Finite action, holographic conformal anomaly and quantum brane-worlds in d5 gauged supergravity

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We report our recent results concerning d5 gauged supergravity (dilatonic gravity) considered on AdS background. The finite action on such background as well as d4 holographic conformal anomaly (via AdS/CFT correspondence) are found. In such formalism the bulk potential is kept to be arbitrary, dilaton dependent function. Holographic RG in such theory is briefly discussed. d5 AdS brane-world Universe induced by quantum effects of brane CFT is constructed. Such brane is spherical, hyperbolic or flat one. Hence, the possibility of quantum creation of inflationary brane-world Universe is shown.

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1 Introduction

AdS/CFT correspondence may be realized in a sufficiently simple form as d5 gauged supergravity/boundary gauge theory correspondence. The reason is very simple: different versions of five-dimensional gauged SG (for example, \( N = 8 \) gauged SG which contains 42 scalars and non-trivial scalar potential) could be obtained as compactification (reduction) of ten-dimensional IIB SG. Then, in practice it is enough to consider 5d gauged SG classical solutions (say, AdS-like backgrounds) in AdS/CFT set-up instead of the investigation of much more involved, non-linear equations of IIB SG. Moreover, such solutions describe RG flows in boundary gauge theory (for a very recent discussion of such flows see \( ^3, ^4, ^44, ^5, ^6, ^7, ^8 \) and refs. therein). To simplify the situation in extended SG one can consider the symmetric (special) RG flows where scalars lie in one-dimensional submanifold of total space. Then, such theory is effectively described as d5 dilatonic gravity with non-trivial dilatonic potential. Nevertheless, it is still extremely difficult to make the explicit identification of deformed SG solution with the dual (non-conformal exactly) gauge theory. As a rule \(^4, ^7, \) only indirect arguments may be suggested in such identification.

From another side, the fundamental holographic principle in AdS/CFT form enriches the classical gravity itself (and here also classical gauged SG). Indeed, instead of the standard subtraction of reference background in making the gravitational action finite and the quasilocal stress tensor well-defined one introduces more elegant, local surface counterterm prescription. Within it one adds the coordinate invariant functional of the intrinsic boundary geometry to gravitational action. Clearly, that does not modify the equations of motion. Moreover, this procedure has nice interpretation in terms of dual QFT as standard regularization. The specific choice of surface counterterm cancels the divergences of bulk gravitational action. As a by-product, it also defines the conformal anomaly of boundary QFT.

Local surface counterterm prescription has been successfully applied to construction of finite action and quasilocal stress tensor on asymptotically AdS space in Einstein gravity \(^12, ^13, ^14, ^15, ^16 \) and in higher derivative gravity \(^17 \).

Such dual theory in massless case is, of course, classically conformally invariant and it has well-defined conformal anomaly. However, among the interacting theories only \( \mathcal{N} = 4 \) SYM is known to be exactly conformally invariant. Its conformal anomaly is not renormalized. For other, d4 QFTs there is breaking of conformal invariance due to radiative corrections which give contribution also to conformal anomaly. Hence, one can call such theories as non-conformal ones or not exactly conformally invariant. The conformal anomalies for such theories are explicitly unknown. Only for few simple theories (like scalar QED or gauge theory without fermions) the calculation of radiative corrections to conformal anomaly has been done up to two or three loops. It is a challenge to find exact conformal anomaly. Presumably, only SG description may help to resolve this problem.
Moreover, the generalization to asymptotically flat spaces is possible as it was first mentioned in ref.\textsuperscript{18}. Surface counterterm has been found for domain-wall black holes in gauged SG in diverse dimensions \textsuperscript{19}. However, actually only the case of asymptotically constant dilaton has been investigated there.

In the current report we present our recent results on the construction of finite action, consistent gravitational stress tensor and dilaton-dependent Weyl anomaly for boundary QFT (from bulk side) in five-dimensional gauged supergravity with single scalar (dilaton) on asymptotically AdS background. Note that dilaton is not constant and the potential is chosen to be arbitrary. The implications of results for the study of RG flows in boundary QFT are presented, in particular, the candidate c-function is suggested. The comparison with holographic RG is done as well.

As an extension, the brane-world solutions in dilatonic gravity are discussed (with quantum corrections). Indeed, after the discovery that gravity on the brane may be localized \textsuperscript{35} there was renewed interest in the studies of higher-dimensional (brane-world) theories. In particular, numerous works \textsuperscript{36} (and refs. therein) have been devoted to the investigation of cosmology (inflation) of brane-worlds. In refs.\textsuperscript{38,37,42} it has been suggested the inflationary brane-world scenario realized due to quantum effects of brane matter. Such scenario is based on large $N$ quantum CFT living on the brane \textsuperscript{38,37}. Actually, that corresponds to implementing of RS compactification within the context of renormalization group flow in AdS/CFT set-up. Note that working within large $N$ approximation justifies such approach to brane-world quantum cosmology as then quantum matter loops contribution is essential.

In the last section we report on the role of quantum matter living on the brane in the study of brane-world cosmology in 5d AdS dilatonic gravity with non-trivial dilatonic potential (bosonic sector of the corresponding gauged supergravity). We are mainly interested in the situation when the boundary of 5d AdS space represents a 4d constant curvature space whose creation (as is shown) is possible only due to quantum effects of brane matter. Thus, the possibility of dilatonic brane-world inflation induced by quantum effects is proved. In different versions of such scenario discussed here the dynamical determination of dilaton occurs as well. This finishes the discussion of our results in the study of AdS/CFT aspects of d5 gauged supergravity (bosonic sector).
2 Holographic Weyl anomaly for gauged supergravity with general dilaton potential

In the present section the derivation of dilaton-dependent Weyl anomaly from gauged SG will be given. This is based on \textsuperscript{23,48}.

We start from the bulk action of \(d+1\)-dimensional dilatonic gravity with the potential \(\Phi\)
\begin{equation}
S = \frac{1}{16\pi G} \int_{M_{d+1}} d^{d+1}x \sqrt{-\hat{G}} \left\{ \hat{R} + X(\phi)(\hat{\nabla}\phi)^2 + Y(\phi)\hat{\Delta}\phi + \Phi(\phi) + 4\lambda^2 \right\}.
\end{equation}

Here \(M_{d+1}\) is \(d+1\) dimensional manifold whose boundary is \(d\) dimensional manifold \(M_d\) and we choose \(\Phi(0) = 0\). Such action corresponds to (bosonic sector) of gauged SG with single scalar (special RG flow). In other words, one considers RG flow in extended SG when scalars lie in one-dimensional submanifold of complete scalars space. Note also that classical vacuum stability restricts the form of dilaton potential \textsuperscript{20}. As well-known, we also need to add the surface terms \textsuperscript{10} to the bulk action in order to have well-defined variational principle. At the moment, for the purpose of calculation of Weyl anomaly (via AdS/CFT correspondence) the surface terms are irrelevant.

We choose the metric \(\hat{G}_{\mu\nu}\) on \(M_{d+1}\) and the metric \(\hat{g}_{\mu\nu}\) on \(M_d\) in the following form
\begin{equation}
ds^2 \equiv \hat{G}_{\mu\nu}dx^{\mu}dx^{\nu} = \frac{l^2}{4}\rho^{-2}d\rho d\rho + \sum_{i=1}^{d} \hat{g}_{ij}dx^i dx^j, \quad \hat{g}_{ij} = \rho^{-1}g_{ij}.\end{equation}

Here \(l\) is related with \(\lambda^2\) by \(4\lambda^2 = d(d-1)/l^2\). If \(g_{ij} = \eta_{ij}\), the boundary of AdS lies at \(\rho = 0\). We follow to method of calculation of conformal anomaly as it was done in refs.\textsuperscript{21,22} where dilatonic gravity with constant dilaton potential has been considered.

The action (1) diverges in general since it contains the infinite volume integration on \(M_{d+1}\). The action is regularized by introducing the infrared cutoff \(\epsilon\) and replacing \(\int d^{d+1}x \rightarrow \int d^d x \int_\epsilon d\rho, \int_{M_d} d^d x \rightarrow \int d^d x \big|_{\rho = \epsilon}\).

We also expand \(g_{ij}\) and \(\phi\) with respect to \(\rho\): \(g_{ij} = g_{(0)ij} + \rho g_{(1)ij} + \rho^2 g_{(2)ij} + \cdots\), \(\phi = \phi_{(0)} + \rho \phi_{(1)} + \rho^2 \phi_{(2)} + \cdots\). Then the action is also expanded as a power series on \(\epsilon\). The subtraction of the terms proportional to the inverse power of \(\epsilon\) does not break the invariance under the scale transformation \(\delta g_{\mu\nu} = 2\delta \sigma g_{\mu\nu}\) and \(\delta \epsilon = 2\delta \sigma \epsilon\). When \(d\) is even, however, the term proportional to \(\ln \epsilon\) appears. This term is not invariant under the scale transformation and the subtraction of the \(\ln \epsilon\) term breaks the invariance. The variation of the \(\ln \epsilon\) term under
the scale transformation is finite when $\epsilon \to 0$ and should be canceled by the variation of the finite term (which does not depend on $\epsilon$) in the action since the original action (1) is invariant under the scale transformation. Therefore the $\ln \epsilon$ term $S_{\ln}$ gives the Weyl anomaly $T$ of the action renormalized by the subtraction of the terms which diverge when $\epsilon \to 0$ ($d = 4$)

$$S_{\ln} = -\frac{1}{2} \int d^4x \sqrt{-g} T.$$  (3)

The conformal anomaly can be also obtained from the surface counterterms, which is discussed in Section 3.

For $d = 4$, by solving $g_{(1)ij}$, $g_{(2)ij}$, $\phi_{(1)}$ and $\phi_{(2)}$ with respect to $g_{(0)ij}$, $\phi_{(0)}$ and by using the equations of motion, we obtain the following expression for the anomaly:

$$T = -\frac{1}{8\pi G} \left[ h_1 R^2 + h_2 R_{ij} R^{ij} + h_3 R^{ij} \partial_i \phi \partial_j \phi + h_4 R g^{ij} \partial_i \phi \partial_j \phi \\
+ h_5 \frac{R}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) + h_6 (g^{ij} \partial_i \phi \partial_j \phi)^2 \\
+ h_7 \left( \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right)^2 + h_8 g^{kl} \partial_k \phi \partial_l \phi \sqrt{-g} \partial_i (\sqrt{-g} g^{ij} \partial_j \phi) \right].$$  (4)

Here

$$h_1 = \left[ 3 \{(24 - 10\Phi)\Phi^6 + (62208 + 22464\Phi + 2196\Phi^2)\Phi^2 + 72\Phi^3 + \Phi^4\} \Phi'' \right] - 2\Phi' \{ (108 + 162\Phi + 7\Phi^2)\Phi'' \\
+ 72 \left( -8 + 14\Phi + \Phi^2 \right) V \} - 2\Phi'^2 \left\{ (6912 + 2736\Phi + 192\Phi^2 + \Phi^3)\Phi'' \right. \\
+ 4(11232 + 6156\Phi + 552\Phi^2 + 13\Phi^3)\Phi''V + 32 \left( -2592 + 468\Phi + 96\Phi^2 \\
+ 5\Phi^3 \right) V^2 \left. \right\} - 3(-24 + \Phi)(6 + \Phi)^2 \Phi'^2 \Phi''' \Phi'' + 8V' \} / \\
\left[ 16(6 + \Phi)^2 \left\{ -2\Phi'^2 + (24 + \Phi)\Phi'' \right\} \left\{ -2\Phi'^2 + (18 + \Phi)(\Phi'' + 8V) \right\}^2 \right]$$

$$h_2 = -\frac{3 \{(12 - 5\Phi)\Phi^2 + (288 + 72\Phi + \Phi^2)\Phi'' \}}{8(6 + \Phi)^2 \left\{ -2\Phi'^2 + (24 + \Phi)\Phi'' \right\}}$$  (5)

and $V(\phi) \equiv X(\phi) - Y'(\phi)$. The explicit forms of $h_3, \cdots h_8$ are given in $^{48}$. This expression which should describe dual d4 QFT of QCD type, with broken SUSY looks really complicated. The interesting remark is that Weyl anomaly is not integrable in general. In other words, it is impossible to construct the anomaly induced action. This is not strange, as it is usual situation for conformal anomaly when radiative corrections are taken into account.

5
In case of the dilaton gravity in \(21\) corresponding to \(\Phi = 0\) (or more generally in case that the axion is included \(24\) as in \(22\)), we have the following expression:

\[
T = \frac{l^3}{8\pi G} \int d^4x \sqrt{-g(0)} \left[ \frac{1}{8} R_{(0)ij} R^{ij} - \frac{1}{24} R^2(0) \right.
\]

\[
- \frac{1}{2} R_{(0)}^{ij} \partial_i \varphi(0) \partial_j \varphi(0) + \frac{1}{6} R(0) g_{(0)}^{ij} \partial_i \varphi(0) \partial_j \varphi(0)
\]

\[
+ \frac{1}{4} \left\{ \frac{1}{\sqrt{-g(0)}} \partial_i \left( \sqrt{-g(0)} g_{(0)}^{ij} \partial_j \varphi(0) \right) \right\}^2 + \frac{1}{3} \left( g_{(0)}^{ij} \partial_i \varphi(0) \partial_j \varphi(0) \right)^2 \right]\] (6)

Here \(\varphi\) can be regarded as dilaton. In the limit of \(\Phi \rightarrow 0\), if one chooses \(V = -2\) and makes AdS/CFT identification of SG parameters one finds that the standard result (conformal anomaly of \(\mathcal{N} = 4\) super YM theory covariantly coupled with \(\mathcal{N} = 4\) conformal supergravity \(25\)) in (6) is reproduced \(21, 7\).

We should also note that the expression (4) cannot be rewritten as a sum of the Gauss-Bonnet invariant \(G\) and the square of the Weyl tensor \(F\), which are given as

\[
G = R^2 - 4 R_{ij} R^{ij} + R_{ijkl} R^{ijkl},\quad F = \frac{1}{3} R^2 - 2 R_{ij} R^{ij} + R_{ijkl} R^{ijkl}.
\]

This is the signal that the conformal symmetry is broken already in classical theory. When \(\phi\) is constant, only two terms corresponding to \(h_1\) and \(h_2\) survive in (4). As \(h_1\) depends on \(V\), we may compare the result with the conformal anomaly from, say, scalar or spinor QED, or QCD in the phase where there are no background scalars and (or) spinors. The structure of the conformal anomaly in such a theory has the following form: \(T = \hat{a}G + \hat{b}F + \hat{c}R^2\), where \(a = \text{constant} + a_1 e^2, b = \text{constant} + a_2 e^2, c = a_3 e^2\). Here \(e^2\) is the electric charge (or \(g^2\) in case of QCD). Imagine that one can identify \(e\) with the exponential of the constant dilaton (using holographic RG \(26, 27\)). \(a_1, a_2\) and \(a_3\) are some numbers. Then we obtain \(\hat{a} = -\hat{b} = \frac{b_2}{16\pi G}, \quad \hat{c} = -\frac{1}{8\pi G} (h_1 + \frac{1}{3} h_2)\). If one assumes \(\Phi(\phi) = ae^{b\phi}\), \(|a| \ll 1\), we find

\[
a_1 = -a_2 = \frac{1}{16\pi G} \cdot \frac{1}{8} \cdot \frac{a^2}{36}, \quad a_3 = -\frac{1}{8\pi G} \cdot \frac{a^2}{24} \cdot \left( -\frac{5}{162} + \frac{b^2}{576V} \right).\] (7)

Here \(V\) should be arbitrary but constant. We should note \(\Phi(0) \neq 0\). One can absorb the difference into the redefinition of \(l\) since we need not to assume \(\Phi(0) = 0\) in deriving the form of \(h_1\) and \(h_2\). Hence, this simple example suggests the way of comparison between SG side and QFT descriptions of non-conformal boundary theory.

Let us discuss the properties of conformal anomaly. In order that the region near the boundary at \(\rho = 0\) is asymptotically AdS, we need to require

\[\text{6}\]
Φ → 0 and Φ′ → 0 when ρ → 0. One can also confirm that $h_1 → \frac{1}{l}$ and $h_2 → -\frac{1}{l}$ in the limit of Φ → 0 and Φ′ → 0 even if Φ″ ≠ 0 and Φ‴ ≠ 0. In the AdS/CFT correspondence, $h_1$ and $h_2$ are related with the central charge c of the conformal field theory (or its analog for non-conformal theory). Since we have two functions $h_1$ and $h_2$, there are two ways to define the candidate c-function when the conformal field theory is deformed:

$$c_1 = \frac{24\pi h_1}{G}, \quad c_2 = -\frac{8\pi h_2}{G}. \quad (8)$$

If we put $V(\phi) = 4\lambda^2 + \Phi(\phi)$, then $l = \left(\frac{12}{V(0)}\right)^{\frac{1}{2}}$. One should note that it is chosen $l = 1$ in (8). We can restore l by changing $h → l^3h$ and $k → l^3k$ and $Φ → lΦ$, $Φ'' → l^3Φ''$ and $Φ''' → l^3Φ'''$ in (4). Then in the limit of Φ → 0, one gets $c_1, c_2 → \frac{\pi}{G} \left(\frac{12}{V(0)}\right)^{\frac{3}{2}}$, which agrees with the proposal of the previous work in the limit. The c-function $c_1$ or $c_2$ in (8) is, of course, more general definition. It is interesting to study the behaviour of candidate c-function for explicit values of dilatonic potential at different limits. It also could be interesting to see what is the analogue of our dilaton-dependent c-function in non-commutative YM theory (without dilaton, see 29).

The definitions of the c-functions in (8), are, however, not always good ones since our results are too wide. They quickly become non-monotonic and even singular in explicit examples. They presumably measure the deviations from SG description and should not be taken seriously. As pointed in 33, it might be necessary to impose the condition $Φ' = 0$ on the conformal boundary. Such condition follows from the equations of motion of d5 gauged SG. Anyway as $Φ' = 0$ on the boundary in the solution which has the asymptotic AdS region, we can add any function which proportional to the power of $Φ' = 0$ to the previous expressions of the c-functions in (8). As a trial, if we put $Φ' = 0$, we obtain

$$c_1 = \frac{2\pi}{3G} \left( \frac{62208 + 22464\Phi + 2196\Phi^2 + 72\Phi^3 + \Phi^4}{(6 + \Phi)^2(24 + \Phi)(18 + \Phi)} \right), \quad c_2 = \frac{3\pi}{G} \left( \frac{288 + 72\Phi + \Phi^2}{(6 + \Phi)^2(24 + \Phi)} \right). \quad (9)$$

instead of (8). We should note that there disappear the higher derivative terms like $Φ''$ or $Φ'''$. That will be our final proposal for acceptable c-function in terms of dilatonic potential. The given c-functions in (9) reproduce the known result for the central charge on the boundary. Since $\frac{dΦ}{dx} → 0$ in the asymptotically AdS region even if the region is UV or IR, the given c-functions in (9) have fixed points in the asymptotic AdS region $\frac{dΦ}{dx} = \frac{dΦ}{dx} = \frac{dΦ}{dx} → 0$, where $U = ρ^{-\frac{1}{2}}$ is the radius coordinate in AdS or the energy scale of the boundary field theory.
We can now check the monotonity in the c-functions. For this purpose, we consider some examples in 6 and 7, where \( V = -2 \). In the classical solutions for the both cases, \( \phi \) is the monotonically decreasing function of the energy scale \( U = \rho^{-\frac{1}{2}} \) and \( \phi = 0 \) at the UV limit corresponding to the boundary. Then in order to know the energy scale dependences of \( c_1 \) and \( c_2 \), we only need to investigate the \( \phi \) dependences of \( c_1 \) and \( c_2 \) in (9). The potentials in 6 and 7, and also \( \Phi \) have a minimum \( \Phi = 0 \) at \( \phi = 0 \), which corresponds to the UV boundary in the solutions in 6 and 7, and \( \Phi \) is monotonically increasing function of the absolute value \( |\phi| \), we only need to check the monotonities of \( c_1 \) and \( c_2 \) with respect to \( \Phi \) when \( \Phi \geq 0 \). From (9), we find \( \frac{d(\ln c_1)}{d\Phi}, \frac{d(\ln c_2)}{d\Phi} < 0 \). Therefore the c-functions \( c_1 \) and \( c_2 \) are monotonically decreasing functions of \( \Phi \) or increasings function of the energy scale \( U \) as the c-function in 4,7. We should also note that the c-functions \( c_1 \) and \( c_2 \) are positive definite for non-negative \( \Phi \).

In 28, another c-function has been proposed in terms of the metric as follows:

\[
c_{\text{GPPZ}} = \left( \frac{dA}{dz} \right)^{-3}, \tag{10}
\]

where the metric is given by \( ds^2 = dz^2 + e^{2A} dx_\mu dx^\mu \). The c-function (10) is positive and has a fixed point in the asymptotically AdS region again and the c-function is also monotonically increasing function of the energy scale. The c-functions (9) proposed in 23,48 are given in terms of the dilaton potential, not in terms of metric, but it might be interesting that the c-functions in (9) have the similar properties (positivity, monotonity and fixed point in the asymptotically AdS region). These properties could be understood from the equations of motion.

We can also consider other examples of c-function for different choices of dilatonic potential. In 30, several examples of the potentials in gauged supergravity are given. They appeared as a result of sphere reduction in M-theory or string theory, down to three or five dimensions. We find, however, that the proposed c-functions have not acceptable behaviour for the potentials in 30. The problem seems to be that the solutions in above models have not asymptotic AdS region in UV but in IR. On the same time the conformal anomaly in (4) is evaluated as UV effect. If we assume that \( \Phi \) in the expression of c-functions \( c_1 \) and \( c_2 \) vanishes at IR AdS region, \( \Phi \) becomes negative. When \( \Phi \) is negative, the properties of the c-functions \( c_1 \) and \( c_2 \) become bad, they are not monotonic nor positive, and furthermore they have a singularity in the region given by the solutions in 30. Thus, for such type of potential other proposal for c-function which is not related with conformal nomaly should be made.
Hence, we discussed the holographic Weyl anomaly from SG side and typical behaviour of candidate c-functions. However, it is not completely clear which role should play dilaton in above expressions as holographic RG coupling constant in dual QFT. It could be induced mass, quantum fields or coupling constants (most probably, gauge coupling), but the explicit rule with what it should be identified is absent. The big number of usual RG parameters in dual QFT suggests also that there should be considered gauged SG with few scalars.

3 Surface Counterterms and Finite Action

Let us turn now to discussion of of surface counterterms which are also connected with holographic Weyl anomaly. As well-known, we need to add the surface terms to the bulk action in order to have the well-defined variational principle. Under the variation over the metric $\hat{g}_{\mu\nu}$ and the scalar field $\phi$, the variation of the action (1) $\delta S = \delta S_{M_{d+1}} + \delta S_{M_d}$ is given by

$$\delta S_{M_{d+1}} = \frac{1}{16\pi G} \int_{M_{d+1}} d^{d+1}x \sqrt{-\hat{G}} \left[ \delta \hat{G}^{\zeta\xi} \left\{ -\frac{1}{2} \hat{G}^{\zeta\xi} \hat{R} ight\}ight.$$

$$+ (X(\phi) - Y'(\phi)) (\hat{\nabla} \phi)^2 + \Phi(\phi) + 4\lambda^2 \right) + \hat{R}_{\zeta\xi} + (X(\phi) - Y'(\phi)) \partial_{\zeta} \phi \partial_{\xi} \phi \right\}$$

$$+ \delta \phi \left\{ (X'(\phi) - Y''(\phi)) (\hat{\nabla} \phi)^2 + \Phi'(\phi) \right\}$$

$$- \frac{1}{\sqrt{-G}} \partial_\mu \left( \sqrt{-\hat{G}} \hat{G}^{\mu\nu} (X(\phi) - Y'(\phi)) \partial_\nu \phi \right) \right]\right].$$

$$\delta S_{M_d} = \frac{1}{16\pi G} \int_{M_d} d^dx \sqrt{-\hat{g}} n_\mu \left[ \partial^\mu \left( \hat{G}_{\zeta\nu} \delta \hat{G}^{\zeta\nu} \right) - D_\nu \left( \delta \hat{G}^{\mu\nu} \right) + Y(\phi) \partial^\mu (\delta \phi) \right].$$

Here $\hat{g}_{\mu\nu}$ is the metric induced from $\hat{G}_{\mu\nu}$ and $n_\mu$ is the unit vector normal to $M_d$. The surface term $\delta S_{M_d}$ of the variation contains $n^\mu \partial_\mu \left( \delta \hat{G}^{\zeta\nu} \right)$ and $n^\mu \partial_\mu \left( \delta \phi \right)$, which makes the variational principle ill-defined. In order that the variational principle is well-defined on the boundary, the variation of the action should be written as $\delta S_{M_d} = \lim_{\rho \to 0} \int_{M_d} d^dx \sqrt{-\hat{g}} \left[ \delta \hat{G}^{\zeta\nu} \left\{ \cdots \right\} + \delta \phi \left\{ \cdots \right\} \right]$ after using the partial integration. If we put $\{ \cdots \} = 0$ for $\{ \cdots \}$, one could obtain the boundary condition corresponding to Neumann boundary condition. We can, of course, select Dirichlet boundary condition by choosing $\delta \hat{G}^{\zeta\nu} = \delta \phi = 0$, which is natural for AdS/CFT correspondence. The Neumann type condition becomes, however, necessary later when we consider the black hole mass etc.
by using surface terms. If the variation of the action on the boundary contains \( n^\mu \partial_\mu (\delta \hat{G}^{\xi\nu}) \) or \( n^\mu \partial_\mu (\delta \phi) \), however, we cannot partially integrate it on the boundary since \( n_\mu \) expresses the direction perpendicular to the boundary. Therefore the “minimum” of the action is ambiguous. Such a problem was well studied in \(^{10}\) for the Einstein gravity and the boundary term was added to the action. It cancels the term containing \( n^\mu \partial_\mu (\delta \hat{G}^{\xi\nu}) \). We need to cancel also the term containing \( n^\mu \partial_\mu (\delta \phi) \). Then one finds the boundary term \(^{21}\)

\[
S_b^{(1)} = -\frac{1}{8\pi G} \int_{M_4} d^4x \sqrt{-\hat{g}} \left[ D_\mu n^\mu + Y(\phi) n_\mu \partial_\mu \phi \right]. \tag{12}
\]

We also need to add surface counterterm \( S_b^{(2)} \) which cancels the divergence coming from the infinite volume of the bulk space, say AdS. In order to investigate the divergence, we choose the metric in the form (2). In the parametrization (2), \( n_\mu \) and the curvature \( R \) are given by

\[
n^\mu = \left( \frac{2\rho}{l}, 0, \cdots, 0 \right), \quad R = \hat{R} + \frac{3\rho^2}{l^2} g^{ij} g^{kl} g_{ij} g_{kl} - \frac{4\rho^2}{l^4} g^{ij} \hat{g}^{kl} g_{ij} g_{kl} - \frac{\rho^2}{l^6} g^{ij} \hat{g}^{kl} g_{ij} g_{kl} \tag{13}\]

Here \( \hat{R} \) is the scalar curvature defined by \( g_{ij} \) in (2).

Expanding \( g_{ij} \) and \( \phi \) with respect to \( \rho \), we find the following expression for \( S + S_b^{(1)} \):

\[
S + S_b^{(1)} = \frac{1}{16\pi G} \lim_{\rho \to 0} \int d^dX \rho^{-\frac{d}{2}} \sqrt{-g_{(0)}} \left[ \frac{2 - 2d}{l^2} - \frac{1}{d} \Phi(\phi_0) \right.
\]

\[
+ \rho \left\{ -\frac{1}{d - 2} R_{(0)} - \frac{1}{l^2} g_{(0)}^{ij} g_{(1)ij} \right.
\]

\[
- \frac{1}{d - 2} \left( X(\phi_0) \left( \nabla_{(0)} \phi_0 \right)^2 + Y(\phi_0) \Delta \phi_0 + \Phi'(\phi_0) \phi_1 \right) \left\} + O(\rho^2) \right]\tag{14}
\]

Then for \( d = 2 \)

\[
S_b^{(2)} = \frac{1}{16\pi G} \int d^dX \sqrt{-\hat{g}} \left[ \frac{2d - 2}{d - 2} \Phi(\phi) \right] \tag{15}
\]

and for \( d = 3, 4, \)

\[
S_b^{(2)} = \frac{1}{16\pi G} \int d^dX \left[ \sqrt{-\hat{g}} \left\{ \frac{2d - 2}{d - 2} - \frac{l}{d - 2} R - \frac{2l^2}{d(d - 2)} \Phi(\phi) \right. \right.
\]

\[
+ \frac{l}{d - 2} \left( X(\phi) \left( \hat{\nabla} \phi \right)^2 + Y(\phi) \Delta \phi \right) \left. \right\} - \frac{l^2}{d(d - 2)} n^\mu \partial_\mu \left( \sqrt{-\hat{g}} \Phi(\phi) \right) \tag{16}
\]
Note that the last term in above expression does not look typical from the AdS/CFT point of view. The reason is that it does not depend from only the boundary values of the fields. Its presence may indicate to breaking of AdS/CFT conjecture in the situations when SUGRA scalars significantly deviate from constants or are not asymptotic constants.

Thus we got the boundary counterterm action for gauged SG. Using these local surface counterterms as part of complete action one can show explicitly that bosonic sector of gauged SG in dimensions under discussion gives finite action in asymptotically AdS space. The corresponding example will be given in below.

Let us turn now to the discussion of deep connection between surface counterterms and holographic conformal anomaly. It is enough to mention only \( d = 4 \). In order to control the logarithmically divergent terms in the bulk action \( S \), we choose \( d - 4 = \epsilon < 0 \). Then \( S + S_b = \frac{1}{\epsilon} S_{\ln} + \) finite terms. Here \( S_{\ln} \) is given in (3). We also find \( g^{ij}_{(0)} \frac{\delta}{\delta g^{ij}_{(0)}} S_{\ln} = - \frac{1}{2} L_{\ln} + O (\epsilon^2) \). Here \( L_{\ln} \) is the Lagrangian density corresponding to \( S_{\ln} : S_{\ln} = \int d^{d+1} L_{\ln} \). Then we obtain the following expression of the trace anomaly:

\[
T = \lim_{\epsilon \to 0^-} \frac{2 g^{ij}_{(0)} \delta (S + S_b)}{\sqrt{-g(0)}} = - \frac{1}{2} L_{\ln} ,
\]

which is identical with the result found in (3). We should note that the last term in (16) does not lead to any ambiguity in the calculation of conformal anomaly since \( g_{(0)} \) does not depend on \( \rho \). If we use the equations of motion, we finally obtain the expression (4). Hence, we found the finite gravitational action (for asymptotically AdS spaces) in 5 dimensions by adding the local surface counterterm. This action correctly reproduces holographic trace anomaly for dual (gauge) theory. In principle, one can also generalize all results for higher dimensions, say, d6, etc. With the growth of dimension, the technical problems become more and more complicated as the number of structures in boundary term is increasing.

Let us consider the black hole or “throat” type solution for the equations of the motion when \( d = 4 \). The surface term (16) may be used for calculation of the finite black hole mass and/or other thermodynamical quantities.

For simplicity, we choose \( X(\phi) = \alpha \) (constant), \( Y(\phi) = 0 \) and we assume the spacetime metric in the following form:

\[
d s^2 = - e^{2\rho} d t^2 + e^{2\sigma} d r^2 + r^2 \sum_{i=1}^{d-1} (d x^i)^2
\]

(18)
and $\rho$, $\sigma$ and $\phi$ depend only on $r$. We now define new variables $U$ and $V$ by $U = e^{\rho + \sigma}$, $V = r^2 e^{\rho - \sigma}$. When $\Phi(0) = \Phi'(0) = \phi = 0$, a solution corresponding to the throat limit of D3-brane is given by $U = 1$, $V = V_0 \equiv \frac{4}{r^4} - \mu$. In the following, we use large $r$ expansion and consider the perturbation around this solution. Then we obtain, when $r$ is large or $c$ is small, one gets

$$U = 1 + c^2 u, \quad u = u_0 + \frac{\alpha \beta}{6} r^{-2\beta},
V = V_0 + c^2 v, \quad v = v_0 - \frac{\tilde{\mu}(\beta - 6)}{6(\beta - 4)(\beta - 2)} r^{-2\beta + 4}. \quad (19)$$

Here $u_0$ and $v_0$ are constants of the integration. Here we choose $v_0 = u_0 = 0$.

The horizon which is defined by $V = 0$ lies at

$$r = r_h \equiv l^+ \mu + c^2 \frac{\tilde{\mu}(\beta - 6)(\beta - 3)}{6(\beta - 4)(\beta - 2)}.$$ (20)

Hawking temperature is

$$T = \frac{1}{4\pi} \left[ \frac{1}{r^2} \frac{dV}{dr} \right]_{r = r_h} = \frac{1}{4\pi} \left\{ 4r^2 \frac{d}{dr} r^2 + c^2 \frac{\tilde{\mu}(\beta - 6)(\beta - 3)}{6(\beta - 4)(\beta - 2)} r^2 \right\}. \quad (21)$$

We now evaluate the free energy of the black hole within the standard prescription \cite{31,32}. The free energy $F$ can be obtained by substituting the classical solution into the action $S$: $F = TS$. Here $T$ is the Hawking temperature. Since we have $0 = \frac{2}{\beta} (\Phi(\phi) + \frac{12}{l^2}) + \tilde{R} + \alpha (\nabla \phi)^2$ by using the equations of motion, we find the following expression of the action (1) after Wick-rotating it to the Euclid signature

$$S = \frac{1}{16\pi G} \cdot \frac{2}{3} \int_{\mathcal{M}_5} d^5 \sqrt{G} \left( \Phi(\phi) + \frac{12}{l^2} \right) = \frac{1}{16\pi G} \cdot \frac{2}{3} \int_{V_0}^{T} drr^3 U \left( \Phi(\phi) + \frac{12}{l^2} \right). \quad (22)$$

Here $V_0$ is the volume of the 3d space ($\int d^5 x \cdots = \beta V_0$, $\int drr^3 \cdots$) and $\beta$ is the period of time, which can be regarded as the inverse of the temperature $T (\frac{1}{T})$. The expression (22) contains the divergence. We regularize the divergence by replacing $\int_{V_0}^{T} dr \rightarrow \int_{V_0}^{T_{\max}} dr$ and subtract the contribution from a zero temperature solution, where we choose $\mu = c = 0$, and the solution corresponds to the vacuum or pure AdS:

$$S_0 = \frac{1}{16\pi G} \cdot \frac{2}{3} \frac{12 V_0}{l^2} \frac{G_{tt}(r = r_{\max}, \mu = c = 0)}{G_{tt}(r = r_{\max})} \int_{r_h}^{r_{\max}} dr r^3. \quad (23)$$
The factor $\sqrt{\frac{Gtt(r=r_{\text{max}}, \mu=0)}{G_{tt}(r=r_{\text{max}})}}$ is chosen so that the proper length of the circles which correspond to the period $\frac{1}{T}$ in the Euclid time at $r_{\text{max}}$ coincides with each other in the two solutions. Then we find the following expression for the free energy $F = \lim_{r_{\text{max}} \to \infty} T(S - S_0)$,

$$F = \frac{V(3)}{2\pi G T^2} \left[ \frac{\ell^2 \mu}{8} + c^2 \mu^{1-\frac{\beta}{2}} \left\{ \frac{(\beta - 1)}{12 \beta (\beta - 4)(\beta - 2)} \right\} + \cdots \right].$$ (24)

Here we assume $\beta > 2$ or the expression $S - S_0$ still contains the divergences and we cannot get finite results. However, the inequality $\beta > 2$ is not always satisfied in the gauged supergravity models. In that case the expression in (24) would not be valid. One can express the free energy $F$ in (24) in terms of the temperature $T$ instead of $\mu$:

$$F = \frac{V(3)}{16\pi G} \left[ -\pi T^4 \ell^6 + c^2 T^8 - 4\beta T^4 - 23 \frac{2 \beta^3 - 15 \beta^2 + 22 \beta - 4}{6 \beta (\beta - 4)(\beta - 2)} \right] + \cdots .$$ (25)

Then the entropy $S = -\frac{dF}{dT}$ and the energy (mass) $E = F + TS$ is given by

$$S = \frac{V(3)}{16\pi G} \left[ 4\pi T^3 \ell^6 + c^2 T^8 - 4\beta T^4 - 23 \frac{2 \beta^3 - 15 \beta^2 + 22 \beta - 4}{3 \beta (\beta - 4)} \right] + \cdots ,
$$

$$E = \frac{V(3)}{16\pi G} \left[ 3\pi T^4 \ell^6 + c^2 T^8 - 4\beta \left( \frac{2 \beta - 3}{(2 \beta - 3)(2 \beta^3 - 15 \beta^2 + 22 \beta - 4)} \right) + \cdots \right] .$$ (26)

We now evaluate the mass using the surface term of the action in (16), i.e. within local surface counterterm method. The surface energy momentum tensor $T_{ij}$ is now defined by

$$\delta S_b^{(2)} = \sqrt{-\hat{g}} \hat{g}^{ij} T_{ij} = \frac{1}{16\pi G} \left[ \sqrt{-\hat{g}} \hat{g}^{ij} \left\{ -\frac{1}{2} \hat{g}_{ij} \left( \frac{6}{7} + \frac{l}{2} R \right) \right\} \right].$$

$\hat{S}$ does not contribute due to the equation of motion in the bulk. The variation of $S + S_b^{(1)}$ gives a contribution proportional to the extrinsic curvature $\theta_{ij}$ at the boundary:

$$\delta \left( S + S_b^{(1)} \right) = \sqrt{-\hat{g}} \left( \theta_{ij} - \theta_{\hat{g}_{ij}} \right) \delta \hat{g}^{ij}. \text{ The contribution is finite even in the limit of } r \to \infty. \text{ Then the finite part does not depend on the parameters characterizing the black hole. Therefore after subtracting the contribution from the reference metric, which could be that of AdS, the contribution from the variation of } S + S_b^{(1)} \text{ vanishes.}$$
Note that the energy-momentum tensor is still not well-defined due to the term containing $n^\mu \partial_\mu$. If we assume $\delta \hat{g}^{ij} \sim O(\rho^{\alpha_1})$ for large $\rho$ when we choose the coordinate system (2), then $n^\mu \partial_\mu (\delta \hat{g}^{ij} \cdot (a_1 + \partial_\mu)) \cdot$. Or if $\delta \hat{g}^{ij} \sim O(\rho^{\alpha_2})$ for large $\rho$ when we choose the coordinate system (18), then $n^\mu \partial_\mu (\delta \hat{g}^{ij} \cdot (a_2 + \partial_\mu)) \cdot$. As we consider the black hole-like object in this section, one chooses the coordinate system (18). Then mass $E$ of the black hole like object is given by

$$E = \int d^{d-1}x \sqrt{\sigma} N \delta T_{tt} (u^t)^2 .$$

(28)

Here we assume the metric of the reference spacetime (e.g. AdS) has the form of $ds^2 = f(r) dr^2 - N^2(r) dt^2 + \sum_{i,j=1}^{d-1} \delta_{ij} dx^i dx^j$ and $\delta T_{tt}$ is the difference of the $(t,t)$ component of the energy-momentum tensor in the spacetime with black hole like object from that in the reference spacetime, which we choose to be AdS, and $u^t$ is the $t$ component of the unit time-like vector normal to the hypersurface given by $t = \text{constant}$. By using the solution in (19), the $(t,t)$ component of the energy-momentum tensor in (27) has the following form:

$$T_{tt} = \frac{3r^2}{16\pi G l^3} \left[ 1 - \frac{r^3 \mu}{r^4} + \frac{\tilde{\mu} c^2}{r^{2\beta}} \left( \frac{1}{12} - \frac{1}{6\beta(\beta - 6)} - \frac{\beta - 6}{6(\beta - 4)(\beta - 2)} - \frac{(3 - \beta)(1 + a_2)}{12} \right) + \cdots \right] .$$

(29)

If we assume the mass is finite, $\beta$ should satisfy the inequality $\beta > 2$, as in the case of the free energy in (24) since $\sqrt{\sigma} N (u^t)^2 = lr^2$ for the reference AdS space. Then the $\beta$-dependent term in (29) does not contribute to the mass and one gets $E = \frac{2\hbar V_{(3)} T^4}{16\pi G}$ and using (21)

$$E = \frac{3r^6 V_{(3)} T^4}{16\pi G} \left\{ 1 - c^2 \mu l^2 - 4\beta (\pi T^4) - \frac{2}{\beta - 4} \frac{(\beta - 6)(2\beta - 3)}{(\beta - 4)(\beta - 2)} \right\} ,$$

(30)

which does not agree with the result in (26). This might express the ambiguity in the choice of the regularization to make the finite action. A possible origin of it might be following. We assumed $\phi$ can be expanded in the (integer) power series of $\rho$ when deriving the surface terms in (16). However, this assumption seems to conflict with the classical solution, where the fractional power seems to appear since $r^2 \sim \frac{\phi}{\rho}$. In any case, in QFT there is no problem in regularization
dependence of the results. In many cases (see example in ref.\textsuperscript{17}) the explicit choice of free parameters of regularization leads to coincidence of the answers which look different in different regularizations. As usually happens in QFT the renormalization is more universal as the same answers for beta-functions may be obtained while using different regularizations. That suggests that holographic renormalization group should be developed and the predictions of above calculations should be tested in it.

As in the case of the c-function, we might drop the terms containing $\Phi'$ in the expression of $S_b^{(2)}$ in (16) but the result of the mass $E$ in (30) does not change.

4 Comparison with other counterterm schemes and holografic RG

In this section we compare the surface counterterms and the trace anomaly obtained here with those in ref.\textsuperscript{34} (flat 4d case) and give generalization for 4d curved space.

We start with the following action:

$$S = \frac{1}{16\pi G} \int_{M_5} d^5 x \sqrt{-\hat{G}} \left( \hat{R} - \frac{1}{2} g_{IJ} G^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J - V(\phi) \right)$$

$$- \frac{1}{8\pi G} \int_{M_4} d^4 x \sqrt{-g} \left( D_\mu n^\mu + L_{c.t} \right).$$

(31)

Here we choose $V(\phi) = -\frac{12}{l^2}$ and $n^\mu$ is given by $n^\mu = (1, 0, \cdots, 0)$, where the first component corresponds to $r$-component. As an extension of \textsuperscript{39,34}, one takes the following metric:

$$ds^2 = dr^2 + e^{2A(r)} \tilde{g}_{ij} dx^i dx^j$$

(32)

Here $\tilde{g}_{ij}$ is the metric of the Einstein manifold, where Ricci tensor $\tilde{R}_{ij}$ given by $\tilde{g}_{ij}$ satisfies the following condition:

$$\tilde{R}_{ij} = k \tilde{g}_{ij}$$

(33)

where $k$ is a constant. The equations of motion from varying (31) with respect to the metric lead to the following form instead of (8), (9) in \textsuperscript{34}.

$$\frac{d^2 A}{dr^2} = - \frac{1}{6} g_{IJ} \frac{d\phi^I}{dr} \frac{d\phi^J}{dr} - \frac{k}{3} e^{-2A}$$

(34)

$$\left( \frac{dA}{dr} \right)^2 = \frac{1}{l^2} + \frac{1}{24} g_{IJ} \frac{d\phi^I}{dr} \frac{d\phi^J}{dr} + \frac{k}{3} e^{-2A}$$

(35)
If \( \frac{d\phi}{dr} \) is not zero, we can treat \( A' \equiv \frac{dA}{dr} \) and \( \phi'^I \equiv \frac{d\phi^I}{dr} \) as functions of \( \phi \) and rewrite (34) as

\[
\frac{\partial A'}{\partial \phi^I} \frac{d\phi^I}{dr} = -\frac{1}{6} g_{IJ} \frac{d\phi^J}{dr} - \frac{k}{3} e^{-2A} .
\] (36)

If we assume the solution of (36) in the following form: \( \frac{d\phi^I}{dr} = f(\phi^K, A) g^{IJ} \frac{\partial A'}{\partial \phi^I} , \) we obtain

\[
f(\phi^K, A) g^{IJ} \frac{\partial A'}{\partial \phi^I} \frac{\partial A'}{\partial \phi^I} = -\frac{1}{6} f(\phi^K, A)^2 g^{IJ} \frac{\partial A'}{\partial \phi^I} \frac{\partial A'}{\partial \phi^I} - \frac{k}{3} e^{-2A} ,
\] (37)

which can be solved with respect to \( f(\phi^K, A) \):

\[
f(\phi^K, A) = -3 \pm \sqrt{9 - \frac{2ke^{-2A}}{g^{IJ} \frac{\partial A'}{\partial \phi^I} \frac{\partial A'}{\partial \phi^I}}} .
\] (38)

Then we find

\[
\frac{d\phi^I}{dr} = \left( -3 \pm \sqrt{9 - \frac{2ke^{-2A}}{g^{KL} \frac{\partial A'}{\partial \phi^K} \frac{\partial A'}{\partial \phi^L}}} \right) g^{IJ} \frac{\partial (\ln A')}{\partial \phi^I} .
\] (39)

In \( \pm \) sign in (39), the \( - \) sign reproduces the result in (34) when \( k = 0 \). We should note that \( \phi^I \) can be regarded as a coupling constant associated with the operator \( O_I \), \( \int d^4x \phi^I O_I \), from the AdS/CFT correspondence. Since \( \ln A \) can be also regarded as a logarithm of the scale, the \( \beta \)-function could be given by

\[
\beta^I \equiv \frac{d\phi^I}{dA} = \frac{1}{A'} \frac{d\phi^I}{dr} = \left( -3 \pm \sqrt{9 - \frac{2ke^{-2A}}{g^{KL} \frac{\partial A'}{\partial \phi^K} \frac{\partial A'}{\partial \phi^L}}} \right) g^{IJ} \frac{\partial (\ln A')}{\partial \phi^I} .
\] (40)

First, we recall the surface terms:

\[
S_{s.t} = -\frac{1}{8\pi G} \int_{M_4} d^4x \sqrt{-g} \left( D_\mu n^\mu + L_{c.t} \right)
\]

\[
= -\frac{1}{8\pi G} \int_{M_4} d^4x \sqrt{-g} \left( \frac{1}{2} g^{ij} g_{ij,r} + L_{c.t} \right)
\] (41)

and varying these terms with respect to the boundary metric \( g_{ij} \) one gets

\[
\begin{split}
\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{ij}} \bigg|_{\text{surface term}} + \frac{1}{\sqrt{-g}} \frac{\delta S_{s.t}}{\delta g^{ij}} \\
= -\frac{1}{8\pi G} \left\{ -\frac{1}{4} g_{ij} g^{kl} g_{kl,r} + \frac{1}{2} g_{ij,r} - \frac{1}{2} g_{ij} L_{c.t} + \frac{\delta L_{c.t}}{\delta g^{ij}} \right\} .
\end{split}
\] (42)
One can take $L_{c,t}$ as in $^{34}$,

$$L_{c,t} = \frac{3}{l}\left(1 - \frac{l^2}{12}R_g\right)$$  \hspace{1cm} (43)

where $R_g = g^{ij}R_{g,ij} = g^{ij}\tilde{R}_{ij} = 4ke^{-2A(r)}$. We denote 4 dimensional curvatures given by $g_{ij}$ and its derivatives with respect to $x^i$ by the suffix $g$. Then the variation of $R_g$ with respect to the boundary metric $g^{ij}$ is given by

$$\frac{\delta L_{c,t}}{\delta g^{ij}} = -\frac{l}{4}R_{g,ij} + \cdots = -\frac{l}{4}ke^{-2A(r)}g_{ij} + \cdots.$$  \hspace{1cm} (44)

Here $\cdots$ expresses total derivative terms. And the equation (42) is rewritten as

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{ij}} \bigg|_{\text{surface term}} + \frac{1}{\sqrt{-g}}\frac{\delta S_{s,t}}{\delta g^{ij}} = -\frac{1}{8\pi G}\left\{ -\frac{1}{4}g_{ij}g^{kl}g_{kl,r} + \frac{1}{2}g_{ij,r} - \frac{l}{4}ke^{-2A(r)}g_{ij} - \frac{3}{2l}g_{ij} \left(1 - \frac{l^2}{12}\cdot4ke^{-2A(r)}\right) \right\}. $$  \hspace{1cm} (45)

Since one can regard $\tilde{g}_{ij}$ as metric of the 4 dimensional spacetime where the field theory lives, we could define trace anomaly by

$$T = \frac{2}{\sqrt{-g}}\frac{\tilde{g}^{ij}\delta S}{\delta \tilde{g}^{ij}} \bigg|_{\text{surface term}} + \frac{2}{\sqrt{-g}}\frac{\delta S_{s,t}}{\delta \tilde{g}^{ij}} = \frac{2e^{4A}}{\sqrt{-g}}\tilde{g}^{ij}\frac{\delta S}{\delta g^{ij}} \bigg|_{\text{surface term}} + \frac{2e^{4A}}{\sqrt{-g}}\tilde{g}^{ij}\frac{\delta S_{s,t}}{\delta \tilde{g}^{ij}}.$$  \hspace{1cm} (46)

Then

$$T = -\frac{e^{4A}}{4\pi G}\left\{ -6A' + lke^{-2A(r)} - \frac{6}{l} \right\}. $$  \hspace{1cm} (47)

We now compare the above result with the one given in this report. The solution of the Einstein equations where the boundary has constant curvature is given in $^{44}$. When the scalar fields vanish, the solution is given by

$$ds^2 = f(y)dy^2 + y \sum_{i,j=0}^3 \hat{g}_{ij}(x^k)dx^idx^j, \quad f = \frac{l^2}{4y^2\left(1 + \frac{l^2}{4y^2}\right)}.$$  \hspace{1cm} (48)
Here the boundary lies at \( y \to \infty \). If we change the coordinate by \( z = \int \frac{dy}{2y\sqrt{1+\frac{k^2}{y^2}}} \), the metric in (48) can be rewritten in the form of (32), where \( e^{2A} = y(z) \). Then the anomaly \( T \) in (47) is

\[
T = -\frac{y^2}{4\pi G} \left\{ -\frac{6}{l} \sqrt{1 + \frac{k^2}{3y}} + \frac{lk}{y^2} - \frac{6}{l} \right\}.
\] (49)

On the boundary, where \( y \to \infty \), \( T \) has the finite value:

\[
T \to -\frac{k^2 l^3}{48\pi G}.
\] (50)

On the other hand, if we use the previous expression (6) of the trace anomaly \( T \) with constant \( \varphi(0) \): \( T = -\frac{\rho^4}{8\pi G} \left( \frac{1}{2\pi} \tilde{R}^2 - \frac{1}{8} \tilde{R}^{ij} \tilde{R}_{ij} \right) \), by substituting (33), we obtain

\[
T = -\frac{k^2 l^3}{48\pi G},
\] (51)

which is identical with (50). Thus, holographic RG consideration gives the same conformal anomaly as in second section.

For simplicity, one can consider the case that the boundary is flat and the metric \( g_{ij} \) in (2) on the boundary is given by \( g_{ij} = F(\rho) \eta_{ij} \). We also assume the dilaton \( \phi \) only depends on \( \rho \): \( \phi = \phi(\rho) \). This is exactly the case of ref.34. Then the conformal anomaly (4) vanishes on such background.

Let us demonstrate that our discussion is consistent with results of ref.34. In 34, the following counterterms scheme is proposed

\[
S^{(2)}_{BGM} = \frac{1}{16\pi G} \int d^4x \sqrt{-\hat{g}} \left\{ \frac{6u(\phi)}{l} \frac{\dot{\rho}^2}{\rho^2} + \frac{l}{2u(\phi)} \hat{R} \right\},
\] (52)

instead of (16). Here \( u \) is obtained in terms of this paper as follows: \( u(\phi)^2 = 1 + \frac{l^2}{12} \Phi(\phi) \). Then based on the counter terms in (52), the following expression of the trace anomaly is given in 34:

\[
T = \frac{3}{2\pi G l} (-2B - u).
\] (53)

Here \( B \equiv \rho \partial_\rho A \). The above trace anomaly was evaluated for fixed but finite \( \rho \). If the boundary is asymptotically AdS, \( F \) goes to a constant \( F \to F_0 \) (\( F_0 \): a constant). Then, we find the behaviors of \( A \) and \( B \) as \( A \to \frac{1}{2} \ln \frac{F_0}{\rho} \), \( B \to -\frac{1}{2} \).
Then from (39), we find $\phi$ becomes a constant. Since we have the following equation of the motion

$$0 = -\frac{l^2}{4\rho^2} \left( \Phi(\phi) + \frac{12}{l^2} \right) + \frac{3}{\rho^2} \frac{3}{F^2} \left( \partial_\rho F \right)^2 - \frac{6}{\rho F} \partial_\rho F - \frac{1}{2} \left( \partial_\rho \phi \right)^2 ,$$

one gets

$$u = \sqrt{1 + \frac{l^2}{12} \Phi(\phi)} \to 1 .$$

(55)

Since $B \to -\frac{1}{2}$, this tells that the trace anomaly (53) vanishes on the boundary. Thus, we demonstrated that trace anomaly of $34$ vanishes in the UV limit what is expected also from AdS/CFT correspondence.

We should note that the trace anomaly (4) is evaluated on the boundary, i.e., in the UV limit. We evaluated the anomaly by expandind the action in the power series of the infrared cutoff $\epsilon$ and subtracting the divergent terms in the limit of $\epsilon \to 0$. If we evaluate the anomaly for finite $\rho$ as in $34$, the terms with positive power of $\epsilon$ in the expansion do not vanish and we would obtain non-vanishing trace anomaly in general. Thus, the trace anomaly obtained in this paper does not not have any contradiction with that in $34$, i.e. with holografic RG.

5 Dilatonic brane-world inflation induced by quantum effects: Constant bulk potential

In this section we consider brane-world solutions in d5 dilatonic gravity following ref.49 when brane CFT is present. We start with Euclidean signature for the action $S$ which is the sum of the Einstein-Hilbert action $S_{EH}$ with kinetic term for dilaton $\phi$, the Gibbons-Hawking surface term $S_{GH}$, the surface counter term $S_1$ and the trace anomaly induced action $W^g$:

$$S = S_{EH} + S_{GH} + 2S_1 + W ,$$

(56)

$$S_{EH} = \frac{1}{16\pi G} \int d^5x \sqrt{g(5)} \left( R(5) - \frac{1}{2} \nabla_\mu \phi \nabla_\mu \phi + \frac{12}{l^2} \right) ,$$

(57)

$$S_{GH} = \frac{1}{8\pi G} \int d^4x \sqrt{g(4)} \nabla_\mu n^\mu ,$$

(58)

$$S_1 = -\frac{3}{8\pi G l} \int d^4x \sqrt{g(4)} ,$$

(59)

For the introduction to anomaly induced effective action in curved space-time (with torsion), see section 5.5 in 40.
\[ W = b \int d^4 x \sqrt{\tilde{g}} \tilde{F} A + b' \int d^4 x \sqrt{\tilde{g}} \left\{ A \left[ 2\Box^2 + \tilde{R}_{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \right] \\
- \frac{4}{3} \tilde{R} \Box^2 + \frac{2}{3} \left( \tilde{\nabla}^\mu \tilde{R} \right) \tilde{\nabla}_\mu \right\} A + \left( \tilde{G} - \frac{2}{3} \Box \tilde{R} \right) A \right. \\
- \frac{1}{12} \left\{ b'' + \frac{2}{3} (b + b') \right\} \int d^4 x \sqrt{\tilde{g}} \left[ \tilde{R} - 6 \Box A - 6 (\tilde{\nabla}_\mu A) (\tilde{\nabla}^\mu A) \right]^2 \\
+ C \int d^4 x A \phi \left[ \Box^2 + 2 \tilde{R}_{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu - \frac{2}{3} \tilde{R} \Box^2 + \frac{1}{3} (\tilde{\nabla}^\mu \tilde{R}) \tilde{\nabla}_\mu \right] \phi. \quad (60) \]

Here the quantities in the 5 dimensional bulk spacetime are specified by the suffices \((5)\) and those in the boundary 4 dimensional spacetime are specified by \((4)\). The factor 2 in front of \(S^1\) in (56) is coming from that we have two bulk regions which are connected with each other by the brane. In (58), \(n^\mu\) is the unit vector normal to the boundary. In (60), one chooses the 4 dimensional boundary metric as

\[ g_{(4)\mu\nu} = e^{2A} \hat{g}_{\mu\nu}, \quad (61) \]

and we specify the quantities given by \(\hat{g}_{\mu\nu}\) by using \(\hat{\cdot}\). \(G (\hat{G})\) and \(F (\hat{F})\) are the Gauss-Bonnet invariant and the square of the Weyl tensor. In the brane effective action (60), we consider the case corresponding to \(N = 4\) \(SU(N)\) Yang-Mills theory, where \(b = \frac{C}{4\pi^2} = \frac{N^2 - 1}{4\pi^2}\). The dilaton field \(\phi\) which appears from the coupling with extended conformal supergravity is in general complex but we consider the case in which only the real part of \(\phi\) is non-zero.

Adopting AdS/CFT correspondence one can argue that in symmetric phase the quantum brane matter appears due to maximally SUSY Yang-Mills theory as above. Note that there is a kinetic term for the dilaton in the classical bulk action but also there is dilatonic contribution to the anomaly induced effective action \(W\). Here, it appears the difference with the correspondent construction in ref.\(^{42}\) where there was no dilaton.

In the bulk, the solution of the equations of motion is given in \(^{44}\), as follows

\[ ds^2 = f(y)dy^2 + y \sum_{i,j=0}^{d-1} \hat{g}_{ij}(x^k) dx^i dx^j, \quad f = \frac{d(d-1)}{4y^d+2\lambda^2 \left( 1 + \frac{c^2}{2\lambda^2 y^2} + \frac{kd}{\lambda^2 y} \right)} \]

\[ \phi = c \int dy \sqrt{\frac{d(d-1)}{4y^d+2\lambda^2 \left( 1 + \frac{c^2}{2\lambda^2 y^2} + \frac{kd}{\lambda^2 y} \right)}}. \quad (62) \]

Here \(\lambda^2 = \frac{d^2}{4}\) and \(\hat{g}_{ij}\) is the metric of the Einstein manifold, which is defined by \(r_{ij} = kr_{ij}\), where \(r_{ij}\) is the Ricci tensor constructed with \(\hat{g}_{ij}\) and \(k\) is a constant.
We should note that there is a curvature singularity at \( y = 0 \). The solution with non-trivial dilaton would presumably correspond to the deformation of the vacuum (which is associated with the dimension 4 operator, say \( \text{tr} F^2 \)) in the dual maximally SUSY Yang-Mills theory.

If one defines a new coordinate \( z \) by

\[
z = \int dy \left( \frac{d(d-1)}{4y^2 \lambda^2 \left( 1 + \frac{c^2}{2\lambda y^2} + \frac{kd}{\lambda^2 y} \right)} \right),
\]

and solves \( y \) with respect to \( z \), we obtain the warp factor \( e^{2\hat{A}(z,k)} = y(z) \). Here one assumes the metric of 5 dimensional space time as follows:

\[
ds^2 = dz^2 + e^{2\hat{A}(z,\sigma)} \tilde{g}_{\mu\nu} dx^\mu dx^\nu, \quad \tilde{g}_{\mu\nu} dx^\mu dx^\nu \equiv l^2 \left( d\sigma^2 + d\Omega^2 \right).
\]

(63)

Here \( d\Omega^2 \) corresponds to the metric of 3 dimensional unit sphere. Then we find

\[
A(z,\sigma) = \hat{A}(z,k = 3) - \ln \cosh \sigma, \quad \text{for unit sphere (}k = 3\text{)}
\]

(64)

\[
A(z,\sigma) = \hat{A}(z,k = 0) + \sigma, \quad \text{for flat Euclidean space (}k = 0\text{)}
\]

(65)

\[
A(z,\sigma) = \hat{A}(z,k = -3) - \ln \sinh \sigma, \quad \text{for unit hyperboloid (}k = -3\text{)}
\]

(66)

We now identify \( A \) and \( \tilde{g} \) in (64) with those in (61). Then we find \( \tilde{F} = \tilde{G} = 0 \), \( \tilde{R} = \frac{\partial}{\partial z} \) etc.

According to the assumption in (64), the actions in (57), (58), (59), and (60) have the following forms:

\[
S_{\text{EH}} = \frac{l^4 V_3}{16\pi G} \int dz d\sigma \left\{ (-8\partial_z^2 A - 20(\partial_z A)^2) e^{4A} + (-6\partial_z^2 A) \right\},
\]

(67)

\[
S_{\text{GH}} = \frac{3l^4 V_3}{8\pi G} \int d\sigma e^{4A}\partial_z A,
\]

(68)

\[
S_1 = \frac{-3l^3 V_3}{8\pi G} \int d\sigma e^{4A},
\]

(69)

\[
W = V_3 \int d\sigma \left[ b' \left( 2\partial_z^2 A - 8\partial_{\sigma}^2 A \right) - 2(b + b') \left( 1 - \partial_{\sigma}^2 A - (\partial_{\sigma} A)^2 \right)^2 
+ C A \phi \left( \partial_z^2 \phi - 4\partial_{\sigma}^2 \phi \right) \right].
\]

(70)

Here \( V_3 \) is the volume or area of the unit 3 sphere: \( V_3 = 2\pi^2 \).
On the brane at the boundary, one gets the following equations

\[ 0 = \frac{48\ell^4}{16\pi G} \left( \partial_z A - \frac{1}{7} \right) e^{4A} + b' \left( 4\partial_y A - 16\partial_y^2 A \right) - 4(b + b') \left( \partial_y A + 2\partial_y^2 A - 6(\partial_y A)^2 \partial_y^2 A \right) + 2C \left( \partial_y^4 \phi - 4\partial_y^2 \phi \right), \]  

(72)

from the variation over \( A \) and

\[ 0 = -\frac{\ell^4}{8\pi G} e^{4A} \partial_y \phi + C \left\{ A \left( \partial_y^4 \phi - 4\partial_y^2 \phi \right) + \partial_y^4 (A\phi) - 4\partial_y^2 (A\phi) \right\}, \]  

(73)

from the variation over \( \phi \). We should note that the contributions from \( S_{\text{EH}} \) and \( S_{\text{GH}} \) are twice from the naive values since we have two bulk regions which are connected with each other by the brane. The equations (72) and (73) do not depend on \( k \), that is, they are correct for any of the sphere, hyperboloid, or flat Euclidean space. The \( k \) dependence appears when the bulk solutions are substituted. Substituting the bulk solution given by (62), (63) and (65), (66) or (67) into (72) and (73), one obtains

\[ 0 = \frac{1}{\pi G} \left( \sqrt{1 + \frac{k\ell^2}{3y_0} + \frac{l^2c^2}{24y_0^2}} - 1 \right) y_0^2 + 8b', \]  

(74)

\[ 0 = -\frac{c}{8\pi G} + 6C\phi_0. \]  

(75)

Here we assume the brane lies at \( y = y_0 \) and the dilaton takes a constant value there \( \phi = \phi_0 \):

\[ \phi_0 = \frac{c}{48\pi GC}. \]  

(76)

Note that eq.(74) does not depend on \( b \) and \( C \). Eq.(75) determines the value of \( \phi_0 \). That might be interesting since the vacuum expectation value of the dilaton cannot be determined perturbatively in string theory. Of course, (76) contains the parameter \( c \), which indicates the non-triviality of the dilaton. The parameter \( c \), however, can be determined from (74). Hence, in such scenario one gets a dynamical mechanism to determine of dilaton on the boundary (in our observable world).

The effective tension of the domain wall is given by

\[ \sigma_{\text{eff}} = \frac{3}{4\pi G} \partial_y A = \frac{3}{4\pi G l} \sqrt{1 + \frac{k\ell^2}{3y_0} + \frac{l^2c^2}{24y_0^2}}. \]  

(77)

One should note that the radial \( (z) \) component of the geodesic equation in the metric (64) is given by \( \frac{\dot{z}^2}{\tau^2} + \partial_z e^{2A} \left( \frac{\dot{\tau}^4}{\tau^2} \right)^2 = 0 \). Here \( \tau \) is the proper
time and we can normalize $e^{2A} \left( \frac{dz}{dt} \right)^2 = 1$ and obtain $\frac{d^2 z^*}{d\tau^2} + \partial^2 z A = 0$. Since the cosmological constant on the brane is given by $\frac{3}{4\pi G}$, $\sigma_{\text{eff}}$ gives the effective mass density: $\frac{3}{4\pi G} \frac{d^2 z^*}{d\tau^2} = -\sigma_{\text{eff}}$.

As in $^{37}$, defining the radius $R$ of the brane as $R_0 \equiv y$, we can rewrite (74) as

$$0 = \frac{1}{\pi G l} \left( \sqrt{1 + \frac{kl^2}{3R^2} + \frac{l^2c^2}{24R^8}} - 1 \right) R^4 + 8b'. \quad (78)$$

Especially when the dilaton vanishes ($c = 0$) and the brane is the unit sphere ($k = 3$), the equation (78) reproduces the result of ref.$^{37}$ for $N = 4$ $SU(N)$ super Yang-Mills theory in case of the large $N$ limit where $b' \to -\frac{N^2}{3(4\pi)^2}$:

$$\frac{R^3}{l^3} \sqrt{1 + \frac{R^2}{l^2}} = \frac{R^4}{l^4} + \frac{GN^2}{8\pi l^3}. \quad (79)$$

Let us define a function $F(R, c)$ as

$$F(R, c) \equiv \frac{1}{\pi G l} \left( \sqrt{1 + \frac{kl^2}{3R^2} + \frac{l^2c^2}{24R^8}} - 1 \right) R^4, \quad (80)$$

It appears in the r.h.s. in (78).

First we consider the $k > 0$ case. Since

$$\frac{\partial (\ln F(R, c))}{\partial R} = \frac{1}{R} \left( \sqrt{1 + \frac{kl^2}{3R^2} + \frac{l^2c^2}{24R^8}} - 1 \right)^{-1} \left( \sqrt{1 + \frac{kl^2}{3R^2} + \frac{l^2c^2}{24R^8}} - 1 \right)^{-1} \times \left( 4 + \frac{kl^2}{R^2} + 4\sqrt{1 + \frac{kl^2}{3R^2} + \frac{l^2c^2}{24R^8}} \right)^{-1} \times \left( \frac{8kl^2}{3R^2} + \frac{k^2l^4}{R^4} - \frac{2l^2c^2}{3R^8} \right). \quad (81)$$

$F(R, c)$ has a minimum at $R = R_0$, where $R_0$ is defined by $0 = \frac{8kl^2}{3R_0^2} + \frac{k^2l^4}{R_0^4} - \frac{2l^2c^2}{3R_0^8}$. When $k > 0$, there is only one solution for $R_0$. Therefore $F(R, c)$ in the case of $k > 0$ (sphere case) is a monotonically increasing function of $R$ when $R > R_0$ and a decreasing function when $R < R_0$. Since $F(R, c)$ is clearly a monotonically increasing function of $c$, we find for $k > 0$ and $b' < 0$ case that $R$ decreases when $c$ increases if $R > R_0$, that is, the non-trivial dilaton makes the radius smaller. Then, since $1/R$ corresponds to the rate of the inflation of the universe, when we Wick-rotate the sphere into the inflationary universe, the large dilaton supports the rapid universe expansion. Hence, we showed that quantum CFT living on the domain wall leads to the creation of inflationary
dilatonic 4d de Sitter-brane Universe realized within 5d AdS bulk space. Of course, such ever expanding inflationary brane-world is understood in a sense of the analytical continuation of 4d sphere to Lorentzian signature. It would be interesting to understand the relation between such inflationary brane-world and inflation in D-branes, for example, of Hagedorn type.

Since one finds \( F(R_0, c) = \frac{klR_0^2}{4\pi G} \), Eq. (78) has a solution if \( \frac{klR_0^2}{4\pi G} \leq -8b' \). That puts again some bounds to the dilaton value. When \( |c| \) is small, one obtains \( R_0^4 \sim \frac{2c^2}{3\pi e} \), \( F(R_0, c) \sim \frac{1}{\sqrt{3\pi G}} \). Therefore Eq. (78) is satisfied for small \( |c| \). On the other hand, when \( c \) is large, we get \( R_0^6 \sim \frac{c^2}{2\pi e} \), \( F(R_0, c) \sim \frac{(k|c|)^2}{4\pi G} \). Therefore Eq. (78) is not always satisfied and we have no solution for \( R \) in (78) for very large \( |c| \). Then the existence of the inflationary Universe gives a restriction on the value of \( c \), which characterizes the behavior of the dilaton.

We now consider the \( k < 0 \) case. When \( c = 0 \), there is no solution for \( R \) in (78). Let us define another function \( G(R, c) \) as follows:

\[
G(R, c) = 1 + \frac{t^2c^2}{24R^8} + \frac{kl^2}{3R^2}.
\] (82)

Since \( G(R, c) \) appears in the root of \( F(R, c) \) in (80), \( G(R, c) \) must be positive. Then \( \frac{\partial G(R, c)}{\partial R} = -\frac{t^2c^2}{36R^8} - \frac{2kl^2}{3R^6} \), \( G(R, c) \) has a minimum \( 1 + \frac{kl^2}{4} \left( -\frac{2k}{e^2} \right)^\frac{1}{2} \) when \( R^6 = -\frac{c^2}{2e} \). Therefore if \( c^2 \geq \frac{4kL^6}{32e^2} \), \( F(R, c) \) is real for any positive value of \( R \). Since \( F(0, c) = \frac{|c|}{\pi e^\sqrt{24}} \), and when \( R \to \infty \), \( F(R, c) \to \frac{k|c|}{6e^\sqrt{24}} < 0 \), there is a solution \( R \) in (78) if \( |c| > 8b' \). If we Wick-rotate the solution corresponding to hyperboloid, we obtain a 4 dimensional AdS space, whose metric is given by

\[
ds_{AdS_4}^2 = dz^2 + e^{\frac{2t}{3}} (-dt^2 + dx^2 + dy^2).
\] (83)

Then there is such kind of solution due to the quantum effect if the parameter \( c \) characterizing the behavior of the dilaton is large enough. Thus we demonstrated that due to the dilaton presence there is the possibility of quantum creation of a 4d hyperbolic wall Universe. Again, some bounds to the dilaton appear. It is remarkable that hyperbolic brane-world occurs even for usual matter content due to the dilaton. One can compare with the case in ref. 42 where a hyperbolic 4d wall could be realized only for higher derivative conformal scalar. Such brane-world quantum inflation for the case of constant dilaton has been presented in refs. 38, 37, 42. In the usual 4d world the anomaly induced inflation has been suggested in ref. 43 (no dilaton) and in ref. 46 when a non-constant dilaton is present.
In summary, in this section for constant bulk potential, we presented the nice realization of quantum creation of 4d de Sitter or 4d hyperbolic brane Universes living in 5d AdS space. The quantum dynamical determination of dilaton value is also remarkable.

One can consider the case that the dilaton field $\phi$ has a non-trivial potential:

$$\frac{12}{l^2} \rightarrow V(\phi) = \frac{12}{l^2} + \Phi(\phi).$$

The surface counter terms when the dilaton field $\phi$ has a non-trivial potential are given in (16), which we write in the following form:

$$S^{(2)} = S_1^{\phi} + S_2^{\phi},$$

$$S_1^{\phi} = -\frac{1}{16\pi G} \int d^4\sqrt{g_4} \left( \frac{6}{l} + \frac{l}{4} \Phi(\phi) \right),$$

$$S_2^{\phi} = -\frac{1}{16\pi G} \int d^4 \left\{ \sqrt{g_4} \left( \frac{l}{2} R_4 - \frac{l}{2} \Phi(\phi) 
- \frac{l}{4} \nabla_4(\phi) \cdot \nabla_4(\phi) - \frac{l^2}{8} n^\mu \partial_4 \left( \sqrt{g_4} \Phi(\phi) \right) \right\}. \right.$$  

Following the argument in 37, if one replaces $\frac{12}{l^2}$ in (57) and $S_1^{\phi}$ in (56) with $V(\phi)$ in (84) and $S_1^{\phi}$ in (85), we obtain the gravity on the brane induced by $S_2^{\phi}$. We now assume the metric in the following form

$$ds^2 = f(y)dy^2 + y \sum_{i,j=0}^{3} \hat{g}_{ij}(x^k)dx^i dx^j,$$

as in 44 and $\phi$ depends only on $y$. As the singularity usually appears at $y = 0$, we investigate the behavior when $y \sim 0$. Here we only consider the case $k > 0$.

First one assumes that there is no singularity. Then $\phi$, $\frac{d\phi}{dy}$, and $\frac{d^2\phi}{dy^2}$ would be finite and we can assume

$$\phi \rightarrow \phi_1 \text{ (constant) when } y \rightarrow 0.$$  

It is supposed the spacetime becomes asymptotically AdS, which is presumably the unique choice to avoid the singularity and to localize gravity on the brane 45. The condition to get asymptotically AdS requires

$$\Phi'(\phi_1) = 0,$$

and one assumes

$$\Phi'(\phi) \sim \beta \phi_2^\alpha \ (\alpha > 0), \quad \phi_2 \equiv \phi - \phi_1.$$
Then from the equation of motion, if we also assume $\phi_2$ behaves as

$$\phi_2 \sim \tilde{b} y^a \ (a > 0) ,$$

one obtains

$$\alpha = 1 - \frac{1}{a} \quad (91)$$

and

$$\beta = -\frac{4k}{3} \tilde{b}^2 a \left( a + \frac{3}{2} \right) . \quad (92)$$

Eq.(91) requires $0 < \alpha < 1$ and/or $a > 1$ and Eq.(92) tells that $\beta$ cannot vanish and $\tilde{b}$ should be positive, which tells, from the equation of motion that $\phi$ increases when $y \sim 0$.

In $^{49}$, it was considered the following example as a toy model:

$$l^2 \Phi(\phi) = -\frac{4}{3} \phi^3 + \frac{3}{4} \phi^4 - \frac{1}{8} \phi^8 + \frac{17}{24} . \quad (93)$$

Using the numerical calculations, it was confirmed that there is no any (curvature) singularity and the gravity on the brane can be localized. Hence, we presented examples of inflationary and hyperbolic brane-worlds as analytical solutions in d5 dilatonic gravity when brane CFT quantum effects are also taken into account.

6 Discussion

In summary, we reported the results on various topics in d5 gauged supergravity with single scalar and arbitrary scalar potential in AdS/CFT set-up. In particular, the surface counterterms, finite gravitational action and consistent stress tensor in asymptotically AdS space is found. Using this action, the regularized expressions for free energy, entropy and mass are derived for d5 dilatonic AdS black hole. From another side, finite action may be used to get the holographic conformal anomaly of boundary QFT with broken conformal invariance. Such conformal anomaly is calculated from d5 gauged SG with arbitrary dilatonic potential with the use of AdS/CFT correspondence. Due to dilaton dependence it takes extremely complicated form. Within holographic RG where identification of dilaton with some coupling constant is made, we suggested the candidate c-function for d4 boundary QFT from holographic conformal anomaly. It is shown that such proposal gives monotonic and positive c-function for few examples of dilatonic potential.

We expect that our results may be very useful in explicit identification of supergravity description (special RG flow) with the particular boundary gauge
theory (or its phase) which is very non-trivial task in AdS/CFT correspond-
dence. We show that on the example of constant dilaton and special form
of dilatonic potential where qualitative agreement of holographic conformal
anomaly and QFT conformal anomaly (with the account of radiative corrections)
from QED-like theory with single coupling constant may be achieved.

The role of brane quantum matter effects in the realization of de Sitter
or AdS dilatonic branes living in d5 (asymptotically) AdS space is reported.
(We are working again with d5 dilatonic gravity). The explicit examples of
such dilatonic brane-world inflation are presented for constant bulk dilatonic
potentials as well as for non-constant bulk potentials. Dilaton gives extra
contributions to the effective tension of the domain wall and it may be de-
termined dynamically from bulk/boundary equations of motion. The main
part of discussion has dealing with maximally SUSY Yang-Mills theory (ex-
act CFT) living on the brane. However, qualitatively the same results may
be obtained when not exactly conformal quantum matter (like classically con-
formally invariant theory of dilaton coupled spinors) lives on the brane. An
explicit example of toy (fine-tuned) dilatonic potential is presented for which
the following results are obtained from the bulk/boundary equations of motion:
1. Non-singular asymptotically AdS space is the bulk space. 2. The brane is
described by de Sitter space (inflation) induced by brane matter quantum ef-
ects. 3. The localization of gravity on the brane occurs. The price to avoid
the bulk naked singularity is the fine-tuning of dilatonic potential and dynam-
ical determination (actually, also a kind of fine-tuning) of dilaton and radius
of de Sitter brane. Note also that in the same fashion as in ref.37 one can show
that the brane CFT strongly suppresses the metric perturbations (especially,
on small scales).

One can easily generalize the results of this report in different directions.
For example, following to brane-world line and taking into account that it is
not easy to find new dilatonic bulk solutions like asymptotically AdS space
presented in this work one can think about changes in the structure of the
boundary manifold. One possibility is in the consideration of a Kantowski-
Sachs brane Universe. Another important question is related with the study of
cosmological perturbations around the founded backgrounds and of details of
late-time inflation and exit from inflationary phase in brane-world cosmology
(eventual decay of de Sitter brane to FRW brane). The number of other
topics on relations between AdS/CFT and brane-world quantum cosmology in
dilatonic gravity maybe also suggested.

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A Remarks on boundary values

From the leading order term in the equations of motion

\[ 0 = -\sqrt{-\hat{G}} \frac{\partial \Phi(\phi_1, \cdots, \phi_N)}{\partial \phi_\beta} - \partial_\mu \left( \sqrt{-\hat{G}} \hat{G}^{\mu\nu} \partial_\nu \phi_\beta \right), \]

which are given by variation of the action (95)

\[ S = \frac{1}{16\pi G} \int_{M_{d+1}} d^{d+1}x \sqrt{-\hat{G}} \left\{ \hat{R} - \sum_{\alpha=1}^{N} \frac{1}{2} (\nabla_\alpha \phi)^2 + \Phi(\phi_1, \cdots, \phi_N) + 4\lambda^2 \right\}. \]

with respect to \( \phi_\alpha \), we obtain

\[ \frac{\partial \Phi(\phi(0))}{\partial \phi(0)_\alpha} = 0. \]

The equation (96) gives one of the necessary conditions that the spacetime is asymptotically AdS. The equation (96) also looks like a constraint that the boundary value \( \phi(0) \) must take a special value satisfying (96) for the general fluctuations but it is not always correct. The condition \( \phi = \phi(0) \) at the boundary is, of course, the boundary condition, which is not a part of the equations of motion. Due to the boundary condition, not all degrees of freedom of \( \phi \) are dynamical. Here the boundary value \( \phi(0) \) is, of course, not dynamical. This tells that we should not impose the equations given only by the variation over \( \phi(0) \). The equation (96) is, in fact, only given by the variation over \( \phi(0) \). In order to understand the situation, we choose the metric in the following form

\[ ds^2 \equiv \hat{G}_{\mu\nu} dx^\mu dx^\nu = \frac{l^2}{4} \rho^{-2} d\rho d\rho + \sum_{i=1}^{d} \hat{g}_{ij} dx^i dx^j, \quad \hat{g}_{ij} = \rho^{-1} g_{ij}, \]

(If \( g_{ij} = \eta_{ij} \), the boundary of AdS lies at \( \rho = 0 \)) and we use the regularization for the action (95) by introducing the infrared cutoff \( \epsilon \) and replacing

\[ \int d^{d+1}x \rightarrow \int d^d x \int_\epsilon d\rho, \quad \int_{M_d} dx^i (\cdots) \rightarrow \int d^d x (\cdots) |_{\rho = \epsilon}. \]
Then the action (95) has the following form:

\[ S = \frac{l}{16\pi G d} \varepsilon^{\frac{d}{2}} \int_{M_4} d^4x \sqrt{-\hat{g}(0)} \left\{ \Phi(\phi_1(0), \ldots, \phi_N(0)) - \frac{8}{l^2} \right\} + \mathcal{O}\left(\varepsilon^{-\frac{d}{2} + 1}\right). \tag{99} \]

Then it is clear that Eq. (96) can be derived only from the variation over \( \phi(0) \) but not other components \( \phi_i(0) \) (\( i = 1, 2, 3, \ldots \)). Furthermore, if we add the surface counterterm \( S_b^{(1)} \)

\[ S_b^{(1)} = -\frac{1}{16\pi G} \frac{d}{2} \varepsilon^{\frac{d}{2}} \int_{M_4} d^4x \sqrt{-\hat{g}(0)}\Phi(\phi_1(0), \ldots, \phi_N(0)) \tag{100} \]

to the action (95), the first \( \phi(0) \) dependent term in (99) is cancelled and we find that Eq. (96) cannot be derived from the variational principle. The surface counterterm in (100) is a part of the surface counterterms, which are necessary to obtain the well-defined AdS/CFT correspondence. Since the volume of AdS is infinite, the action (95) contains divergences, a part of which appears in (99). Then in order that we obtain the well-defined AdS/CFT set-up, we need the surface counterterms to cancel the divergence.

References

9. G. ’t Hooft, gr-qc/9310026.
12. V. Balasubramanian and P. Kraus, hep-th/9902121.
34. V. Balasubramanian, E. Gimon and D. Minic, hep-th/0003147.


47. S. Abel, K. Freese and I. Kogan, hep-th/0005028.