THEORETICAL ASPECTS OF THE BEAM-BEAM INTERACTION

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ABSTRACT

Experimental data on the beam-beam limit in e^+e^- storage rings are compared to predictions from two models: the coherent beam-beam effect and the fourth-order coupling resonance excited by the beam-beam interaction. Better tunes for existing e^+e^- storage rings are recommended.

1. INTRODUCTION

Experimental data on the maximum luminosity achieved in e^+e^- storage rings should not be interpreted uniquely in terms of a stochasticity limit until all other simpler possibilities for explaining the observations are exhausted (Occam's Razor*). In this paper, attention is drawn to two particular phenomena which might indeed limit the luminosity at lower values than stochasticity, namely the coherent beam-beam effect and a fourth-order coupling resonance driven by the beam-beam interaction itself. In either case, it is recommended to change systematically the tunes of several e^+e^- storage rings. The effect of these recommendations on their performance can - and should - be checked.

2. COHERENT BEAM-BEAM EFFECT

The term coherent beam-beam effect describes a phenomenon associated with the transverse motion of k rigid bunches circulating in a storage ring in one direction, colliding with the k bunches of the opposite beam in 2k interaction regions.

In the simplest case with just one bunch in each beam, this phenomenon has been used to measure\(^1,2\) the linear small-amplitude tune shift \(\Delta Q\) in one interaction region. The difference in frequency is measured between the only two modes of coherent oscillation possible in this case, the 0-mode and the \(\pi\)-mode. In the 0-mode, the bunches oscillate in phase and the coherent frequency is equal to the product of tune \(Q\) and revolution frequency \(f\). In the \(\pi\)-mode, the bunches oscillate out of phase. The coherent frequency \(Q_f\) is higher than \(Q\) when the beam-beam force is attractive. It is given by

\[
\cos \pi Q_f = \cos \pi Q - 4 \pi \Delta Q \sin \pi Q
\]

For \(Q\) not too close to an integer, and for \(\Delta Q < 1\), \(Q_f = Q + 4 \Delta Q\).

In the simplest form of the theory\(^3,4,5\), the transverse forces between the bunches are assumed to be proportional to the impact parameter between their centres of gravity. In this case, the phenomenon can be formulated as an eigenvalue problem for a symplectic matrix of order \(4k\). When all eigenvalues are on the unit circle in the complex plane, all modes of coherent beam-beam oscillations are stable. If an eigenvalue has an absolute value larger than unity, an unstable mode with exponential growth occurs. A discussion of the behaviour at large amplitudes is outside the scope of this linear theory.

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*Willian of Occam (c. 1290 - 1350), English theologian, propounded the principle that for purposes of explanation things not known to exist should not, unless it is absolutely necessary, be postulated as existing (The Oxford English Dictionary, Vol. VIII).
Fig. 1: Coherent beam-beam limit ΔQ vs. tune Q for 1 to 4 bunches.

The maximum permissible value of the linear beam-beam tune shift ΔQ is a periodic function of the tune Q with period k. It has a characteristic sawtooth behaviour. For attractive forces as in e^+e^- machines, the maximum values of ΔQ vanish just below all integral values of the tune Q, while they reach their maximum values just above integral values of Q. This is shown in Figure 1. For repulsive forces the situation is reversed. Compared to an earlier version of these stability diagrams^4) no unstable regions just below half-integral tunes are shown. There are no such unstable regions in a machine in which the betatron phase advances between all 2k interaction regions are the same. However, there may be narrow stopbands originating at half-integral tunes in a machine with small differences in the phase advances. An example of this general phenomenon for k = 1 was given by Chao and Keil^5).

The beam-beam limits which can be calculated from this effect for several machines are shown in Table 1. In a fair fraction of all machines, limiting values well below the canonical value ΔQ = 0.06 are obtained, in particular when operating with more than one bunch in each beam.

Table 1
Coherent beam-beam limits for various machines

<table>
<thead>
<tr>
<th>Machine</th>
<th>Plane</th>
<th>Tune Q</th>
<th>Number of Bunches k</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>ADONE^6)</td>
<td>H,V</td>
<td>3.1</td>
<td>&gt; 0.1</td>
</tr>
<tr>
<td>CESSR^7)</td>
<td>H</td>
<td>9.43</td>
<td>0.099</td>
</tr>
<tr>
<td>DC^6)</td>
<td>V</td>
<td>0.60</td>
<td>0.026</td>
</tr>
<tr>
<td>FEP^8)</td>
<td>V</td>
<td>18.77</td>
<td>0.021</td>
</tr>
<tr>
<td>PETRA^6)</td>
<td>V</td>
<td>23.3</td>
<td>0.049</td>
</tr>
<tr>
<td>SPEAR^6)</td>
<td>H</td>
<td>5.27</td>
<td>&gt; 0.1</td>
</tr>
</tbody>
</table>

If the coherent beam-beam limit is to occur above the limit given by stochasticity, the tunes Q of a machine should be chosen in the following ranges

\[ Q = k \cdot n + m + \epsilon \]  

where 0 < m < k - 1 and n are integers, and 0 < \epsilon < 1. The best choice for m is m = 0. For ΔQ = 0.05 and k ≫ 1 it is the only choice as can be seen from Figure 1.
In a discussion of feedback systems against the coherent beam-beam effect it is important to know the order of magnitude of the largest eigenvalue $|\lambda|_{\text{max}}$ of the transformation matrix for one turn, shown in Figure 2, which may be large compared to unity. This implies that the most unstable mode of coherent oscillations will grow by a factor of the order of $|\lambda|_{\text{max}}$ in one turn. This rate of growth is probably too fast for reasonable feedback systems. The growth will stop eventually because of the non-linearity of the forces.

![Graph showing $|\lambda|_{\text{max}}$ vs. tune](image)

**Fig. 2:** Largest eigenvalue $|\lambda|_{\text{max}}$ vs. tune for 4 bunches and $\delta Q = 0.03$.

![Graph showing $(Q_y - Q_x) \mod k$ vs. $k\Delta Q$](image)

**Fig. 3:** Limiting values of $(k\Delta Q)$ in the vicinity of the coupling resonance $2Q_x - 2Q_y = 2kn$.

3. **COUPLING RESONANCES DRIVEN BY THE BEAM-BEAM INTERACTION**

The head-on collisions between $k$ bunches in each of two counter-rotating beams, colliding in $2k$ interaction regions, systematically drive the following non-linear resonances:

$$2p_x Q_x \pm 2p_y Q_y = 2kn$$

(3)

Here $p_x$, $p_y$ and $n$ are integers. The horizontal and vertical tunes are called $Q_x$ and $Q_y$, respectively. The coefficients of $Q_x$ and $Q_y$ must be even because of the assumed symmetry of the head-on collisions. Odd coefficients only arise from off-centre collisions. Difference resonances are basically stable\(^9\), and only lead to an exchange between horizontal and vertical emittance. However, the design of $e^+e^-$ storage rings with low-beta insertions usually relies on the vertical emittance being much smaller than the horizontal one. If this condition is not fulfilled a drastic reduction in luminosity occurs, and possibly also a reduction in lifetime because of a vertical aperture limitation.

The resonance of lowest order has $p_x = p_y = 1$. The beam behaviour in its neighbourhood has been studied by Monticelli\(^10\). Its consequences are shown in Figure 3. The abscissa is $(Q_y - Q_x) \mod k$, the ordinate is the product $(k\Delta Q)$ of the number of bunches times the beam-beam tune shift $\Delta Q$. Emittance exchange is expected to occur above the lines. The most striking feature of Figure 3 is its asymmetry. For $(Q_y - Q_x) \mod k < 0$, the coupling is easily avoided, while for $(Q_y - Q_x) \mod k > 0$, coupling is very hard to avoid. The values of $(Q_y - Q_x) \mod k$ are also shown in Figure 3 for several machines with small design coupling\(^7,8,11\).
4. CONCLUSIONS

From the considerations presented, better tunes for the operation of PEP and PETRA can be recommended.

1) The tunes in PEP (Q_x = 21.84, Q_y = 18.77) should each be lowered by about half a unit. This would bring PEP into the best possible operating range for the coherent beam-beam effect with one and with three bunches. The coupling resonance should be avoided by keeping the fractional part of Q_y lower than that of Q_x, as it is now.

2) The tunes in PETRA (Q_x = 25.2, Q_y = 23.3) should be changed such that the fractional part of Q_y is smaller than that of Q_x, in order to avoid the coupling resonance. Whether or not a change of the integral part of the tunes is also required, depends on the number of bunches in one beam. For one bunch, the present choice is good; for two bunches it is marginal. For operation with four bunches, different tunes should be adopted. The best choice is Q_x and Q_y just above a multiple of four, as given in (2) with m = 0 or possibly m = 1 (the present choice of Q_y with m = 3 is bad).

It seems important to follow up on these recommendations in PEP and PETRA, the only multi-bunch machines presently in operation. This is so not only because it might further our understanding of the beam-beam limit, but also because of the substantial gains in performance to be expected if the explanations offered here are correct. It goes without saying that these considerations should also be applied to future machines such as LEP.

Acknowledgement

I should like to thank B.W. Montague for helpful discussions and Occam's Razor.

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2) M. Tigner et al., private communication (1980).
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