HARMONIC SPINORS OF DIRAC OPERATOR OF CONNECTION WITH TORSION IN DIMENSION 4

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Abstract

It is shown that the existence of parallel (-)-spinor with respect to a metric connection
with totally skew-symmetric torsion requires more local restriction than the existence of parallel
(+) -spinor. It is proved that every harmonic spinor with respect to the Dirac operator of this
connection on a compact four dimensional spin Riemannian manifold is parallel with respect to
a naturally arising metric connection with totally skew-symmetric torsion and all these spaces
are classified up to a conformal transformations.

MIRAMARE - TRIESTE
June 2000
1 Introduction

Linear connections preserving a Riemannian metric and having totally skew-symmetric torsion are
have recently been of interest mainly for physical reasons. These connections arise in a natural
way in theoretical and mathematical physics. For example, the target space of supersymmetric
sigma models with the Wess-Zumino term carries a geometry of a metric connection with skew-
symmetric torsion [7, 16, 17] (see also [22] and references therein); in the supergravity theories,
the geometry of moduli space of a class of black holes is carried out by a metric connection
with skew-symmetric torsion [14], the geometry of NS-5 brane solution of type II supergravity
theories is generated by a metric connection with skew-symmetric torsion [24, 25, 23].

The existence of parallel spinors with respect to a metric connection with skew-symmetric
torsion on a spin Riemannian manifold is important in string theory since parallel spinors are
associated with some string solitons (BPS solitons) [23].

On a 4-dimensional Riemannian spin manifold \((M, g)\) the existence of parallel spinors with
respect to a metric connection \(\nabla\) with skew-symmetric torsion forces the (restricted) holonomy
group of \(\nabla\) to be contained in \(SU(2)\) [23]. As shown in [18] this is equivalent to the local existence
of a HKT structure i.e. a hypercomplex structure parallel with respect to \(\nabla\). It is well known
consequence of the integrability theorem in [4] that the self-dual Weyl tensor must vanish if there
exists (local) hypercomplex structure (see e.g. [13] and references therein). If \(M\) is compact it is
shown in [18], relying on the results in [12, 2], that the holonomy of \(\nabla\) is contained in \(SU(2)\) iff
\(M\) is either a Calabi-Yau or a Hopf surface. We note that there are Hopf surfaces which do not
admit any (global) hyperhermitian structure [10] but the holonomy of \(\nabla\) is contained in \(SU(2)\).

In the present note we are interested in the space of harmonic spinors with respect to the
Dirac operator naturally associated to a metric connection \(\nabla\) with skew-symmetric torsion \(T\)
on a compact 4-dimensional Riemannian spin manifold \((M, g)\). If \(dT = 0\) the structure \((g, \nabla)\)
is called strong and if \(d^*T = 0\) it is called closed. On a spin Riemannian manifold the spinor
bundle \(\Sigma M\) decomposes into two parts \(\Sigma M = \Sigma_+ M \oplus \Sigma_- M\) of \((+)\)-spinors and \((-)\)-spinors. We
discovered that the existence of \(\nabla\)-parallel \((-)\)-spinor induced more (local) restrictions on the
geometry than the existence of \(\nabla\)-parallel \((+)\)-spinor (see Theorem 4.2 below).

Following [19] we consider the associated Weyl structure i.e. the unique torsion-free linear
connection \(\nabla^W\) preserving the conformal structure and determined by \(\nabla^W g = *T \otimes g\), where
* is the Hodge star operator. The corresponding scalar curvature \(k\) of \(\nabla^W\) is called conformal
scalar curvature and has conformally invariant sign [10]. We define 'conformal change' of the
connection \(\nabla\) i.e. a natural transformation of \(\nabla\) induced by a conformal rescaling of the metric
and show that the dimension of the space of harmonic spinors i.e. the dimension of the kernel of
the associated Dirac operator, is a conformal invariant. This allows us to apply the Gauduchon
gauge and prove our main

Theorem 1.1 Let \((M, g, \nabla, T)\) be an oriented four-dimensional compact spin Riemannian man-
ifold equipped with a metric connection with torsion 3-form $T$ and of non-negative conformal scalar curvature $k$. Then

i) There exists non-zero $\nabla$-harmonic $(-)$ spinor iff $k = 0$ and $(M, g, \nabla)$ is a conformally equivalent to a flat torus with its hyper-Kähler metrics or a coordinate quaternionic Hopf surface with its standard flat, strong and closed HKT structure and the metric is of constant scalar curvature in the conformal class of the standard locally conformally flat metric.

ii) There exists non-zero $\nabla$-harmonic $(+)$ spinor iff $k = 0$ and $(M, g, \nabla)$ is conformally equivalent either to one of the spaces listed in i) or to a K3-surface with its hyper-Kähler metrics.

iii) There exists $\nabla$-harmonic spinor of generic type iff there exists $\nabla$-harmonic $(-)$ spinor.

The proof relies on the Lichnerowicz formula for Dirac operator of general connection with torsion proved in [5] (see also [1]).

If $M$ is not compact the existence of $\nabla$-parallel $(+)$-spinor does not imply the vanishing of the Ricci tensor of $\nabla$ as can be seen in the examples in the end of the note. However, Theorem 1.1 shows that in the compact case the existence of parallel spinor forces the Ricci tensor to be zero.

2 Preliminaries

Let $(M, g)$ be an $n$-dimensional ($n > 1$) Riemannian manifold. Let $\nabla$ be a linear connection. Denote by $T$ the torsion of $\nabla$, $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$. The connection $\nabla$ has skew-symmetric torsion if $T(X, Y, Z) := g(T(X, Y), Z)$ is a 3-form. In the sequel, we shall call a Riemannian manifold $(M, g, \nabla, T)$ with a linear connection of skew-symmetric torsion Riemannian manifold with torsion (briefly RT). If the torsion is closed then the manifold is called strong RT manifold.

We note that in the physics literature the geometry of a metric connection with torsion is often called Riemann-Cartan geometry. Here we prefer RT manifold to emphasize on the skew-symmetric property of the torsion.

Any metric connection $\nabla$ with skew-symmetric torsion $T$ is connected by the Levi-Civita connection $\nabla^g$ of the metric $g$ by

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} T(X, Y, Z).$$

Let $R = [\nabla, \nabla] - [\nabla, [\nabla]]$ be the curvature tensor of type $(1,3)$ of $\nabla$. We denote the curvature tensor of type $(0,4)$ $R(X, Y, Z, V) = g(R(X, Y)Z, V)$ by the same letter.

The Ricci tensor $Ric^\nabla$ and the scalar curvature $s^\nabla$ are defined by

$$Ric^\nabla(X, Y) = \sum_{i,j=1}^{n} R(e_i, X, Y e_i), \quad s^\nabla = \sum_{i,j=1}^{n} R(e_j, e_i, e_i, e_j).$$

Here and henceforth $e_1, e_2, ..., e_{2n}$ is an orthonormal basis of the tangential space.
By (2.1) the Ricci tensor \( \text{Ric} \) and the scalar curvature \( s \) of the Levi-Civita connection are connected by \( \text{Ric}^\nabla \) and \( s^\nabla \) as follows

\[
(2.2) \quad \text{Ric}(X,Y) = \text{Ric}^\nabla(X,Y) + \frac{1}{2} d^*T(X,Y) - \frac{1}{4} \sum_{i=1}^n g(T(X,e_i),T(Y,e_i)), \quad s = s^\nabla + \frac{3}{2} |T|^2,
\]

where \( |\cdot|^2 \) denotes the norm of a p-form.

**Remark 1.** It is well known (see e.g. [27]) that if \( \nabla \) is flat i.e. with zero curvature, then \( (M, g, \nabla, T) \) carries (locally) a structure of a compact Lie group, \( g \) is a biinvariant metric and \( \nabla \) is the canonical invariant connection with torsion given by the Lie bracket except some structures on seven dimensional spheres.

We recall the notion of HKT structures. A Hyperhermitian structure on a 4n-dimensional manifold \( M \) is a triple \( (J_\alpha), \alpha = 1, 2, 3, \) of complex structures satisfying the quaternionic identities and a Riemannian metric \( g \) which is hermitian with respect to each complex structure \( J_\alpha, \alpha = 1, 2, 3 \). A Hyperhermitian manifold with torsion (briefly HKT) is a Hyperhermitian manifold equipped with a linear connection \( \nabla \) with totally skew-symmetric torsion which preserves the metric and each of the complex structures \( J_\alpha, \alpha = 1, 2, 3 \). It turns out that a Hyperhermitian manifold is HKT if and only if the Bismut connections of the three complex structures \( J_\alpha, \alpha = 1, 2, 3 \) coincide [17, 14, 11]. In particular the three Lee forms also coincide [13, 19] and the common Lee form \( \theta \) is called the Lee form of the HKT structure. If each \( J_\alpha, \alpha = 1, 2, 3 \) is parallel with respect to the Levi-Civita connection then the structure is Hyper Kähler.

In dimension four every hyperhermitian structure is a HKT structure with Bismut connection of torsion \( T = -*\theta \) [13]. If the Lee form is closed then the HKT structure is closed since \( d^*T = d\theta \).

Let \( (M, g, J) \) be a spin manifold with spinor bundle \( \Sigma M \) and let \( \mu : T^*M \otimes \Sigma M \rightarrow \Sigma M \) be the Clifford multiplication. Identifying \( T^*M \) and \( TM \) via the metric we shall also consider \( \mu \) as a map from \( TM \otimes \Sigma M \) into \( \Sigma M \). We shall usually write the Clifford multiplication by juxtaposition, i.e.

\[
\mu(\alpha \otimes \psi) = \alpha \psi, \quad \mu(X \otimes \psi) = X\psi, \quad \alpha \in T^*M, \quad X \in TM, \quad \psi \in \Sigma M.
\]

The volume form \( \omega_g \) of the metric \( g \) acts on \( \Sigma M \) as an involution with (-1)-eigenspace \( \Sigma_+ M \) and (+1)-eigenspace \( \Sigma_- M \) and \( \Sigma M \) decomposes as

\[
\Sigma M = \Sigma_+ M \oplus \Sigma_- M, \quad \Sigma_\pm M = \{ \psi | \omega_g \psi = \mp \psi \}.
\]

Any metric connection \( \nabla \) with skew-symmetric torsion in the tangent bundle gives rise to a metric connection in the spinor bundle and it is a easy consequence of (2.1) that

\[
(2.3) \quad \nabla_X \psi = \nabla^g_X \psi + \frac{1}{4} \sum_{i<j} T(X, e_i, e_j)e_i.e_j, \quad X \in \Gamma(TM), \psi \in \Gamma(\Sigma M).
\]
The Dirac operator $D$ of $\nabla$ is defined by $D = \mu \circ \nabla$ or equivalently $D \psi = \sum_{i=1}^{n} e_i \nabla_{e_i} \psi$, $\psi \in \Gamma(\Sigma M)$. As a consequence of (2.3) we get the formula

$$D = D^g + \frac{3}{4} \sum_{i<j<k} T(e_i, e_j, e_k) e_i.e_j.e_k$$

where $D^g$ is the Dirac operator of the Levi-Civita connection. From (2.4) it follows that $D$ is self-adjoint.

The Lichnerowicz formula for $D$, proved in [5], states that

$$D^2 = (\nabla^3)^* \nabla^3 + \frac{1}{4} \Delta - \frac{3}{4} dT - \frac{9}{8} |T|^2,$$

where the connection $\nabla^3$ is a metric connection on $TM$ with torsion $3T$ and its action on $\Sigma M$ is given by

$$\nabla_X \psi = \nabla^g_X \psi + \frac{3}{4} \sum_{i<j} T(X, e_i, e_j) e_i.e_j$$

Let $\nabla^{1/3}$ be a metric connection on $TM$ with torsion $\frac{1}{3} T$. The action of the corresponding Dirac operator $D^{1/3}$ on $\Sigma M$ is

$$D^{1/3} = D^g + \frac{1}{4} \sum_{i<j<k} T(e_i, e_j, e_k) e_i.e_j.e_k$$

The operator $D^{1/3}$ is self-adjoint and the Lichnerowicz formula (2.5) gives

$$(D^{1/3})^2 = \nabla^* \nabla + \frac{1}{4} \Delta + \frac{1}{4} dT - \frac{1}{8} |T|^2,$$

It is shown in [5] that, in case of complex manifold, the operator $D^{1/3}$ coincides with the Dolbeault operator $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ provided that the torsion $T$ is the torsion of the unique Bismut connection determined by any hermitian structure.

3 Conformal transformations of RT manifolds in dimension four

From now on $(M, g, \nabla, T)$ is a 4-dimensional RT manifold with torsion 3-form $T$.

Let $*$ be the Hodge star operator. We define the torsion 1-form $t$ as the 1-form corresponding to $T$ via $*$-operator i.e.

$$T = -* t.$$

Let $\tilde{g} = e^f g$ be a conformal rescaling of the metric. In general, in higher dimensions there is no natural transformation of the connection $\nabla$. However, in case of a QKT structure, i.e. when there is a quaternionic structure parallel with respect to $\nabla$ and the torsion is $(1,2)+(2,1)$-form with respect to any compatible local almost complex structure (see e.g. [22]), a natural 'conformal transformation' of the connection is defined in [19]. It is well known that every 4-dimensional oriented Riemannian manifold carries a hermitian quaternionic structure and therefore every
4-dimensional RT manifold is a QKT space in a natural way. Following [19] we define the conformally transformed connection \( \tilde{\nabla} \) by the equation

\[
\tilde{g}(\tilde{\nabla} X Y, Z) = \tilde{g}(\tilde{\nabla}_X Y, Z) - \frac{1}{2} \ast_{\tilde{g}}^\ast (t + df)(X, Y, Z),
\]

where \( \tilde{\nabla} \tilde{g} \) and \( \ast_{\tilde{g}} \) are the Levi-Civita connection and Hodge \( \ast \)-operator of \( \tilde{g} \) i.e. the torsion 1-form \( \tilde{t} \) of \( \tilde{\nabla} \) is given by \( \tilde{t} = t + df \).

We shall call \((M, g, \nabla)\) and \((M, \tilde{g}, \tilde{\nabla})\) conformally related RT manifolds.

If \( M \) is compact, we may apply the theorem of Gauduchon for the existence of a Gauduchon metric on a compact hermitian or Weyl manifold [8, 9, 10] to get the following partial result of [19]

**Proposition 3.1** Let \((M, g, \nabla, T)\) be a compact 4-dimensional RT manifold. In the conformal class of \( g \) there exists a unique (up to homothety) conformal metric \( g_C \) and a unique strong RT manifold \((M, g_C, \nabla_C, T_C)\) conformally related to \((M, g, \nabla, T)\).

**Proof.** The existence of Gauduchon metric \( g_C \) means \( d^* t_C = 0 \) which is equivalent to \( d T_C = 0 \) by (3.9).

Q.E.D.

The conformal scalar curvature \( k \) is expressed in terms of \( s \) and the torsion 1-form \( t \) by [10]

\[
k = s - \frac{3}{2} |t|^2 - 3 d^* t
\]

and is a conformal invariant of weight -1 i.e. if \( \tilde{g} = e^f g \) then

\[
\tilde{k} = e^{-f} k.
\]

Using (2.2) we find from (3.11) \( k = s^\nabla - 3 d^* t \). Thus, \( k = s^\nabla \) if and only if the RT structure is strong.

4 Dirac operator in 4-dimensional RT manifold

From now on \((M, g, T)\) is a 4-dimensional spin RT manifold.

We shall abbreviate \((+, -)\)-spinors (resp. \((-,-)\)-spinors by \( \psi_+ \) (resp. \( \psi_- \)). Thus \( \psi_\pm \) means \( \psi_\pm \in \Gamma(\Sigma_\pm M) \). Direct computations using (3.9) yield

\[
\sum_{i < j}^4 T(X, e_i, e_j) e_i e_j = - \sum_{i < j}^4 \ast t(X, e_i, e_j) e_i e_j = - (X.t + t(X))\omega.
\]

We apply (4.13) to (2.3), (2.4) (2.7) and the Lichnerowicz formula (2.8) to get

\[
\nabla_X \psi_+ = \nabla\nabla_X^2 \psi_+ + \frac{1}{4} (X.t + t(X))\psi_+,
\]

\[
\nabla_X \psi_- = \nabla\nabla_X^2 \psi_- - \frac{1}{4} (X.t + t(X))\psi_-,
\]

\[
D \psi_+ = D^g \psi_+ - \frac{1}{4} t \psi_+,
\]

\[
D \psi_- = D^g \psi_- + \frac{1}{4} t \psi_-.
\]
\[
D^{1/3} \psi_+ = D \psi_+ + \frac{1}{2} t \psi_+, \quad D^{1/3} \psi_- = D \psi_- - \frac{1}{2} t \psi_-
\]
(4.16)

\[
(D^{1/3})^2 = \nabla^\alpha \nabla_{\alpha} + \frac{1}{4} s + \frac{1}{4} d^* t \omega - \frac{1}{8} |T|^2,
\]
(4.17)

since \(dT = d^* t \omega\) by (3.9).

Recall (cf. [15]) that under a conformal change of the metric \(\bar{g} = e^f g\) of a spin manifold \((M, g)\) there is an identification \(\tilde{\Sigma}^\pm M\) of the spinor bundles \(\Sigma^\pm M\) of \((M, g)\) and the spinor bundle \(\tilde{\Sigma}^\pm M\) of \((M, \bar{g})\) such that

\[
\tilde{X} \tilde{\psi} = X \psi, \quad \tilde{\alpha} \tilde{\psi} = \tilde{\alpha} \psi, \quad \tilde{\nabla}_X \tilde{\psi} = \nabla_X \psi - \frac{1}{4} Xdf \tilde{\psi} - \frac{1}{4} Xf \tilde{\psi},
\]
(4.18)

\[
\tilde{D} \tilde{\psi} = e^{- \frac{1}{2} h} \tilde{D} \psi + \frac{1}{4} df \tilde{\psi},
\]
(4.19)

where \(\psi \in \Gamma(\Sigma M)\), \(\tilde{\psi} \in \Gamma(\tilde{\Sigma} M)\) is the spinor corresponding to \(\psi\), \(X \in TM\), \(\tilde{X} = e^{- \frac{1}{2} h} X\), \(\alpha \in T^* M\), \(\tilde{\alpha} = e^{\frac{1}{2} \alpha}\).

**Remark 2.** It follows from (4.14) and (4.18) that two conformally related RT structures \((g, \nabla)\) and \((\bar{g}, \tilde{\nabla})\) have common parallel (+)-spinors while a \(\nabla\)-parallel (-)-spinor is never \(\tilde{\nabla}\)-parallel provided the function \(f\) is not a constant.

**Proposition 4.1** Let \((M, g, \nabla)\) be a 4-dimensional RT manifold. The dimension of the kernel of the Dirac operator \(D\) of \(\nabla\) is a conformal invariant.

**Proof:** Since \(D(\Gamma(\Sigma^\pm M)) \subset \Gamma(\Sigma^\pm M)\), it is enough to show that \(\dim \ker D|_{\Sigma^\pm M}\) is a conformal invariant. The torsion 1-form \(\tilde{t}\) of the structure \((\bar{g}, \tilde{\nabla})\) is \(t + df\) by (3.10). Then (4.15), (4.18) and (4.19) yield \(\tilde{D} \tilde{\psi}_+ = e^{- \frac{1}{2} h} \tilde{D} \psi_+\), \(\tilde{D}(e^{\frac{1}{2} f} \tilde{\psi}_-) = e^{- 2f} \tilde{D} \tilde{\psi}_-\), and these equalities prove Proposition 4.1.

Q.E.D.

In the following we shall denote by \(<, \cdot, \cdot>\) and \(|\cdot|\) pointwise inner products and norms and by \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\) - the global ones, respectively. The self-dual (resp. anti-self-dual) part of a 2-form \(\alpha\) with respect to Hodge \(*\)-operator we abbreviate by \(\alpha^+\) (resp. \(\alpha^-\)).

As we have already mentioned in the introduction the existence of parallel spinor on a RT manifold implies that the holonomy of \(\nabla\) is contained in \(SU(2)\) [23]. In the next local observation we show that the existence of parallel (-)-spinor requires the RT manifold to be strong, closed and flat. We have

**Theorem 4.2** Let \((M, g, \nabla, T)\) be an oriented four-dimensional spin RT manifold admitting a non-zero parallel spinor. Then the holonomy of \(\nabla\) is contained in \(SU(2)\) and therefore, there (locally) exists a hypercomplex structure \((J_a), a = 1, 2, 3\) such that \((g, (J_a), \nabla)\) is a HKT structure. In particular \(g\) is anti-self-dual, \(k = 0\) and \(dt^+ = 0\). In this case there always exists a parallel (+)-spinor.

i) If there exists a parallel (-)-spinor then in addition \(dT = 0\), \(d^* T = 0\), \(\nabla\) is flat and therefore it is a 4-dimensional group manifold with its standard strong, closed and flat HKT
structure which is locally conformally hyperkähler. In this case there always exists a parallel \((-\cdot\)-spinor.

ii) There exists a parallel spinor of generic type iff there exists a parallel \((-\cdot\)-spinor.

Proof. We use some elementary facts from representation theory. Let $\mathcal{Cl}(4)$ be the Clifford algebra of a 4-dimensional Euclidean vector space. Then $\mathcal{Cl}(4)\cong \mathcal{H} \oplus \mathcal{H}$, where $\mathcal{H}$ is the space of quaternions. The space of spinors $S^4 = S_+ \otimes S_-$ is a reducible $\mathcal{Cl}(4)$-module, considered as a representation of the group $\text{Spin}(4)$ where $S_+$ and $S_-$ are irreducible. The action of group $\text{Spin}(4)\cong SU(2) \times SU(2)\cong Sp(1) \times Sp(1)$ on $S^4 \cong \mathcal{H} \oplus \mathcal{H}$ is $\{A, B\} \{s_+, s_-\} = \{As_+, Bs_-\}$. If there is a parallel spinor then the holonomy of the connection should be a subgroup of the stabilizer of the action of $\text{Spin}(4)$. If there exists a parallel spinor of generic type the stabilizer is trivial and if there exists a parallel half spinor then the stabilizer is $\text{SU}(2)\cong Sp(1)$. Hence, $\nabla$ is flat in the first case and its holonomy is contained in $\text{SU}(2)$ in the second. The latter is equivalent to the existence of local $\nabla$-parallel hyper almost complex structure which turns out to be a hypercomplex structure as it is shown in [18]. In this case $\nabla$ is the Bismut connection and the Weyl connection $\nabla^W$ coincides with the Obata connection [13] which is flat [26]. This implies $k = 0$.

Using (4.14) we see that $\nabla$, restricted to sections of $\Sigma^+M$, coincides with the Weyl connection considered in [21]. Hence, as proved in [21], $dt^+ = 0$ and a parallel spinor in $\Sigma^+M$ with respect to $\nabla$ gives rise to a hyper-Hermitian structure $(J_\omega)$ on $M$ and vice versa.

To prove i) we shall use the Lichnerowicz formula (4.17). We compute, using (4.15) and (4.16),

\begin{equation}
(D^{1/3})^2 \psi_\pm = \left(D^2 \mp \frac{1}{2}(tD - Dt) \mp \frac{1}{4}|t|^2\right) \psi_\pm.
\end{equation}

We apply (4.15) to the general identity $D\theta t + tD\theta = dt + d^*t - 2\nabla^\theta t$ to get

\begin{equation}
tD + Dt = dt + d^*t - 2\nabla^\theta t,
\end{equation}

since $\nabla^\theta t = \nabla t$ because of (4.13).

Substituting $Dt$ from (4.21) into (4.20), we derive

\begin{equation}
(D^{1/3})^2 \psi_\pm = \left(D^2 \mp tD \mp \frac{1}{2}(dt + d^*t) \mp \nabla t + \frac{1}{4}|t|^2\right) \psi_\pm.
\end{equation}

Inserting (4.22) into the Lichnerowicz formula (4.17), we obtain by use of (3.11) that

\begin{equation}
(D^2 - tD - \nabla t - \nabla^* \nabla + \frac{1}{2}dt - \frac{1}{4}k) \psi_+ = 0,
\end{equation}

\begin{equation}
(D^2 + tD + \nabla t - \nabla^* \nabla - \frac{1}{2}dt - \frac{1}{4}k - \frac{3}{2}d^*t) \psi_- = 0.
\end{equation}

Assume that there exists non-zero parallel \((-\cdot\)-spinor $\psi_-$. Then (4.24) reduces to

\[2dt\psi_- - (k + 6d^*t)\psi_- = 0\]

and consequently $2 < dt\psi_- , \psi_- > - (k + 6d^*t)|\psi_-|^2 = 0$. The real and imaginary parts of the last identity imply $k + 6d^*t = 0$, $dt^- = 0$. Since the holonomy of $\nabla$
is contained in SU(2) the first part of the theorem shows that there exist a parallel (+)-spinor and therefore \( dt^+ = k = 0 \). Hence, \( 0 = d^*t = dT, \quad d^*T = dt = dt^+ + dt^- = 0 \) and \( \nabla \) is flat because of the existence of parallel spinor of generic type. Hence, \((M, g, \nabla)\) is a 4-dimensional group manifold with the standard HKT structure. The existence of parallel spinor of generic type on 4-dimensional HKT group manifold is clear and the proof is complete. Q.E.D.

Applying the classification of compact hypercomplex surfaces of [6] (see also [21]), we get

Corollary 4.3 Let \((M, g, \nabla, T)\) be an oriented four-dimensional compact spin RT manifold.

i) There exists non-zero parallel (-) spinor iff \((M, g, \nabla, T)\) is a flat torus with its hyper-Kähler metrics or a coordinate quaternionic Hopf surface with its standard flat, strong and closed HKT structure and the metric is of constant scalar curvature in the conformal class of the standard locally conformally flat metric.

ii) There exists non-zero parallel (+) spinor iff \((M, g, J_\alpha)\) is either one of the spaces listed in i) or a K3-surface with its hyper-Kähler metrics.

iii) There exists parallel spinor of generic type iff there exists parallel (-)-spinor.

5 Proof of Theorem 1.1

We take the global inner product in (4.23), (4.24) and (4.21) to get

\[
||D\psi_+||^2 - (t D\psi_+, \psi_+) - (\nabla_t \psi_+ , \psi_+ ) = ||\nabla \psi_+||^2 - \frac{1}{2}(dt \psi_+, \psi_+) + \frac{1}{4}(k \psi_+, \psi_+),
\]

\[
||D\psi_-||^2 + (t D\psi_-, \psi_-) + (\nabla_t \psi_- , \psi_- ) = ||\nabla \psi_-||^2 + \frac{1}{2}(dt \psi_-, \psi_-) + \frac{(k + 6d^*t)\psi_- , \psi_-}{4},
\]

\[
(t \psi, D\psi) - (D\psi, t \psi) = (dt \psi, \psi) + (d^* t \psi, \psi) - (2 \nabla_t \psi, \psi).
\]

The left side of (5.27) is an imaginary complex number and therefore the real part of the right side must vanish, i.e.

\[
Re(\nabla_t \psi, \psi) = (d^* t \psi, \psi).
\]

Inserting (5.28) into (5.25) and (5.26), we derive

\[
||D\psi_+||^2 - (t D\psi_+, \psi_+) = ||\nabla \psi_+||^2 - \frac{1}{2}(dt \psi_+, \psi_+) + \frac{1}{4}((k + 4d^*t)\psi_+, \psi_+) = 0,
\]

\[
||D\psi_-||^2 + (t D\psi_-, \psi_-) = ||\nabla \psi_-||^2 + \frac{1}{2}(dt \psi_-, \psi_-) + \frac{1}{4}((k + 2d^*t)\psi_- , \psi_-) = 0,
\]

Let \( KerD \neq 0 \). It follows from (3.12), Proposition 3.1 and Proposition 4.1 that we can assume that \((M, g, \nabla, T)\) is a strong RT manifold. Let \( 0 \neq \psi_+ \in KerD \) (resp. \( 0 \neq \psi_- \in KerD \)). Then, by (5.29) and (5.30), we get

\[
||\nabla \psi_+||^2 + \frac{1}{4}(k \psi_+, \psi_+) = 0, \quad ||\nabla \psi_-||^2 + \frac{1}{4}(k \psi_- , \psi_-) = 0,
\]

since \((dt \psi, \psi)\) is an imaginary complex number. Hence, \( \nabla \psi_+ = 0 \) (resp. \( \nabla \psi_- = 0 \)) by (5.31) and \( k \geq 0 \). The Theorem 1.1 follows from Corollary 4.3. Q.E.D.
Example 1. The next examples, coming originally from [20] and which have been considered in [21], show that there exist (locally) a strong non-closed and non-flat HKT structures and therefore admitting parallel (+)-spinors but do not admit parallel (-)-spinors. We take the description from [21].

Let \( u : U \subset \mathbb{C}^2 \rightarrow \mathbb{C} \) be a holomorphic function and consider the Riemannian metric \( g_u = dz^1 \bar{d}z^1 + |u|^2 dz^2 \bar{d}z^2 \). A hyperhermitian structure \((J_1, J_2, J_3)\) compatible with the opposite orientation is defined by \( J_1 + \sqrt{-1} J_2 = udz^1 \bar{d}z^1 + udz^2 \bar{d}z^2 \), \( \sqrt{2} J_3 = dz^1 \bar{d}z^1 - |u|^2 dz^2 \bar{d}z^2 \). The common Lee form is \( \theta = \frac{1}{4} \frac{\partial u}{\partial z^2} dz^2 \). It is not closed provided \( \frac{\partial u}{\partial z^1} \neq 0 \) and it is an easy computation to show \( d^* \theta = 0 \). Thus, \((U, g, (J_1, J_2, J_3))\) is a HKT manifold with respect to the Bismut connection \( \nabla \) admitting \( \nabla \)-parallel (+)-spinor. By (2.2) \( 2 \text{Skew}(Ric^\nabla) = d^* T = d \theta \neq 0 \).

Hence, the Ricci tensor is not zero and therefore there are no non-zero \( \nabla \)-parallel (-)-spinor by Theorem 4.2.

More examples of this type are supplied by non-closed HKT structures with triholomorphic Killing vector field described in [13].

Example 2. We take the next example from [23]. Let \( \mathcal{H} \) be the space of quaternions. Consider the metric \( h ds^2(\mathcal{H}), h = 1 + \frac{1}{|q|^2}, q \in \mathcal{H} - \{0\} \) and \( ds^2(\mathcal{H}) \) is the flat metric on \( \mathcal{H} \), the 3-form \( T = -* dh \), where \( * \) is the Hodge *-operator with respect to \( ds^2(\mathcal{H}) \). The geometry of the RT manifolds \((\mathcal{H} - \{0\}, h ds^2(\mathcal{H}), \nabla^\pm, \pm T)\) is a part of the NS-5-brane soliton of IIA supergravity [23]. These RT structures are in fact HKT with respect to the hyperhermitian structure defined by right and left multiplication by the imaginary unit quaternions, respectively [23]. It is shown in [23] that the holonomy of \( \nabla^\pm \) is SU(2) and therefore there are only \( \nabla^\pm \)-parallel (+)-spinor and \( \nabla^- \)-parallel (-)-spinor by Theorem 4.2.

6 Eigenvalue estimate

On a RT manifold \((M, g, \nabla, T)\) particularly important is the Dirac operator \( D^{1/3} \) of the metric connection with torsion \( \frac{1}{3} T \). As we have already mentioned in the case of complex manifold this operator coincides with the Dolbeault operator \( \sqrt{2}(\bar{\partial} + \bar{\partial}^*) \) provided that \( \nabla \) is the Bismut connection of the Hermitian structure [5]. An eigenvalue estimate for the Dolbeault operator on compact hermitian spin surface is given in [3]. We show below that the same estimate is valid for any compact RT manifold.

Proposition 6.1 Let \((M, g, \nabla, T)\) be an oriented four dimensional compact spin RT manifold with non-negative conformal scalar curvature. Then the first eigenvalue \( \lambda \) of the Dirac operator \( D^{1/3} \) satisfies the inequality

\[
\lambda^2 \geq \frac{1}{6} \inf_{\mathcal{M}} s.
\]

There is an equality in (2.1) if and only if there exists a \( \nabla \)-parallel spinor in \( \Sigma_+ M \) (hence the conformal scalar curvature is identically zero), the scalar curvature is constant and \( M \) is one of the spaces listed in Theorem 1.1.
Proof. Let $\phi$ be an eigenspinor of $D^{1/3}$ with eigenvalue $\lambda$ i.e. $D^{1/3}\phi = \lambda\phi$. Then $\phi = \phi_+ + \phi_-$ and $(D^{1/3})^2 = \lambda^2\phi_+$. We obtain from (2.8), using (3.11), that

$$\lambda^2 |\phi_+|^2 = |\nabla\phi_+|^2 + \frac{1}{6} \left( s + \frac{k}{2} \right) |\phi_+|^2. \quad (6.33)$$

Hence, if $k \geq 0$ we get the estimate (6.32) and the equality is attended iff $\nabla\phi_+ = 0$. The assertion follows from Corollary 4.3. Q.E.D.

Note that for the Gauduchon metric $g_G$ the expression (3.11) yields $s_G = k_G + \frac{3}{2} |t_G|^2$. Now, the condition $k \geq 0$ implies $s_G \geq 0$. In the Gauduchon metric (6.33) holds also for (-)-spinors. The dimension of the kernel of $D^{1/3}$ is conformally invariant by applying Proposition 4.1 for the metric connection with torsion $\frac{1}{4} T$. Hence, $Ker D^{1/3} = 0$ and by (6.33) the eigenvalue $\lambda^2 > 0$ provided $t$ is not identically zero.

Acknowledgments.

The research was done during the visit of the second author (S.I.) at the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, Spring 2000. S.I. thanks the Abdus Salam ICTP for support and the excellent environment. The authors thank G. Papadopoulos for his interest in this work. S. Ivanov is supported by Contract MM 809/1998 with the Ministry of Science and Education of Bulgaria, Contract 238/1998 with the University of Sofia “St. Kl. Ohridski” and the EDGE Contract with the EC.

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