Reduced Models and Noncommutative Gauge Theories

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Abstract

This is a short review of the relation between the Reduced Models and Noncommutative Yang–Mills Theories (NCYM) based on the works [13, 14, 15].

Contents:
1. Twisted Eguchi–Kawai model (TEK),
2. Mapping onto NCYM,
3. Morita equivalence,
4. Fundamental matter,
5. Wilson loops in NCYM,


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1 Introduction

The recent interest in noncommutative gauge theories has been inspired by the work [1] on compactification of Matrix Theory [2, 3]. Matrix Theory is generically of the type of reduced models [4] where (infinite) matrices are space-independent while the space-dependence appears when expanding around a classical vacuum. In order for a reduced model to be equivalent to the 't Hooft limit of a large-$N$ quantum field theory on the continuum space-time, it should be either quenched [5] or twisted [6, 7, 8, 9].

It has been recently recognized [10] that there exists a limit of the twisted reduced models when they describe quantum field theories on the noncommutative space represented by operators $\hat{x}^\mu$ with the commutation relation

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \quad (1.1)$$

The multiplication of the fields is given by the star-product

$$\phi_1(x) \ast \phi_2(x) \overset{\text{def}}{=} \phi_1(x) e^{\frac{i}{2} \hat{\partial}_\mu \theta^{\mu\nu} \hat{\partial}_\nu} \phi_2(x) \quad (1.2)$$

while the action of noncommutative $U_\theta(1)$ theory is

$$S = \frac{1}{4e^2} \int F^2 \quad (1.3)$$

with $F$ being the noncommutative field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i (A_\mu \ast A_\nu - A_\nu \ast A_\mu). \quad (1.4)$$

The description [8] of the planar limit of the $U(N)$ Yang–Mills theory is associated with $\theta \to \infty$. This relation between the twisted reduced models and noncommutative gauge theories has been further elaborated in Refs. [11, 12, 13, 14, 15]. In particular, the proper observables of noncommutative gauge theories have been constructed.

I review in this talk the relation between the twisted reduced models and noncommutative gauge theories. A particular emphasis is given to equivalences between some noncommutative and ordinary (or “commutative”) gauge theories known as Morita equivalence [16]. The simplest example is the above-mentioned equivalence of the large-$\theta$ limit of noncommutative $U_\theta(1)$ gauge theory and the large-$N$ limit of ordinary $U(N)$ Yang–Mills. Another example is that noncommutative $U_\theta(1)$ gauge theory in a box with periodic boundary conditions is equivalent at some rational values of $\theta$ to ordinary Yang–Mills theory in a smaller box with twisted boundary conditions representing the 't Hooft flux. These results are obtained starting from the twisted reduced model at finite $N$ which is mapped onto noncommutative gauge theory on a lattice of finite spatial extent thereby providing rigorous results for a regularized quantum theory. An extension of Morita equivalence to the case when fundamental matter is incorporated is described using the proper twisted reduced model [17]. The path-integration over the fundamental matter determines observables in noncommutative gauge theory which are expressed via both closed and open Wilson loops. A description of Morita equivalence as T-duality in the D-brane language in presented.
2 Twisted Eguchi–Kawai model

2.1 The definition [8]

The twisted Eguchi–Kawai model (TEK) is built out of $D N \times N$ unitary matrices $U_{\mu i}^{ij}$ ($\mu = 1, \ldots, D$). The partition function

$$Z_{\text{TEK}} = \int \prod_{\mu} dU_{\mu} e^{\frac{i}{2g^{2}} \sum_{\mu \neq \nu} Z_{\mu \nu}^* \text{tr} U_{\mu} U_{\nu} U_{\mu}^T U_{\nu}^T + \text{h.c.}}$$

is of the type of Wilson’s lattice gauge theory on a unit hypercube with twisted boundary condition. The twisting factor $Z_{\mu \nu}$ is

$$Z_{\mu \nu} = e^{4\pi i n_{\mu \nu}/N} \in \mathbb{Z}_N \quad \text{(integral } n_{\mu \nu} = -n_{\nu \mu})$$

where we assume $N$ to be odd.

The twisted Eguchi–Kawai model possesses the symmetries

$$\text{gauge : } U_{\mu} \rightarrow \Omega U_{\mu} \Omega^\dagger,$$

$$Z_{N}^{D} : U_{\mu} \rightarrow Z_{\mu} U_{\mu} \quad (Z_{\mu} \in \mathbb{Z}_N).$$

The vacuum state is given modulo a gauge transformation by

$$U_{\mu}^{\text{cl}} = \Gamma_{\mu}$$

where $\Gamma_{\mu}$ are twist eaters obeying the Weyl–’t Hooft commutation relation

$$\Gamma_{\mu} \Gamma_{\nu} = Z_{\mu \nu} \Gamma_{\nu} \Gamma_{\mu}.$$ (2.6)

They are known explicitly for any $n_{\mu \nu}$.

The simplest twist is when

$$n_{\mu \nu} = L^{D/2-1} \varepsilon_{\mu \nu}, \quad \varepsilon_{\mu \nu} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

and $N = L^{D/2}$. The group $SU(N)$ can then be decomposed in the direct product $SU(N) \supset \prod_{i=1}^{D/2} \otimes SU(L)$ so that $\Gamma_{i}$, $\Gamma_{i+1}$ ($i = 1, \ldots, D/2$) can be chosen to be Weyl’s clock and shift matrices for one of $SU(L)$’s. Then $\Gamma_{\mu}^T = 1$ for this simplest twist.
2.2 Continuum limit of TEK [9]

The continuum limit of TEK is reached when the lattice spacing \( a \to 0 \) \((N \to \infty)\) so that
\[
U_\mu = e^{iaA_\mu}, \quad \Gamma_\mu = e^{ia\gamma_\mu}, \tag{2.8}
\]
where \( A_\mu \) and \( \gamma_\mu \) are Hermitean. The relation (2.6) turns into the Heisenberg commutator
\[
[\gamma_\mu, \gamma_\nu] = iB_{\mu\nu}, \quad B_{\mu\nu} = \frac{4\pi n_{\mu\nu}}{N a^2}. \tag{2.9}
\]

The action of continuum TEK becomes
\[
S = \frac{1}{4g^2} \text{tr} \left( [A_\mu, A_\nu] - iB_{\mu\nu} \right)^2 \tag{2.10}
\]
and the vacuum configuration reads
\[
A^\text{cl}_\mu = \gamma_\mu \tag{2.11}
\]
modulo a gauge transformation \( A_\mu \to \Omega A_\mu \Omega^\dagger \). The Wilson loops of large-\( N \) Yang–Mills theory are represented by
\[
W(C) = \left\langle \frac{1}{N} \text{tr} P e^{-i \int_C d\xi^\mu \gamma_\mu} \frac{1}{N} \text{tr} P e^{i \int_C d\xi^\mu A_\mu} \right\rangle_{\text{TEK}}. \tag{2.12}
\]
They are nontrivial since \( A_\mu \)'s do not commute. The first trace on the right-hand side of Eq. (2.12) vanishes for open loops. For a closed loop it represents the flux of \( B_{\mu\nu} \) through a surface bounded by the contour.

2.3 Compactification of reduced models

A compactification to a D-torus \( \mathbb{T}^D \) can be described [1] by imposing on \( A_\mu \) the quotient condition
\[
A_\mu + 2\pi R_\mu \delta_{\mu\nu} = \Omega_\nu A_\mu \Omega^\dagger_\nu \tag{2.13}
\]
where \( \Omega_\nu \) are unitary transition matrices. Taking the trace of Eq. (2.13), we see that a solution exists only for infinite matrices (= Hermitean operators).

Exponentiating \( A_\mu \) according to Eq. (2.8) with a dimensionful parameter \( a \), we get
\[
e^{2\pi i a\delta_{\mu\nu} R_\nu} U_\mu = \Omega_\nu U_\mu \Omega^\dagger_\nu \tag{2.14}
\]
where \( U_\mu \) is unitary. This equation is a \( N \times N \) matrix discretization of Eq. (2.13) and has solutions (described below) for finite \( N \).

Taking the trace of Eq. (2.14), we conclude that \( U_\mu \) should be traceless which is the case for the twist eaters. Taking the determinant of Eq. (2.14), we conclude that \( aR_\mu N \) should be integral. The consistency of Eq. (2.14) also requires
\[
\Omega_\mu \Omega_\nu = z \Omega_\nu \Omega_\mu \tag{2.15}
\]
with \( z \in \mathbb{Z}_N \). The quotient condition (2.14) is compatible with the gauge symmetry (2.3) if \( \Omega \) commutes with the transition matrices \( \Omega_\nu \)'s.
2.4 Finite-N solution [13]

To describe a solution to Eq. (2.14), let us first introduce the Weyl basis on $gl(N)$:

$$J_k = \Gamma_1^{k_1} \cdots \Gamma_D^{k_D} e^{2\pi i \frac{k}{N} \sum_{\mu, \nu} n_{\mu, \nu} k_{\mu} k_{\nu}},$$

(2.16)

where the last factor provides a symmetric product and $J_{L-k} = J_{k}^\dagger$. These generators obey

$$J_k J_q = J_{k+q} e^{2\pi i \frac{1}{N} \sum_{\mu, \nu} n_{\mu, \nu} q_{\mu} q_{\nu}},$$

(2.17)

which results finally in noncommutativity.

Let us choose

$$\Omega_\mu = \prod_\nu \Gamma_{\mu \nu}^{m \epsilon_{\mu \nu}}$$

(2.18)

where $m$ is an integer. Then a particular solution to Eq. (2.14) is given by

$$U_{\mu}^{(0)} = \Gamma_{\mu}$$

(2.19)

while a general solution is

$$U_{\mu} = V_{\mu} \Gamma_{\mu},$$

(2.20)

where $V_{\mu}$ obeys

$$V_{\mu} = \Omega_{\nu} V_{\mu} \Omega_{\nu}^\dagger.$$  

(2.21)

The solution to Eq. (2.21) reads

$$V_{\mu}^{ij} = \sum_{k \in \mathbb{Z}_m} (J_k^{\mu})^{ij} U_{\mu}(k)$$

(2.22)

where $n = \frac{L}{m}$ is an integer. Here $k$ runs from 1 to $m$ since $\Gamma_{\mu}^L = 1$. This $V_{\mu}$ obviously commutes with $\Omega_{\nu}$.

Given the c-number coefficients $U_{\mu}(k)$ which describe dynamical degrees of freedom, we can make a Fourier transformation to get the field

$$U_{\mu}(x) = \sum_{k \in \mathbb{Z}_m} e^{2\pi i \frac{k}{am}} U_{\mu}(k)$$

(2.23)

which is periodic on a $m^D$ lattice (or equivalently on a discrete torus $T^D_m$). The spatial extent of the lattice is therefore $l = am$. The field $U_{\mu}(x)$ describes the same degrees of freedom as the (constraint) $N \times N$ matrix $U_{ij}^{\mu}$ while the unitarity condition $U_{\mu} U_{\mu}^\dagger = 1$ can be rewritten as

$$U_{\mu}(x) \star U_{\mu}^*(x) = 1,$$

(2.24)

where $U_{\mu}^*$ stands for complex conjugation and the lattice star-product is given by

$$f(x) \star g(x) = \sum_{x,y} e^{2i(\theta^{-1})_{\mu, \nu} y_{\mu} x_{\nu}} f(x+y) g(x+z)$$

(2.25)

with

$$\theta_{\mu, \nu} = \frac{a^2 mn}{\pi} \epsilon_{\mu, \nu}.$$  

(2.26)

These formulas follow from comparing expansions (2.22) with (2.23) and using Eq. (2.17). As $a \to 0$, Eq. (2.25) recovers Eq. (1.2) for the star-product in the continuum.
The twisted Eguchi–Kawai model (2.1) (in general with the quotient condition (2.14)) can be identically rewritten as a noncommutative $U_\theta(1)$ lattice gauge theory. Given the relations (2.22) and (2.23) between matrices and fields, we rewrite the action of TEK as

$$S = \frac{1}{2e^2} \sum_{x \in \mathbb{T}_m} \sum_{\mu \neq \nu} \mathcal{U}_\mu(x) \ast \mathcal{U}_\nu(x + a\hat{\mu}) \ast \mathcal{U}_\mu^\dagger(x + a\hat{\nu}) \ast \mathcal{U}_\nu^\dagger(x),$$

where $\hat{\mu}$ is a unit vector in the direction $\mu$ and the coupling constant $e^2 = g^2 N$. Analogously, the (constraint) measure $d\mathcal{U}_\mu$ turns into the Haar measure

$$\prod_\mu d\mathcal{U}_\mu \Rightarrow \prod_{x,\mu} d\mathcal{U}_\mu(x).$$

The action (3.1) is invariant under the star-gauge transformations

$$\mathcal{U}_\mu(x) \rightarrow \omega(x) \ast \mathcal{U}_\mu(x) \ast \omega^*(x + a\hat{\mu}),$$

where $\omega(x)$ is star-unitary ($\omega \ast \omega^* = \omega^* \ast \omega = 1$). Eq. (3.3) is the counterpart of Eq. (2.3).

The usual TEK (without the quotient condition) is associated with $n = 1$. Then $\Omega_\mu = \Gamma_\mu = 1$ and Eq. (2.14) becomes trivial. The results of Ref. [8] are reproduced in this case as $N \rightarrow \infty$ at finite $a$ since $\theta \rightarrow \infty$ according to Eq. (2.26). This limit is associated with the 't Hooft limit of large-$N$ Yang–Mills theory where only planar diagrams survive.

Alternatively, one can approach the continuum limit of the usual TEK (with $n = 1$) keeping $\theta$ fixed which requires $a \sim 1/\sqrt{m} = N^{-1/D}$ as $N \rightarrow \infty$. The period $l = am \sim \sqrt{m} = N^{1/D} \rightarrow \infty$ in this limit so that a noncommutative gauge theory on $\mathbb{R}^D$ is reproduced [10].

For $n > 1$ (that is TEK with the quotient condition), the noncommutativity parameter

$$\theta_{\mu\nu} = \frac{l^2}{\pi} \frac{n}{m} \varepsilon_{\mu\nu}$$

can be kept finite as $N \rightarrow \infty$ even for a finite $l$ if the dimensionless noncommutativity parameter $\Theta = n/m$ is kept finite. This means that the noncommutative theory lives on a torus [1]. The case of finite $N$ corresponds [13] to the noncommutative lattice gauge theory (3.1) which is a lattice regularization of the continuum theory. Since the spatial extent of the lattice $l = am$ is finite, one gets the relation

$$p_{\text{max}} \theta p_{\text{min}} = 2\pi n$$

between the UV cutoff $p_{\text{max}} = \pi/a$ and the IR cutoff $p_{\text{min}} = 2\pi/l$ which is similar to that discovered in Ref. [18] for $\mathbb{R}^4$ (associated with $n = 1$ as is discussed in the previous paragraph).
The continuum noncommutative gauge theory with rational values of the dimensionless noncommutativity parameter $\Theta$ has an interesting property known as Morita equivalence [16]. We shall describe it for the lattice regularization associated with the simplest twist (2.7) assuming that the ratio $m/n = \tilde{p}$ is an integer. Then the noncommutative $U_\theta(1)$ gauge theory on a $m^D$ periodic lattice is equivalent to ordinary $U(p)$ Yang–Mills theory with $p = \tilde{p}^D/2$ on a $n^D$ lattice with twisted boundary conditions (representing the 't Hooft flux) and the coupling constant $g^2 = e^2/p$ (where $e^2$ is the coupling of the $U_\theta(1)$ theory).

The twisted boundary conditions read
\begin{equation}
\tilde{V}_\mu(\tilde{x} + an\tilde{\nu}) = \tilde{\Gamma}_\nu \tilde{V}_\mu(\tilde{x}) \tilde{\Gamma}_\nu^\dagger \tag{4.1}
\end{equation}
where $\tilde{\Gamma}_\nu$ are $p \times p$ twist eaters obeying
\begin{equation}
\tilde{\Gamma}_\mu \tilde{\Gamma}_\nu = \tilde{Z}_{\mu\nu} \tilde{\Gamma}_\nu \tilde{\Gamma}_\mu, \quad \tilde{Z}_{\mu\nu} = e^{4\pi i \epsilon_{\mu\nu}/\tilde{p}} \tag{4.2}
\end{equation}
($\tilde{p}$ is also assumed to be odd). Here $\tilde{Z}_{\mu\nu} \in \mathbb{Z}_p$ is not removable since $\tilde{\Gamma}_\mu$ are $SU(p)$ matrices. It represents the non-Abelian 't Hooft flux.

We have discussed in the previous Section the equivalence of TEK (with the quotient condition in general) with $N = (mn)^D/2$ and the noncommutative $U_\theta(1)$ gauge theory on $\mathbb{T}_m^D$. Both theories have same $m^D$ degrees of freedom which are described either by Eq. (2.22) or Eq. (2.23). In the matrix language, the noncommutativity emerges since
\begin{equation}
J^n_k J^n_q = J^n_{k+q} e^{2\pi i \frac{k\mu \epsilon_{\mu\nu} q_\nu}{m}} \tag{4.3}
\end{equation}
as it follows from the general Eq. (2.17) for the given simplest twist. In the noncommutative language, the noncommutativity resides in the star-product
\begin{equation}
e^{2\pi i 1/\tilde{p}} \star e^{2\pi i 1/\tilde{p}} = e^{2\pi i \frac{(k+q)_\mu x_\nu}{\tilde{p}}} e^{2\pi i \frac{m}{m} \epsilon_{\mu\nu} q_\nu} \tag{4.4}
\end{equation}
as it follows from the definition (2.25).

When $m = \tilde{p} n$, a third equivalent model exists where the same dynamical degrees of freedom are described by a $p \times p$ matrix field
\begin{equation}
\tilde{\mathcal{V}}_{\mu}^{ab}(\tilde{x}) = \sum_{k \in \mathbb{Z}_m} \tilde{J}_{k}^{ab} e^{2\pi i \frac{k\mu}{m} \epsilon_{\mu\nu} q_\nu} \tag{4.5}
\end{equation}
Here
\begin{equation}
\tilde{J}_{k} = \prod_{\mu} \tilde{\Gamma}_{k\mu} e^{2\pi i \frac{1}{p} \sum_{\mu>\nu} \epsilon_{\mu\nu} k_\mu k_\nu} \tag{4.6}
\end{equation}
similar to Eq. (2.16). The number of degrees of freedom $n^D p^2 = m^D$ for $p = \tilde{p}^D/2$. The noncommutativity now resides in the matrix part rather than the $\tilde{x}$-dependent part since
\begin{equation}
\tilde{J}_{k+q} \cdot \tilde{J}_{q} = \tilde{J}_{k+q} e^{2\pi i \frac{1}{p} \epsilon_{\mu\nu} q_\nu}. \tag{4.7}
\end{equation}
The action of the third model is just the ordinary Wilson’s lattice action
\[ S = \frac{p}{2e^2} \sum_{\tilde{x} \in \tilde{T}_n^D} \sum_{\mu \neq \nu} \text{tr} \left( p \tilde{V}_\mu(\tilde{x}) \tilde{V}_\nu(\tilde{x} + a \mu) \tilde{V}_\mu^\dagger(\tilde{x} + a \nu) \tilde{V}_\nu^\dagger(\tilde{x}) \right). \] (4.8)

The field \( \tilde{V}_\mu(\tilde{x}) \) is quasi-periodic on \( \tilde{T}_n^D \) and obeys the twisted boundary conditions (4.1) since
\[ \tilde{\Gamma}_\mu \tilde{J}_k \tilde{\Gamma}_\mu^\dagger = \tilde{J}_k e^{2\pi i k/\tilde{p}} \] (4.9)

For \( n = 1 \) when \( \tilde{p} = m \) and \( p = N \), the third model lives on a unit hypercube with twisted boundary conditions and coincides with TEK. It is seen after the change of variables \( U_\mu = \tilde{V}_\mu \tilde{\Gamma}_\mu \). In fact this formulation was the original motivation of Ref. [8] for TEK. Therefore, the derivation of noncommutative gauge theories from TEK is a simplest example of Morita equivalence.

In the continuum limit \( (N \to \infty) \) when TEK is formulated via operators, the noncommutative \( U_\theta(1) \) gauge theory lives on \( \mathbb{T}^D \) with period \( l \) and is Morita equivalent at a rational value of \( \Theta \) to the ordinary \( U(p) \) gauge theory on a smaller torus \( \tilde{T}_D^D \) with twisted boundary conditions and \( \tilde{l} = l/\tilde{p} \). The lattice regularization makes these results rigorous [14]. Arbitrary (irrational or rational) value of \( \Theta \) can be obtained for the most general twist. As is also shown in Ref. [15], the theories on tori can be obtained from TEK by choosing a sophisticated twist instead of imposing the quotient condition.

5 Fundamental matter [14]

The results of two previous Sections can be extended to the presence of matter. Let \( \phi(x) \) is a scalar matter field in the fundamental representation of \( U_\theta(1) \). The matter part of the action is
\[ S_{\text{matter}} = -\sum_{x,\mu} \phi^*(x) \ast U_\mu(x) \ast \phi(x + a \mu) + M^2 \sum_x \phi^*(x) \phi(x) \] (5.1)
and is invariant under the star-gauge transformation
\[ \phi(x) \to \omega(x) \ast \phi(x), \quad \phi^*(x) \to \phi^*(x) \ast \omega^*(x) \] (5.2)
and Eq. (3.3) for \( U_\mu(x) \).

At a rational value of \( \Theta \), the action (5.1) on a torus is Morita equivalent to
\[ S_{\text{matter}} = -\sum_{\tilde{x},\mu} \text{tr}_{(p)} \Phi^*(\tilde{x}) \tilde{V}_\mu(\tilde{x}) \Phi(\tilde{x} + a \mu) + M^2 \sum_{\tilde{x}} \text{tr}_{(p)} \Phi^*(\tilde{x}) \Phi(\tilde{x}) \] (5.3)
where the notations of the previous Section are used and the \( p \times p \) matrix field \( \Phi^{ij}(\tilde{x}) \) obeys the twisted boundary conditions
\[ \Phi(\tilde{x} + \tilde{l} \nu) = \tilde{\Gamma}_\nu \Phi(\tilde{x}) \tilde{\Gamma}_\nu^\dagger \] (5.4)
similar to Eq. (4.1) for the gauge field. The index \( i \) of \( \Phi^{ij} \) plays the role of color while \( j \) plays the role of flavor (labeling species). The color symmetry is local while the flavor symmetry is global. In particular, the model (5.3) reduces for \( n = 1 \) to TEK for fundamental matter of Ref. [17].

The continuum limit of the above formulas is obvious. The continuum \( U_\theta(1) \) gauge theory with fundamental matter (noncommutative QED) is reproduced as \( N \to \infty \). For \( \theta \to \infty \) it is equivalent to large-\( N \) QCD on \( \mathbb{R}^D \) in the the Veneziano limit when the number of flavors of fundamental matter is proportional to the number of colors so the matter survives in the large-\( N \) limit. Again these results are rigorous since they are obtained in a regularized theory.

6 Wilson loops in NCYM

Observables in ordinary Yang–Mills theory can be expressed via the Wilson loops. The standard way to derive proper formulas is to integrate over matter fields doing the Gaussian path integral. This strategy can be repeated for noncommutative gauge theory with fundamental matter given by the action (5.1). On the lattice one can use the large-mass expansion in \( 1/M^2 \). We describe in this Section what kinds of Wilson loops then emerge.

A lattice contour \( C \) consisting of \( J \) links is defined by the set of unit vectors \( \hat{\mu}_j \) associated with the direction of each link \( j \) \((j = 1, \ldots, J)\) forming the contour. The parallel transporter from the point \( x \) to the point \( x + \ell \) \((\ell = a \sum_j \hat{\mu}_j)\) along \( C \) is given by

\[
U_\mu(x; C) = U_{\mu_1}(x) \star U_{\mu_2}(x + a\hat{\mu}_1) \star \cdots \star U_{\mu_J}(x + a \sum_{j=1}^{J-1} \hat{\mu}_j). \tag{6.1}
\]

It is star-gauge covariant:

\[
U_\mu(x; C) \to \omega(x) \star U_\mu(x; C) \star \omega^*(x + \ell) \tag{6.2}
\]

under the star-gauge transformation (3.3).

Given a function \( S_\ell(x) \) with the property

\[
S_\ell(x) \star \omega(x) \star S^*_\ell(x) = \omega(x + \ell) \tag{6.3}
\]

for arbitrary \( \omega(x) \), it is easy to show that

\[
W(C) = \sum_x S_\ell(x) \star U(x; C) \tag{6.4}
\]

is star-gauge invariant. The solution to Eq. (6.3) is

\[
S_\ell(x) = e^{i\ell_\mu \theta^{-1}_{\mu \nu} x_\nu}, \tag{6.5}
\]

where \( \ell_\mu = an j_\mu \) with integer-valued \( j_\mu \) (modulo possible windings).
The continuum limit of Eq. (6.4) defines star-gauge invariant Wilson loops in non-commutative gauge theory. In addition to closed loops, there exist on $\mathbb{R}^D$ open loops given by Eq. (6.4) with an arbitrary value of $\ell$ [11]. On $\mathbb{T}^D$ the open Wilson loops are star-gauge invariant only for discrete values of $\ell$ measured in the units of $\pi\theta/l$ [13]. The closed Wilson loops appear in the sum-over-path representation of the matter correlator $\langle \phi^*(x) \star \phi(x) \rangle_\phi$ while the open Wilson loops appear for $\langle \phi^*(x) \star S_\ell(x) \star \phi(x + \ell) \rangle_\phi$ [14]. For integral $m/n = \tilde{p}$, the open Wilson loops in noncommutative $U(1)$ gauge theory become the Polyakov loops winding around $\mathbb{T}^D$ in the Morita equivalent $U(p)$ Yang–Mills theory with twisted boundary conditions.

### 7 D-brane interpretation [19]

The results about the Morita equivalence have a simple interpretation via T-duality in the D-brane language. Let us consider the case of a more general twist in $D = 4$ when $SU(p)$ is decomposed as $SU(p) \supset 1_{\tilde{p}_0} \otimes SU(\tilde{p}_1) \otimes SU(\tilde{p}_2)$ ($p = \tilde{p}_0 \tilde{p}_1 \tilde{p}_2$). The simplest twist (2.7) above corresponds to $\tilde{p}_0 = 1$, $\tilde{p}_1 = \tilde{p}_2 = \tilde{p}$.

Let us consider the system of $\tilde{p}_0$ D3-branes populated by $\tilde{p}_0 \tilde{p}_1$ D1-branes localized in the 1,2-plane, $\tilde{p}_0 \tilde{p}_2$ D1-branes localized in the 3,4-plane, and $p = \tilde{p}_0 \tilde{p}_1 \tilde{p}_2$ D-instantons. It is associated with noncommutative $U(\tilde{p}_0)$ gauge theory whose dimensionless noncommutativity parameters equal [20]

$$
\Theta_{12} = \frac{\#D3}{\#D1} = \frac{1}{\tilde{p}_1}, \quad \Theta_{34} = \frac{\#D3}{\#D1} = \frac{1}{\tilde{p}_2}.
$$  

(7.1)

After the T-duality transformation both in the 1,2 and 3,4 planes, we get the system of $p$ D3-branes populated by $\tilde{p}_0 \tilde{p}_1$ D1-branes oriented in the 1,2-plane, $\tilde{p}_0 \tilde{p}_2$ D1-branes oriented in the 3,4-plane, and $\tilde{p}_0$ D-instantons. Now

$$
\tilde{\Theta}_{12} = \frac{\#D3}{\#D1} = \tilde{p}_1, \quad \tilde{\Theta}_{34} = \frac{\#D3}{\#D1} = \tilde{p}_2
$$  

(7.2)

represent magnetic fluxes in the ordinary $U(p)$ gauge theory. The period matrix becomes $\tilde{\Sigma} = \text{diag}(l/\tilde{p}_1, l/\tilde{p}_2)$ due to the fluxes.

It is interesting to note the presence of $\tilde{p}_0$ D-instantons in the T-dual theory. They provide vanishing of the topological charge $Q = \#D3\#D(-1) - \#D1\#D1 = 0$ as it should for the given twist to provide zero-action vacuum configuration.

It is not clear how to extend the interpretation of this Section to the case when the fundamental matter is incorporated.

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