the fixed points [8].

We make a remark on the naturalness of this cancellation. If we regard the models as low-energy effective descriptions of higher-dimensional theories, the fields extended into higher dimension seem relevant to larger supersymmetry, e.g. $N \geq 2$ SUSY. It is hence possible that the leading power-law contributions from such fields cancel out because of the larger symmetry. This line of thought could justify a realization of finiteness condition, starting from a superstring theory [9]. Such a kind of condition generally imposes severe restrictions in constructing realistic models.

We could further consider more generic situations where various gauge fields live in various dimensions, that is, we have different gauge group in each step of compactifications. In this case, the beta-function coefficient is written as follows:

$$\beta(\alpha) \propto \left\{-3C_2(G_V) + \sum_M T(R_M)\right\} \left(\frac{\Lambda}{\mu}\right)^{\delta_V}$$

$$+ \left\{-3C_2(G_{V-1}) + \sum_{M'} T(R_{M'})\right\} \left(\frac{\Lambda}{\mu}\right)^{\delta_V-1} + \cdots.$$  \hspace{1cm} (3.2)

Here, $G_x$ denotes the gauge group in the higher dimension ($G_V \supset G_{V-1} \supset \cdots$). If the coefficients of power-law terms vanish in each curly bracket, the resultant $\delta_g$ is quite reduced and we could have hierarchical structures of Yukawa fixed points between larger number of generations. This type of assumption also has strong constraints on model building.

4 Generation mixing

Until now, we have assumed that the Yukawa matrix takes a diagonal form. In this section, we analyze further the fixed-point structure of Yukawa matrices including the generation-mixing elements. We show that, under Conditions I and II in section 2, a hierarchical structure of Yukawa matrices is realized even if we include generation mixing effect. As an example, we here take the scenario presented in section 3.1.
4.1 Fixed-point structures

First, we deal with a two-generation case for simplicity. Let us suppose the following Yukawa couplings,

\[ W = y_{ij} \tilde{\Psi}_L^i \tilde{\Psi}_R^j H, \quad (i, j = 1, 2). \]  

(4.1)

In addition, we assume that the ‘left-handed’ fields \( \tilde{\Psi}_L \) and Higgs \( H \) can propagate through the extra dimensions while, the ‘right-handed’ fields \( \tilde{\Psi}_R \) reside in our four-dimensional world. All the Yukawa couplings \( y_{ij} \) are expected to take \( O(1) \) values at the high-energy scale \( \Lambda \). As explained in section 3.1, with this configuration the anomalous dimension of the Higgs field receives only logarithmic corrections from the Yukawa terms due to the charge conservations associated with the extra dimensions. On the other hand, we obtain the anomalous dimensions of \( \tilde{\Psi}_{L,R} \) neglecting sub-dominant corrections,

\[ \gamma_{L_{ij}} = \frac{c_i}{16 \pi^2} g^2 \left( \frac{\Lambda}{\mu} \right)^{\delta_i} \delta_{ij}, \]  

(4.2)

\[ \gamma_{R_{ij}} = - \sum_{k=1,2} a_k \frac{16 \pi^2}{\mu^2} (y^T)_{ik} y_{kj} \left( \frac{\Lambda}{\mu} \right)^{\delta_k}. \]  

(4.3)

The anomalous dimension \( \gamma_L \) receives the KK contributions from gauge multiplets but not from the Yukawa terms, \( y_{ij} \). As for \( \gamma_R \), the opposite situation occurs. Each contribution corresponds to the diagram depicted in Fig. 2. In the above expressions of \( \gamma \)'s, the coefficients \( a_i \) and \( c_i \) are order one quantities including the group indices and the volume factors. In the following, we will neglect these coefficients as before.

To see the infrared fixed-point structure, let us write down the beta-functions of the Yukawa couplings:

\[ \frac{dy_{ij}}{dt} = \frac{1}{16 \pi^2} \beta_{ij} = \sum_k \left( \gamma_{L_{ik}} y_{kj} + y_{ik} \gamma_{R_{jk}} \right), \]  

(4.4)

or more explicitly,

\[ \beta_{11} = \left[ y_{11} g^2 \left( \frac{\Lambda}{\mu} \right)^{\delta_1} - y_{11} \left( y_{11}^2 + y_{12}^2 \right) \left( \frac{\Lambda}{\mu} \right)^{\delta_1} - y_{21} \left( y_{11} y_{21} + y_{12} y_{22} \right) \left( \frac{\Lambda}{\mu} \right)^{\delta_2} \right], \]

\[ \beta_{12} = \left[ y_{12} g^2 \left( \frac{\Lambda}{\mu} \right)^{\delta_1} - y_{12} \left( y_{11}^2 + y_{12}^2 \right) \left( \frac{\Lambda}{\mu} \right)^{\delta_1} - y_{22} \left( y_{11} y_{21} + y_{12} y_{22} \right) \left( \frac{\Lambda}{\mu} \right)^{\delta_2} \right], \]

\[ \beta_{21} = \left[ y_{21} g^2 \left( \frac{\Lambda}{\mu} \right)^{\delta_2} - y_{11} \left( y_{11} y_{21} + y_{12} y_{22} \right) \left( \frac{\Lambda}{\mu} \right)^{\delta_1} - y_{21} \left( y_{21}^2 + y_{22}^2 \right) \left( \frac{\Lambda}{\mu} \right)^{\delta_2} \right], \]

14
\[ \beta_{22} = \left[ y_{22} g^2 \left( \frac{\Lambda}{\mu} \right)^{\delta_2^g} - y_{12} \left( y_{11} y_{21} + y_{12} y_{22} \right) \left( \frac{\Lambda}{\mu} \right)^{\delta_1^g} \right] - y_{22} \left( y_{21}^2 + y_{22}^2 \right) \left( \frac{\Lambda}{\mu} \right)^{\delta_2^g}. \]  

(4.5)

The beta-functions are symmetric under an exchange of the right-handed-type indices 1 and 2 because no particular assumption is made on the right-handed fields \( \Psi_R \). Though the beta-functions seem to have complicated structures, as seen below we can find the fixed-point solutions for several definite combinations of Yukawa couplings. In what follows, we assume that the beta-function of the gauge coupling do not have larger power of \( (\Lambda/\mu)^t \) than \( \delta_\chi \). Note that this condition guarantees the strong convergence of Yukawa couplings to the infrared fixed points (see section 2). In this case, neglecting lower-order terms of \( (\Lambda/\mu) \), the fixed-point solutions are derived from the Yukawa beta-functions themselves without including the gauge beta functions.

Now consider the following combinations of Yukawa couplings:

\[ X_1 \equiv y_{11}^2 + y_{12}^2, \quad X_2 \equiv y_{21}^2 + y_{22}^2, \quad Z \equiv y_{11} y_{21} + y_{12} y_{22}. \]  

(4.6)

The beta-functions for \( X_1, X_2 \) and \( Z \) are given by

\[ \frac{dX_1}{dt} = \frac{2}{16\pi^2} \left[ \left( \frac{\Lambda}{\mu} \right)^{\delta_{g1}} g^2 X_1 - \left( \frac{\Lambda}{\mu} \right)^{\delta_1} X_1^2 - \left( \frac{\Lambda}{\mu} \right)^{\delta_2} Z^2 \right], \]  

(4.7)

\[ \frac{dX_2}{dt} = \frac{2}{16\pi^2} \left[ \left( \frac{\Lambda}{\mu} \right)^{\delta_{g2}} g^2 X_2 - \left( \frac{\Lambda}{\mu} \right)^{\delta_1} X_2^2 - \left( \frac{\Lambda}{\mu} \right)^{\delta_2} Z^2 \right], \]  

(4.8)

\[ \frac{dZ}{dt} = \frac{1}{16\pi^2} Z \left[ \left( \frac{\Lambda}{\mu} \right)^{\delta_{g1}} g^2 + \left( \frac{\Lambda}{\mu} \right)^{\delta_{g2}} g^2 - 2 \left( \frac{\Lambda}{\mu} \right)^{\delta_1} X_1 - 2 \left( \frac{\Lambda}{\mu} \right)^{\delta_2} X_2 \right]. \]  

(4.9)

From these, we find the fixed-point solutions at the KK threshold scale \( \mu_0 \),

\[ X_1^* = g^2 \left( \frac{\Lambda}{\mu_0} \right)^{\delta_{g1} - \delta_1}, \quad X_2^* = g^2 \left( \frac{\Lambda}{\mu_0} \right)^{\delta_{g2} - \delta_2}, \quad Z^* = 0. \]  

(4.10)

It is easily shown that these solutions are infrared stable against small fluctuations about the solutions. We thus find a hierarchy between the Yukawa couplings as the infrared fixed-point predictions with one parameter \( \theta \),

\[ y_{ij} \simeq \begin{pmatrix} x \cos \theta & x \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \left( \frac{\mu_0}{\Lambda} \right)^{(\delta_2 - \delta_1)/2} g, \]  

(4.11)

\[ x \equiv \left( \frac{\mu_0}{\Lambda} \right)^{(\delta_2 - \delta_1 - \delta_1 + \delta_1)/2}. \]  

(4.12)
The parameter $\theta$ and the relative sign in the matrix are to be fixed by initial values of the Yukawa couplings.

In this way, we find a particular form of the low-energy Yukawa matrix as the fixed-point solutions. For this, we have only assumed that the anomalous dimensions of the left-handed fields, $\gamma_L$, dominate the beta-functions of Yukawa couplings. An interesting feature is that on the fixed point the Yukawa matrix realizes a hierarchy of order $x$ even when the matrix elements do not have any hierarchies in initial conditions. The hierarchy is induced by the RG evolution with different values of $\delta_i$’s, i.e. the numbers of extra dimensions which the corresponding fields ($\Psi_L$ in the present case) can feel. After diagonalizing the matrix on the fixed point, one obtains the eigenvalues with a ratio $y_1 : y_2 = x : 1$, where the larger eigenvalue is of order $$(\mu_0/\Lambda)^{(\delta_2-\delta_g)^2/2}.$$

Another interesting point is the mixing of generations. It is found that the mixing angles of the left-handed fields $\Psi_{L_i}$ are small in the infrared region. (In the particular form of matrix (4.11), the mixing angle of $\Psi_L (= (Z/X_2)^2)$ is zero.) This is a remarkable result: that gives a natural explanation why the observed quark mixing angles are so small. The mixing angles are expected to be of order one at high energy but they are reduced to very small values under the RG evolution with large KK-mode effects. More interestingly, the smallness of mixing angles is realized on the infrared stable fixed point and does not depend on details of models (symmetry properties, etc.). This novel explanation of the small CKM angles has a geometrical origin and is different from other approaches; the Higgs mixing, the higher-dimensional operators, etc.

Exactly speaking, with a simple field configuration discussed above, the mixing angle of the left-handed fields is almost zero on the infrared fixed point. However, a more complicated structure, for example, where the right-handed fields $\Psi_R$ also feel some number of extra dimensions, can induce nonzero mixing angles between the generations. We leave this issue to future investigations, and in this paper, only do order-of-magnitude estimations of each matrix element.

We comment on the mixing angles of the right-handed fields $\Psi_R$. With the matrix form (4.11), the right-handed mixing is given by $\tan \theta$. It cannot be settled on the infrared fixed-point values but it is natural to expect that it is of order one. Note that the above analysis can be applied to the Yukawa couplings of the fields with strong gauge interactions. When one considers the grand unification scenarios, the lepton mixing angles are also predicted by the gauge symmetry. In particular, in the $SU(5)$ formalism, the right-handed down quarks
and the left-handed charged leptons are contained in a single multiplet 5* and their mixing angles are closely connected. Both mixing angles therefore become large if one assumes that the behavior of down-quark Yukawa matrix (i.e. 10 5* H_d couplings) is described by the above mechanism. The large mixing angles give the boundary conditions at the grand unification scale. This scenario could explain the large lepton mixing recently observed in the Superkamiokande experiment [10].

4.2 A toy model

We here adopt the mechanism for generating hierarchies among the Yukawa couplings discussed in the previous section, and construct a model (field configurations) which leads to the realistic up and down-quark mass hierarchies. We only perform the order-of-magnitude estimations.

The essential points of the mechanism are: (i) the left-handed fields propagate through the extra dimensions in order to obtain mass hierarchies besides the small generation mixing. (ii) the overlapping of the left-handed fields with the Higgs field determines the sizes of hierarchies. From (i), the configuration of the SU(2) doublet quarks Q_i (i = 1, 2, 3) sets the mass hierarchies between the generations. In this case, one may naively wonder that the up and down quarks have only the same order of hierarchies. However, notice that the hierarchical factors are determined by the overlap of Q_i and the Higgs fields. That is, if the up-type Higgs H_u and the down-type one H_d feel different numbers (different directions) of extra dimensions, one can realize different hierarchical structure between the up and down parts. Let us consider the field configuration shown in Table 4. We can easily extend the

<table>
<thead>
<tr>
<th></th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q_1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q_2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q_3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>H_u</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>H_d</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4: An example of field configuration.

previous result of matrix form (4.11) to the three generation case and find similar fixed-point

17
structures for several combinations of Yukawa couplings such as the determinant, the mixing angles, the ratios, and so on. As a result, we obtain the following hierarchical forms of the up and down Yukawa couplings at $\mu_0$:

$$
\begin{align*}
 y_u & \sim \begin{pmatrix} \epsilon^6 & \epsilon^6 & \epsilon^6 \\ \epsilon^3 & \epsilon^3 & \epsilon^3 \\ 1 & 1 & 1 \end{pmatrix} \epsilon^{-\delta'_g} g, \\
y_d & \sim \begin{pmatrix} \epsilon^3 & \epsilon^3 & \epsilon^3 \\ \epsilon^2 & \epsilon^2 & \epsilon^2 \\ 1 & 1 & 1 \end{pmatrix} \epsilon^{-\delta'_g} g,
\end{align*}
$$

(4.13)

(4.14)

where $\epsilon \equiv (\mu_0/\Lambda)^{1/2}$ and we have assumed all the gauge contributions are same order (denoted by $\delta'_g$), for simplicity. The Yukawa couplings are denoted in the basis of $y_{ij} \Psi_L^i \Psi_R^j$ form. By diagonalizing the above matrices, we obtain the mass hierarchies given by

$$
\begin{align*}
 m_u : m_c : m_t & \sim \epsilon^6 : \epsilon^3 : 1, \\
m_d : m_s : m_b & \sim \epsilon^3 : \epsilon^2 : 1.
\end{align*}
$$

(4.15)

(4.16)

With an assumption $\epsilon \sim O(1/10)$, the obtained structures are roughly consistent with the experimental data. Note that the predictions are derived from the infrared fixed points and then independent of the high-energy initial values. It is also interesting that a larger hierarchy in the up-quark sector than in the down-quarks is accomplished during the RG evolution. This setup predicts a large value of $\tan \beta$ and $\delta'_g = 0$ for the top quark mass.

In this way, we can obtain various types of hierarchical forms of Yukawa couplings on the infrared fixed points under the RG evolution with the power-running effects. The essential point to have various hierarchies is how the fields in the models spread in the extra dimensions.

5 Summary and discussion

In this paper, we have investigated a way to realize the generation mass hierarchies in models with extra spatial dimensions. We have exploited the enhanced (power-law) RG running effects originated from KK excited modes. The power-law running effects significantly change the behaviors of Yukawa couplings and induce hierarchically different order of couplings as infrared fixed-point values. It is interesting to determine precise low-energy
values of couplings from the fixed points since the predictions become independent of the initial condition at high-energy scale. When applying this approach to reproduce realistic spectrum, we have found several requirements to be satisfied. The clarification of these conditions is one of the main parts of this paper. First, the definite infrared fixed-point solution should exist. That gives several constraints on the beta-function coefficients and hence restricts the possible forms of higher-dimensional models. Second, the fixed-point values should be flavor-dependent. It turns out to be a rather non-trivial problem to obtain the flavor-dependent and suppressed Yukawa couplings. Naive constructions of the models result in flavor-independent ('democratic') type of mass matrices. This is mainly because the Higgs anomalous dimensions enter in all the Yukawa beta-functions. We have presented several mechanisms to avoid this problem and given a simple example for each case.

We have also studied about the generation-mixing Yukawa couplings. Under the RG evolution down to compactification scale, the off-diagonal elements of the Yukawa matrix receive power-law corrections from KK-mode contributions. We have analyzed a two-generation case by utilizing one of the suppression mechanisms presented in section 3. There only the left-handed fields as well as the Higgs field are assumed to live in the extra dimensions while the right-handed fields do not. We have shown that this assumption indeed generates mass hierarchies on the stable fixed points. More interestingly, the low-energy mixing angles of the left-handed fields become very small even when their initial values are large at high-energy region. This is due to the fact that the fields spread into the extra dimensions and therefore can never be accomplished in the usual four-dimensional scenarios. If we apply it to the quark sector, we can explain why the observed CKM angles are very small. On the other hand, the right-handed mixing angles are of order one. When one imagines a grand unified framework, the large lepton mixing, which has recently been observed, is related to the quark sector and could be explained in the same way.

In addition to the Yukawa hierarchy problem, supersymmetric models generally involve another flavor problem coming from sfermion mixing. They induce new sources of flavor violation such as flavor-changing neutral currents and may give severe additional bounds on the flavor structure of models. However, it is recently shown in Ref. [11] that when all Yukawa couplings are fixed by their fixed-point values as supposed in this paper, the squark flavor structure is aligned with that of the quarks and no additional flavor problem arises. Though we have not constructed explicit models, the obtained criteria give some hints in constructing realistic models.
Acknowledgments

The authors would like to thank T. Kugo for valuable discussions and comments. M. B. is supported in part by the Grants-in-Aid for Scientific Research No. 12047225(A2) and 12640295(C2) from the Ministry of Education, Science, Sports and Culture, Japan.
References


